Algebra II

JProf.-Dr. Christoph Wockel

28. Februar 2013

Modules over Rings

Lecture 1 (October 16)

- Def.: Ring (morphism), left- right and bimodule (morphism)
- Ex. (rings): $\mathbb{Z}$, $k[x]$ (polynomials), continuous, smooth or analytic functions
- Ex. (morphism): initial and terminal morphism $\mathbb{Z} \to R$ and $R \to 0$
- Ex. (module): R over itself, vector spaces
- Prop.: $\mathrm{Hom}_R(M, N)$ is naturally an abelian group and if R is commutative, then it also is naturally an R-module.

Lecture 2 (October 19)

- Opposite ring, anti-homomorphis, exchanging left- and right modules
- Ex. (modules): $M \cong \mathrm{Hom}_R(R, M)$ (if R commutative),
  abelian groups as $\mathbb{Z}$-modules, rings as $\mathbb{Z}$-algebras
- Def.: R-algebra (e.g., $k[x]$)
- Rem.: A is R-alg. $\triangleq$ $\Phi\colon R \to A$ with $\Phi(r)a = a\Phi(r)$
- Lem.: Modules over R $\triangleq$ morphisms $R \to \mathrm{End}(M)$
- Def.: Polynomial ring/algebra (A, X) over R
- Prop.: Universal property of (A, X)
- Rem.: Evaluation homomorphism
- Lem.: Modules over $R[X]$ $\triangleq$ R-modules + R-linear map
- Def.: Group ring/group algebra $R[G]$

Lecture 3 (October 23)

- Lem.: $R[G]$ is a ring (an R-algebra if $R = R^{\mathrm{op}}$)
- Def.: Representation $\rho_V := (\rho, V)$ of G on a k-vector space V
- Rem.: $(V, \rho) \triangleq$ left action on V by linear maps;
  canonical and trivial rep., rep. of $\mathbb{Z}$ and $\mathbb{Z}_2$
- Lem.: Rep. of G (over k) $\triangleq$ $k[G]$-modules
- Def.: Morphisms of representations, submodules (of general R-modules)
- Rem.: Subgroups, ideals and subrep. are submodules; kernels and
  images are submodules; quotient modules, relation to ideals
- Lem.: Restriction of scalars (pull-back)
- Def.: Generated submodule, generating system, finitely generated
  and cyclic module
- Prop.: Fundamental Homomorphism Theorems

Lecture 4 (October 26)

- Def.: Annihilator, faithful module, torsion element,
  torsion submodule $\operatorname{Tor}(M)$, torsion free module
- Prop.: $M/\operatorname{Tor}(M)$ is torsion free if R is an integral domain.
- Def.: Direct product $(M, (\pi_i\colon M \to M_i)_{i\in I})$ and direct sum
  $(N, (\iota_i\colon M_i \to N)_{i\in I})$ of a family $(M_i)_{i\in I}$ of R-modules
- Uniqueness up to unique isomorphism, existence
- Sum of submodules, direct (internal) sum of submodules
- Direct sum of representations and of $k[G]$-modules
- Example via Chinese Remainder Theorem: $\mathbb{Z}_6 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_3$
- Def.: Tensor product $M \otimes_A N$ of A-modules (A commutative)
- Lem./Prop.: Uniqueness and existence of the tensor product

Lecture 5 (October 30)

Note: The section on the tensor product followed closely Section VII.10 in the book "Algebra" of Jantzen and Schwermer, cf. http://www.springerlink.com/content/978-3-540-21380-2.

- Lem./Prop.: Uniqueness and existence of the tensor product

- Rem.: Notation $(M \otimes_A N, \otimes_A)$ for "the" tensor product,
  universal properties in terms of bijections of Hom-sets,
  tensor product of module morphisms,
  properties of the tensor product: $0 \otimes M \cong 0$, $A \otimes_A M \cong M$,
  $M \otimes N \cong N \otimes M$ and $M \otimes (N \otimes P) \cong (M \otimes N) \otimes P$

- Ex.: $A^n \otimes_A A^m \cong A^{nm}$, $A[X] \otimes_A A[Y] \cong A[X,Y]$, $\mathbb{Z}_n \otimes \mathbb{Z}_m \cong \mathbb{Z}_{\gcd(n,m)}$
  $\mathrm{Tor}(A \otimes_{\mathbb{Z}} \mathbb{Q}) \cong 0$ and $\mathrm{Tor}(A) \otimes_{\mathbb{Z}} \mathbb{Q} \cong 0$

