Exercises for Higher Structures in Differential Geometry SS 2013

Sheet 06

Exercise 29

Let $\pi\colon Y \to Z$ be a principal G-bundle. Show that $Y \times_Z Y$ is isomorphic to the trivial principal G-bundle over Y. **Hint:** $(y, y') \in Y \times_Z Y$ gives rise to a unique $g \in G$ with $y' = y \cdot g$.

Exercise 30

Let X be a lcs such that the topology of X is induced by a countable family of semi-norms $(p_n)_{n\in\mathbb{N}}$.

a) Show that if we set $p'_n := \sum_{i\leq n} p_i$, we obtain another family of semi-norms inducing the same topology on X. We may thus w.l.o.g. assume that the family satisfies $p_n \leq p_{n+1}$.

b) Show that the following statements are equivalent conditions for a sequence (x_k) in X and $p \in X$:

   i) $(x_k) \xrightarrow{k\to\infty} p$ in the topology of X.

   ii) $(d(x_k, p)) \xrightarrow{k\to\infty} 0$, where d is the metric $d(x, y) := \sum_{n\in\mathbb{N}} 2^{-n} \frac{p_n(x-y)}{1+p_n(x-y)}$.

   iii) $(p_n(x_k - p)) \xrightarrow{k\to\infty} 0$ for each n.

   Moreover, if the family satisfies $p_n \leq p_{n+1}$, then show that any of these conditions is implied by

   iv) $(p_k(x_k - p)) \xrightarrow{k\to\infty} 0$.

Exercise 31

Show that for a Hausdorff space X and a metrisable space Y the topology of compact convergence equals the compact open topology. **Hint:** This involves various typical compactness arguments.

Exercise 32

Show that a morphism of diffeological spaces is continuous for the d-topology.

Exercise 33

Show that if M, N are manifolds, M is compact, N is locally metrisable and $O \subseteq N$ is open, then $C^\infty(M, O)$ is open in the d-topology on $C^\infty(M, N)$.