Exercises for Higher Structures in Differential Geometry SS 2013

Sheet 01

Exercise 01

If $\mathcal{C}$ and $\mathcal{D}$ are categories and $\mathcal{C}$ is small, show that functors $F, G\colon \mathcal{C} \to \mathcal{D}$, together with natural transformations $\alpha\colon F \Rightarrow G$ form a category. It is part of the exercise to determine the composition and identity morphism.

Exercise 02

Show that in **Ab** the equaliser of two arbitrary morphisms f and g always exists and is given by the kernel of $f - g$. Then show that also in **Set** equalisers always exist.

Exercise 03

Let $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$ be two functors and let $\varepsilon\colon \mathrm{id}_{\mathcal{C}} \to GF$ and $\delta\colon FG \to \mathrm{id}_{\mathcal{D}}$ be natural transformations such that the compositions

$$F \xrightarrow{C \mapsto F(\varepsilon(C))} FGF \xrightarrow{C \mapsto \delta(F(C))} F \quad \text{and} \quad G \xrightarrow{D \mapsto \varepsilon(G(D))} GFG \xrightarrow{D \mapsto G(\delta(D))} G$$

are the identity transformations on F and G respectively. Then show that

$$\eta(C, D)\colon \mathcal{D}(F(C), D) \to \mathcal{C}(C, G(D)), \quad \varphi \mapsto G(\varphi) \circ \varepsilon(C)$$

is an adjunction $\eta\colon F \dashv G$.

Exercise 04

Determine the initial and terminal objects in **Set** and **Ab**. Conclude that right adjoints do not preserve colimits in general.

Exercise 05

Show that if $F\colon \mathcal{C} \to \textbf{Set}$ is representable, then the object of $\mathcal{C}$ that represents F is uniquely determined up to isomorphism.