- Tensor products over non-commutative rings, tensor products of bimodules

- Def.: Linearly independent elements of a module, basis, free module

Lecture 6 (November 2)

- Rem.: Modules over fields are free, but not over $\mathbb{Z}$ (e.g. $\mathbb{Z}_n$),
  M is free iff $M \cong \bigoplus_{x \in X} R$ for some X,
  $|X| = n < \infty \Rightarrow M \cong R^n$,
  Homomorphism between free modules as matrices
- Prop.: Different bases of a free module M over a *commutative* ring have the same cardinality (called *rank* of M).
- Rem.: For R non-commutative $R^n \cong R^m \not\Rightarrow m = n$ (in general)
- Prop.: Each module is quotient of a free module.
- Prop.: $f\colon M \to F$ surjective and F free $\Rightarrow M \cong \ker f \oplus F$
- Cor.: N submod. of M with M/N free $\Rightarrow N$ complemented
- Def.: Sequence, chain complex, exact sequence, *short* exact sequence

Lecture 7 (November 6)

- Prop.: Equivalent conditions for a *split* short exact sequence
- Examples of short exact sequences: vector spaces (always split), abelian groups (not always split), $k[\mathbb{Z}]$-modules (not always split) and $k[G]$ modules for G finte gp. (always split)
- Def.: Push-forward, pull-back of morphisms
- Lem.: $\mathrm{Hom}(M, \cdot)$ preserves "left exactness" of short exact sequences.

- Prop.: Equivalent conditions for $\mathrm{Hom}(M, \cdot)$ to preserve "right exactness":

  1. For each diagram

$$\begin{tikzcd} & M \arrow[dl, dotted] \arrow[d] & \\ N_1 \arrow[r] & N_2 \arrow[r] & 0 \end{tikzcd}$$

     with exact bottom row there exists a lift (i.e. a morphism along the dotted arrow making the diagram commute).
  2. Each short exact sequence $N' \to N \to M$ splits
  3. The exists Q such that $M \oplus Q$ is free
  4. For each short exact sequence $T' \xrightarrow{\iota} T \xrightarrow{\pi}$ the sequence

$$\mathrm{Hom}(M, T') \xrightarrow{\iota_*} \mathrm{Hom}(M, T) \xrightarrow{\pi_*} \mathrm{Hom}(M, T'')$$

     is also exact.

- Def.: Projective module

Lecture 8 (November 6)

- Prop.: Equivalent conditions for $\mathrm{Hom}(\cdot\,, M)$ to preserve "right exactness":
  1. For each diagram
  $$\begin{tikzcd} 0 \arrow[r] & N_1 \arrow[r] \arrow[d] & N_2 \arrow[dl, dotted] \\ & M & \end{tikzcd}$$
  with exact top row there exists a lift.
  2. Each short exact sequence $M \to N \to N''$ splits
  3. For each short exact sequence $T' \xrightarrow{\iota} T \xrightarrow{\pi} T''$ the sequence
  $$\mathrm{Hom}(T'', M) \xrightarrow{\pi^*} \mathrm{Hom}(T, M) \xrightarrow{\iota^*} \mathrm{Hom}(T', M)$$
  is also exact.
- Def.: Injective module
- Prop.: Baer's Criterion for injectivity of a module

From now on: $M :=$ R^{op}-module.

- Prop.: $M \otimes_R \cdot$ preserves "right exactness" of short exact sequences.
  M projective $\Rightarrow$ $M \otimes_R \cdot$ also preserves "left exactness".

- Def.: M is *flat* if $M \otimes_R \cdot$ also preserves "left exactness", i.e., if

$$\iota\colon N' \to N \text{ injective } \Rightarrow \mathrm{id}_M \otimes \iota \text{ injective.}$$

- Def.: *divisible* module (if $m \mapsto r \cdot m$ is surjective for all $r \in R$).

- Prop.: Over a pid (principal ideal domain) divisibility and flatness
  are equivalent.

Finiteness and Simplicity

Note: Large parts of the material of this section is taken from Chapter VII and VIII of the book "Algebra" of Jantzen and Schwermer.

