Exercises for Higher Structures in Differential Geometry SS 2013

Sheet 04

Exercise 16

Show the only if part of Proposition I.35: If each representable functor on a site $(\mathcal{C}, K)$ is a sheaf, then the site is subcanonical.

Exercise 17

Show the only if part of Lemma I.36: If the covers of a coverage consist only of singletons $\{f\colon D \to C\}$, then the morphism $f\colon D \to C$ is a colimit (or coequaliser) for the two canonical morphism $D \times_C D \rightrightarrows D$ if the site $(\mathcal{C}, K)$ is subcanonical.

Exercise 18

Show that open covers (as in Example I.32 b) and Definition I.33) yield indeed a coverage on **Top** and **Euc**. Moreover, show that these sites are subcanonical.

Exercise 19

Show that a sequence $A \to B \to C$ of abelian group, such that the composition $A \to C$ is 0, is short exact if and only if the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & 0 \arrow[d] \\ B \arrow[r] & C \end{tikzcd}$$

is cartesian and cocartesian.

Exercise 20

More generally than in Definition I.29 one defines a coverage to be a function K that assigns to each object C of $\mathcal{C}$ a collection $K(C)$ of C-families of morphisms, called *covers* of C, such that

> If $\{f_i\colon D_i \to C \mid i \in I\} \in K(C)$ and $\varphi\colon X \to C$ is any morphism, then there exists a cover $\{g_j\colon Y_j \to X \mid j \in J\}$ of X such that each for each $j \in J$ there exists some $i \in I$ and $\psi\colon Y_j \to D_i$ such that $\varphi \circ g_j = f_i \circ \psi$.

A matching family for a presheaf $F\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$ with respect to $R \in K(C)$ is then an element $(x_i) \in \prod_{i\in I} F(D_i)$ such that for each pair $\pi\colon E \to D_i$ and $\rho\colon E \to D_j$ satisfying $f_i \circ \pi = f_j \circ \rho$ we have $F(\pi)(x_i) = F(\rho)(x_j)$. With this notion of matching family, F is defined to be a sheaf if for each object C and each cover $R \in K(C)$ each matching family has a unique amalgamation.

a) Let K be a coverage of $\mathcal{C}$ in the sense of Definition I.29. For each object C and $R = \{f_i\colon D_i \to C \mid i \in I\}$ in $K(C)$ set

$$\overline{R} := \{f\colon D \to C \mid \text{there exists } i \in I \text{ and } \varphi\colon D \to D_i \text{ such that } f = f_i \circ \varphi\}$$

Then show that $\overline{K}(C) := \{\overline{R} \mid R \in K(C)\}$ defines a coverage in the above sense.

b) Show that $F \in \mathbf{PSh}_{\mathcal{C}}$ is a sheaf (in the sense of Definition I.30) with respect to K if and only if it is a sheaf (in the sense of the above definition) with respect to $\overline{K}$.

Exercise 21

Show that the category **Man**, together with the surjective submersion coverage is a subcanonical site.

Exercise 22

Show that the forgetful functor **Man** $\to$ **Top** does in general not preserve pull-backs. **Hint:** If $f\colon \mathbb{R} \to \mathbb{R}$ is smooth such that $f^{-1}(0) = \{\frac{1}{n} \mid n \in \mathbb{N}^{+}\} \cup \{0\}$, then the pull-back of f and $\{0\} \hookrightarrow \mathbb{R}$ in **Man** is $\{\frac{1}{n} \mid n \in \mathbb{N}^{+}\} \cup \{0\}$ with the *discrete* smooth structure.