Lecture 9 (November 13)

Unless stated otherwise: R : ring, M, N : R-modules

- Prop.: TFAE (for M an R-module)

  a) Each increasing sequence $N_1 \subseteq N_2 \subseteq \cdots$ of submodules becomes stationary.

  b) Each non-empty set of submodules has a maximal element.

  c) Each submodule is finitely generated.

- Def.: Noetherian module and ring

- Ex.: pids and finite-dimensional k-algebra modules are Noetherian.

- Prop.: Noetherian is an *extension property*, i.e., if $N' \to N \to N''$ is a short exact sequence, then N is Noeth. iff N', N'' are so.

- Prop.: If R is Noetherian, then M Noeth. $\Leftrightarrow$ M fin. gen.

- Hilbert's Basis Theorem: R noetherien $\Rightarrow$ $R[X]$ Noetherian.

- Def.: Finitely cogenerated: $\bigcap_{i \in I} N_i = \{0\} \Rightarrow \bigcap_{i \in F} N_i = \{0\}$ for some $|F| < \infty$.
- Prop.: (dually to above) TFAE
  a) Each decreasing sequence $N_1 \supseteq N_2 \supseteq \cdots$ of submodules becomes stationary.
  b) Each non-empty set of submodules has a minimal element.
  c) Each submodule is finitely cogenerated.
- Def.: Artinian module and ring
- Prop.: Artinian is an extension property.
- Each left Artinian ring is also left noetherian, but $\mathbb{Z}$ is not Artinian!
- Ex.: finite-dimensional k-algebra modules are Artinian.

Lecture 10 (November 16)

- Def.: simple and indecomposable module (and representation).
- Ex.: simple and indecomposable modules over $R = k$ a field, $G = \mathbb{Z}$, $R = k[\mathbb{Z}]$ and $k[G]$ for $|G| < \infty$
- Lem.: M simple $\Leftrightarrow$ $M = \langle x \rangle$ for all $x \in M$. For arbitrary M, $x \in M$ and $\varphi(r) := r \cdot m$ we have $\langle x \rangle$ simple $\Leftrightarrow$ $\ker(\varphi)$ maximal ideal.
- Lem.: M : simple, N : arbitrary

  a) each $\varphi\colon M \to N$ is either injective or zero

  b) each $\varphi\colon N \to M$ is either surjective or zero

  c) each $0 \neq \varphi \in \mathrm{End}_R(M)$ is invertible.
- Lem. (Schur): k : alg. closed field, A : k-alg. M : simple A-module with $\dim_k(M) < \infty$. Then

$$k \to \mathrm{End}_A(M), \quad \lambda \mapsto \lambda \cdot \mathrm{id} \qquad \text{is an isomormism.}$$

- Def.: composition series: $M = M_0 \supset M_1 \supset \cdots \supset M_r = 0$ with M_i/M_{i+1} simple. M of finite length $\Leftrightarrow \exists$ composition series.
- Lem.: M of finite length $\Leftrightarrow$ M Artinian and Noetherian.
- Def.: Equivalence and refinements of sequences of submodules.
- Lem. (Schreier): Each two sequences have refinement$s$ that are equivalent.
- Prop. (Jordan-Hölder): Each two composition series are equivalent.
- Cor.: M: finite length, $N \leqslant M \Rightarrow l(M) = l(N) + l(M/N)$
- Cor.: R: of finte length ofer itself $\Rightarrow$ each simple R-modules is quotient of R for each composition series $R = R_0 \supset \cdots \supset R_r = 0$ we have $M \cong R_i/R_{i+1}$ for some i.
- Cor.: k : field, $k \subseteq R$, $\dim_k(R) < \infty \Rightarrow \exists$ up to isomorphism only finitely many simple R-modules.
- Cor.: G : finite group $\Rightarrow \exists$ up to isomorphism only finitely many simple $k[G]$-modules.

Lecture 11 (November 20)

- Def.: Semi-simple module (direct sum of free R-modules)
- Ex.: vector spaces, $\mathbb{Z}_2$, $\mathbb{Z}_2 \times \mathbb{Z}_2$ are semi-simple, $\mathbb{Z}$ and $\mathbb{Z}_4$ are not.
- Lem.: $N \leqslant M$, $(M_i)_{i \in I}$: family of submodules of M

  a) $\sum_{i \in I} M_i$ direct $\Leftrightarrow \sum_{j \in F} M_j$ direct for each $F \subseteq I$ finite

  b) Each M_i simple and $N + \sum_{i \in I} M_i = M \Rightarrow \exists J \subseteq I$ s.th.

$$M = N \oplus \bigoplus_{j \in J} M_j.$$

- Prop.: TFAE:

  a) M is semi-simple.

  b) M is sum of simple modules.

  c) Each submodule of M has a complement.

- Cor.: Submodules and quotients of semi-simple modules are so.

- Def.: Simple and semi-simple ring.
- Ex.: Fields are semi-simple, products of semi-simple rings are so.
- Ex.: $R = M_n(D)$ for D a division ring is semi-simple and each semi-simple ring is isomorphic to a product of such.
- Note: R pid, not a filed $\Rightarrow$ R not semi-simple
- Prop.: R semi-simple
  a) Each R-module is semi-simple
  b) There are (up to isom.) only finitely many simple R-modules.
- Def.: $\mathrm{rad}(M) := \bigcap\{N \leqslant M \mid N \text{ is maximal submodule}\}$
- Rem.:
  a) If no maximal submod. exist, then $\mathrm{rad}(M) = 0$.
  b) $\mathrm{rad}(M) = \bigcap\{\ker(\alpha) \mid \alpha\colon M \to E, \text{ with } E \text{ simple}\}$.
  c) M semi-simple $\Rightarrow$ $\mathrm{rad}(M) = 0$.
  d) $\mathrm{rad}(\mathbb{Z}) = 0$.

Lecture 12 (November 23)

- Lem.: morphisms, direct sums and quotients are compatible with the radical, $\mathrm{rad(M/\,rad(M))} = 0$.
- Cor.: M Artinian $\Rightarrow M/\operatorname{rad}(M)$ is semi-simple.

From now on let R be a pid and M, N be R-modules of finite rank.

- M free, $N \leqslant M \Rightarrow N$ is free.
- Thm. (Elementary Divisor Theorem): $n = \operatorname{rk}(M)$, $N \leqslant M$. Then $\exists$ basis $v_1, ..., v_n$ of M and $a_1, ..., a_n$ s.th. $a_1 \mid \cdots \mid a_n$ and $V = \sum Ra_iV_i$.
- Cor.: $M \cong R/(a_1) \oplus \cdots \oplus R/(a_m)$ for some $a_1, ..., a_m \in R$ s.th. $a_i \notin R^*$ and $a_1 \mid \cdots \mid a_m$.
- Rem.: $\mathcal{P}$: rep. system of prime elt.'s modulo units $\Rightarrow$

$$M \cong R^{n_0} \oplus \bigoplus_{p\in\mathcal{P}} \bigoplus_{r>0} (R/(p^r))^{n(p,r)}$$

with only finitely many $n(p,r)$ non-zero and n_0 and $n(p,r)$ unique.

Lecture 13 (November 27)

- Def.: Categories
- Ex.: **Set** (sets), **Gp** (groups), **R-Mod** (R-modules), **R-S-Bimod**, $\mathbf{Alg_R}$, **Top**, $\varnothing$, $*$, pair groupoid $\mathcal{P}_X$ of a set X, category from a poset
- Def.: $\mathcal{C}^{\mathrm{op}}$, $\mathcal{C} \coprod \mathcal{D}$, $\mathcal{C} \times \mathcal{D}$
- Def.: Functors
- Ex.: Forgetful functors, duals and double duals of vector spaces, $\coprod$, $\otimes_R$, $\mathrm{Hom}_{\mathcal{C}}(X, \cdot)\colon \mathcal{C} \to \mathbf{Set}$, $\mathrm{Hom}_{\mathcal{C}}(\cdot\,, X)\colon \mathcal{C} \to \mathbf{Set}^{\mathrm{op}}$.

Lecture 14 (November 30)

- Def.: Isomorphism of categories (note: is a very rigid concept)
- Ex.: $\mathbf{R\text{-}Mod} \cong \mathbf{Mod\text{-}R^{op}}$; $k[X] - \mathbf{Mod} \cong \mathbf{k\text{-}Mod}$+lin. End.
- Def.: Natural transformation $\alpha \colon F \Rightarrow G$ btw. functors, natural isomorphism
- Ex.: $(\iota \colon V \to V^{**}) \colon \ \mathrm{id}_{\mathbf{k\text{-}Mod}} \Rightarrow (\cdot)^{**}$, morphisms betw. seq. of obj.
- Def.: Equivalence of categories (note: this means "essentially equal")
- Def.: fully faithful and essentially surjective functor
- Prop.: $F \colon \mathcal{C} \to \mathcal{D}$ is an equivalence if and only F is fully faithful and essentially surjective.
- Ex.: $\mathbf{k\text{-}Mod}^{\mathrm{fin}} \simeq$ (natural numbers + Matrices).

Lecture 15 (December 4)

- Categorical description of products and coproducs as functors
$$\prod, \coprod \colon \prod \mathcal{C} \to \mathcal{C}$$
Expression of the universal property of $\prod$ and $\coprod$ as
$$\mathrm{Hom}_{\mathcal{C}}(\coprod(c_i), d) \cong \mathrm{Hom}_{\prod \mathcal{C}}((c_i), \Delta(d))$$
and
$$\mathrm{Hom}_{\mathcal{C}}(d, \prod(c_i)) \cong \mathrm{Hom}_{\prod \mathcal{C}}(\Delta(d), (c_i)).$$

- Def.: Adjoint functors ($F \dashv G$:⇔ existence of natural bijections $\mathrm{Hom}_{\mathcal{D}}(F(x), y) \cong \mathrm{Hom}_{\mathcal{C}}(x, G(y))$).

- Ex.: Forgetful and free R-module functors; forgetful **Fields** → **Set** does not have left adjoint; $I \colon \mathbf{Ab} \to \mathbf{Gp}$ has $G \mapsto G^{\mathrm{ab}} := G/[G, G]$ as left adjoint; scalar extension (induction) and coinduction **S-Mod** → **R-Mod**.

- Def.: unit and counit of an adjunction $F \dashv G$
- Prop.: $F \dashv G \Leftrightarrow \exists\ \eta\colon\ \mathrm{id}_{\mathcal{C}} \Rightarrow G \circ F$ and $\varepsilon\colon F \circ G \Rightarrow \mathrm{id}_{\mathcal{D}}$ such that
  $G(\varepsilon) \circ \eta(G) = \mathrm{id}_G$ and $\varepsilon(F) \circ F(\eta) = \mathrm{id}_F$.

Lecture 16 (December 7)

- Def.: Representable functor $h^X : \mathcal{C} \to \mathbf{Set}$; representing object
- Lem. (Yoneda): The natural transformations from a representable functor h^X to $F: \mathcal{C} \to \mathbf{Set}$ are in bijection with $F(X)$ (Exercise!).
- Rem.: Embedding of $\mathcal{C}$ into $\mathrm{Fun}(\mathcal{C}, \mathbf{Set})$; uniqueness of representing object
- Prop.: Left adjoint functors commute with taking products and coproducts.
- Prop.: Uniqueness of left- and right adjoint functors (up to natural isomorphism).
- Def.: universal initial und terminal morphism.
- Adjoint functors in terms of universal initial and terminal morphisms.

Lecture 17 (December 11)

- Def.: Additive Category (Hom-sets abelian groups, comp. bilinear, existence of finite products and coproducts); additive functor
- Rem.: Existence of initial and terminal object (agree to give the zero object 0); isomorphism btw. product and coproduct.
- Def.: Kernel and cokernel of a morphism in an additive category.
- Def.: Monomorphism and epimorphism in an arbitrary category.
- Lem.: kernels are mono; cokernels are epi.
- Def.: Abelian category (additive + each morphism has kernel and cokernel, + $\iota = \ker(\mathrm{coker}(\iota))$ for ι mono, $p = \mathrm{coker}(\ker(p))$ for p epi)
- Def.: Image and coimage of a morphism.
- Rem.: Uniqueness of image and coimage; (short) exact sequences in arbitrary abelian categories.

- Ex.: fin. generated free abelian groups (not abelian), **R-Mod** (abelian), $\mathcal{C}$ abelian $\Rightarrow \mathcal{C}^{\mathrm{op}}$ abelian.

Lecture 18 (December 14)

- Def.: Additive functor, exact, left-, right- and half-exact functor
- Ex.: $\coprod, \prod$ exact; $M \otimes_R \cdot$ exact $\Leftrightarrow$ M flat;
  Hom$(M, \cdot)$ exact $\Leftrightarrow$ M proj.; Hom$(\cdot\,, M)$ exact $\Leftrightarrow$ M inj.
- Def.: Projective, injective objects in arbitrary categories
- Rem.: In **Set** all objects are injective (also projective iff AOC holds).
- Def.: Limit of a functor $\mathcal{J} \to \mathcal{C}$ (for $\mathcal{J}$ small), pull-back
- Rem.: Pull-back pictorially:

$$\begin{tikzcd}
U \arrow[dr, dotted, "\exists!"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& X \times_Z Y \arrow[r] \arrow[d] & Y \arrow[d] \\
& X \arrow[r] & Z
\end{tikzcd}$$

- Rem.: pull-back unique up to isom.; pull-back in **Set**, **Ab**, **Top**, **R-Mod** given by $\{(x, y) \mid f(x) = g(y)\}$; pull-back is a functor $\mathcal{C}^{\mathcal{I}} \to \mathcal{C}$; pull-back diagram; compatibility of those
- Ex.: $X \times_* Y \cong X \times Y$ if $\mathcal{C}$ has terminal object $*$.
- Def.: push-out (dually to pull-back)
- Rem.: push-outs in **Set** and **Ab**, amalgamated sum (product).

Lecture 19 (December 18)

Throughout: $\mathcal{C}, \mathcal{D}$ denote abelian categories, F, G additive functors.

- Def.: enough projectives, enough injectives
- Prop.: $F \dashv G$ and F exact $\Rightarrow G$ preserves injectives; $F \dashv G$ and G exact $\Rightarrow F$ preserves projectives
- Cor.: $\prod c_i$ injective $\Leftrightarrow$ each c_i injective (dual for projectives).
- Ex.: **R-Mod** has enough proj. and inj.; $\mathbf{Ab}^{\text{fin}}$ has no proj. or inj.; $\mathbf{Ab}^{\text{f.g}}$ has enough proj. but no inj.
- Def.: Projective resolution of an object M in $\mathcal{C}$:

$$P_i \to P_{i-1} \to \cdots \to P_1 \to P_0 \to M \to 0$$

  exact with each P_i projective (dually: injective resolution).
- Lem.: $\mathcal{C}$ has enough projectives $\Rightarrow$ each object has proj. resolution.
- Def.: $\mathbf{Ch}_{\mathcal{C}}$; cycles, boundaries and homology $H_i(C_\bullet, d_\bullet)$ of a chain complex; $(C_\bullet, d_\bullet)$ acyclic chain complex

- Rem.: H_i is a functor $\mathbf{Ch}_{\mathcal{C}} \to \mathcal{C}$; $P_\bullet \to M$ proj. resolution $\Rightarrow$ $H_0(P_\bullet) \cong M$; acyclic vs. exact chain complex; relation to topology; cohomology
- Def.: Quasi-Isomorphism: $f_\bullet \colon C_\bullet \to C'_\bullet$ with $H_i(F_\bullet)$ iso $\forall\ i$.

Lecture 20 (December 21)

- Def.: Chain homotopy $h_i\colon C_i \to D_{i+1}$ between $f_\bullet, g_\bullet\colon C_\bullet \to D_\bullet$; homotopy equivalent chain complexes.
- Rem.: Relation to equivalences of categories.
- Prop.: Chain homotopic maps induce the same morphisms on homology; homotopic chain complexes have isomorphic homology.
- Lem. (Fundamental Lemma of Homological Algebra): Uniqueness of projective/injective Resolutions up to chain homotopy
- Def.: Left- and right-derived functor $L_iF(M)$ and $R_iF(M)$ of an additive functor F on an object M of $\mathcal{C}$.
- Rem.: Left-derived functors vanish on projective objects (dually right-derived on injectives); uniqueness up to isomorphism of $L_iF(M)$ and $R_iF(M)$; functoriality
- Lem.: F right exact $\Rightarrow L_0F = F$; F left-exact $\Rightarrow R_0F = F$

Lecture 21 (January 8)

- Snake Lemma: If

$$\begin{tikzcd} & M' \arrow[r] \arrow[d, "f'"] & M \arrow[r] \arrow[d, "f"] & M'' \arrow[r] \arrow[d, "f''"] & 0 \\ 0 \arrow[r] & N' \arrow[r] & N \arrow[r] & N'' & \end{tikzcd}$$

  has exact rows, then there is an exact sequence

$$\ker f' \to \ker f \to \ker f'' \xrightarrow{\partial} \operatorname{coker} f' \to \operatorname{coker} f \to \operatorname{coker} f''.$$

- Rem.: Naturality of the *connecting homomorphism* ∂

- Prop.: $0 \to M'_\bullet \to M_\bullet \to M''_\bullet \to 0$ exact $\Rightarrow \exists$ long exact seq.

$$\cdots \to H_i(M_\bullet) \to H_i(M''_\bullet) \xrightarrow{\partial} H_{i-1}(M'_\bullet) \to H_i(M_\bullet) \to \cdots$$

- Prop.: $0 \to M' \to M \to M'' \to 0$ exact $\Rightarrow \exists$ long exact sequence for left and right derived functors:

$$0 \to R^0F(M') \to \cdots \to R^iF(M) \to R^iF(M'') \xrightarrow{\partial} R^{i+1}F(M') \to \cdots$$

$$\cdots \to L_iF(M'') \xrightarrow{\partial} L_{i-1}F(M') \to L_{i-1}F(M) \to \cdots \to L_0F(M'') \to 0$$

- Rem.: vanishing of L_1F (resp. R^1F) is equivalent to exactness; naturality of ∂
- Def.: Tor is the derived functor of $Y \mapsto X \otimes Y$ (for X fixed); Ext is the derived functor of $Y \mapsto \mathrm{Hom}(Y, X)$ (for X fixed).

Lecture 21 (January 11)

- Ex.: $\mathrm{Tor}_0^{\mathbb{Z}}(\mathbb{Z}_m, \mathbb{Z}_n) \cong \mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}_m, \mathbb{Z}_n) \cong \mathbb{Z}_{(m,n)}$ and $\mathrm{Tor}_i^{\mathbb{Z}} \equiv 0$ for $i > 1$
  since $\mathbb{Z}$ is pid; $\mathrm{Tor}_1^{\mathbb{Z}}(\mathbb{Z}, \mathbb{Z}_n) = 0$; similar for Ext;
  $R = \mathbb{Z}[t]/(1 - t^n)$: exercises.

- Def.: double complex $X_{\bullet\bullet}$, total degree, morphism of double
  complex, total complexes $|X_{\bullet\bullet}|$ and $\mathrm{Tot}\, X_{\bullet\bullet}$

- Def./Ex.: $(P \otimes Q)_{\bullet\bullet}$ and $\mathrm{Hom}(P, Q)$ for $P_\bullet$ and $Q_\bullet$,
  $\widetilde{\mathrm{Tor}}_n^R(X, Y) := H_i(|P \otimes Q|)$ (symmetric Tor).

- Acyclic Assembly Lemma (exactness of $|X_{\bullet\bullet}|$ and $\mathrm{Tot}\, X_{\bullet\bullet}$ from row or column exactness)

Lecture 21 (January 15)

- Rem.: Acyclic Assembly Lemma also works if diagonals are appropriately bounded.
- Prop.: $\mathrm{Tor} \cong \widetilde{\mathrm{Tor}}$.
- Prop.: same as above for Ext.
- Def.: the bar complex

$$\cdots \to \beta_{n+1}(R;M) \xrightarrow{\sum(-1)^i d_i} \beta_n(R;M) \xrightarrow{\sum(-1)^i d_i} \beta_{n-1}(R;M) \to \cdots$$

with $\beta_n(R;M) := R^{\otimes_{\mathbb{Z}}^{(n+1)}} \otimes_{\mathbb{Z}} M$, $r.(r_0|\cdots|r_{n+1}) := (rr_0|r_1|\cdots|r_{n+1})$ and

$$d_i(r_0|\cdots|r_{n+1}) := r_0|\cdots|r_i r_{i+1}|\cdots|r_{n+1}$$

- Prop.: $\beta_\bullet(R;M)$ is a resolution of M as R-module.

Lecture 22 (January 18)

- Lem.: Extension of scalars of free modules is free.
- Cor.: Conditions s.th. $\beta_\bullet(R; M)$ is free (e.g. R, M free $\mathbb{Z}$-mod).
- Ex.: Description of $\mathrm{Ext}^1_R(M, N)$ in terms of cocycles

  $$f\colon R \times M \to N \quad \text{s.th} \quad rf(s, m) + f(r, sm) = f(rs, m)$$

  and coboundaries.
- Def.: extension of modules, equivalence of extensions ($\mathrm{Ex}^n(M, N)$: equiv. classes of extensions of length n)
- Prop.: In R-mod: $\mathrm{Ex}^1(M, N) \cong \mathrm{Ext}^1_R(M, N)$ if M, R are free $\mathbb{Z}$-mod.

Lecture 23 (January 22)

Throughout G denotes a group and M a $\mathbb{Z}[G]$-module (shortly denoted G-module). If not specified otherwise, $\mathbb{Z}$ is the trivial G-module.

- Rem.: $\mathrm{Ex}^1(M,N) \cong \mathrm{Ext}^1_R(M,N)$ is true in **R-Mod** in general. Moreover, $\mathrm{Ex}^n(M,N)$ can be endowed with structure such that $\mathrm{Ex}^n(M,N) \cong \mathrm{Ext}^n_R(M,N)$ is an isomorphism of functors to **Ab**.

- Def.: Invariants M^G and coinvariants M_G of M, functors

$$(\cdot)^G, (\cdot)_G \colon \mathbb{Z}\mathbf{[G]\text{-}Mod} \to \mathbf{Ab}.$$

- Lem.: $(\cdot)^G \cong \mathrm{Hom}_{\mathbb{Z}[G]}(\mathbb{Z}, \cdot)$ and $(\cdot)_G \cong \mathbb{Z} \otimes_{\mathbb{Z}[G]} (\cdot)$

- Def.:

$$H_n(G,M) := (L_n(\cdot)_G)(M) \quad \text{is the } n\text{-th } \textit{group homology}$$
$$H^n(G,M) := (R_n(\cdot)^G)(M) \quad \text{is the } n\text{-th } \textit{group cohomology}$$

- Ex.: Homology and cohomology of $\mathbb{Z}_n$ and of $\mathbb{Z}$ (via ad-hoc choices of resolutions)
- Rem.: $H_n^R(G, M)$ and $H_R^n(G, M)$ if M is (moreover) an $R[G]$-module.
- Lem.: If $m := \mathrm{ord}(G) < \infty$, k : field with $\mathrm{char}(k) \nmid m$, then $H_k^n(G, M) = 0$ for $n \geqslant 1$ and each $k[G]$-module M.

Lecture 24 (January 25)

- Thm. (Maschke): If $m := \mathrm{ord}(G) < \infty$, k : field with $\mathrm{char}(k) \nmid m$, then each $k[G]$-module M is semi-simple.

- Functoriality of the group homology and cohomology: H^n and H_n are actually functors on the category **GpMod** of pairs (G, M) of a group G and a G-module M with $(\alpha, f)\colon (G, M) \to (H, M) :\Leftrightarrow f\colon M \to \alpha^* N$.

- H^n and H_n do in general *not* admit long exact sequences in G.

- Def.: Extensions $A \to \widehat{G} \to G$ of groups (with A abelian) and induced G-module structure on A.

- Ex.: A: G-module $\Rightarrow A \to A \rtimes G \to G$ is extension (the "trivial")

- Prop.: Splittings of $A \rtimes G \to G$ (or crossed homomorphisms) are up to equivalence classified by $H^1(G, A)$.

- Thm.: Extensions $A \to \widehat{G} \to G$ are (up to equivalence) classified by $H^2(G, A)$.

Lecture 25 (January 29)

- Thm.: Extensions $A \to \widehat{G} \to G$ are (up to equivalence) classified by $H^2(G, A)$ (proof thereof).
- Lem.: $H^n(G, A) \cong H^n_R(G, A)$.
- Ex. (from Topology): The universal covering of a topological group as a central extension $\pi_1(G) \to \widetilde{G} \to G$ (and description of a cocycle thereof).
- Ex.: Classification of the groups G of order 255 (there is only $\mathbb{Z}_{255}$)