Exercises for Algebra II, WS 12/13

Sheet 10 – Solutions

Exercise 43

a) Yes, subgroups of free groups are free.

b) Yes, likewise.

c) No, the homology of $\cdots \to 0 \to \mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z} \to 0 \to \cdots$ is $\mathbb{Z}_n$.

Exercise 44

a)

$$d(d(x,y)) = d(-d(x), f(x) + d(y)) = (d^2(x), f(-d(x)) + d(f(x) + d(y))) = 0$$

b) If $h_{n-1} + f_n \colon C_{n-1} \oplus C_n \to D_n$ extends $f_n \colon C_n \to D_n$ and is a chain map, then $h_n :=: C_n \to D_{n+1}$ satisfies

$$d(h+f)(x,y) = dhx + dfy \overset{!}{=} (h+f)(d(x,y)) = -hdx + fx + fdy,$$

which is equivalent to

$$dhx + hdx = fx.$$

Thus the data needed for an extension is exactly the same as data of a chain homotopy to 0.

c) That the morphisms commute with the differential is the case by construction. Obviously $D_\bullet \to E(f)_\bullet$ is a monomorphism and $E(f)_\bullet \to C[-1]_\bullet$ is an epimorphism. Since $E(f)_\bullet \to C[-1]_\bullet$ is simply the projection to the second factor in each degree it is also clear that the sequence is exact at $E(f)_\bullet$.

d) From the very definition it follows that $H_n(C[-1]_\bullet) = H_{n-1}(C_\bullet)$.

e) The connecting homomorphism is constructed from the diagram

$$\begin{tikzcd}
0 \arrow[r] & D_n \arrow[r] \arrow[d, "d"] & C_{n-1} \oplus D_n \arrow[r] \arrow[d, "d"] & C_{n-1} \arrow[r] \arrow[d, "d"] & 0 \\
0 \arrow[r] & D_{n-1} \arrow[r] & C_{n-2} \oplus D_{n-1} \arrow[r] & C_{n-2} \arrow[r] & 0
\end{tikzcd}$$

by taking an element $c \in \ker(C_{n-1} \overset{d}{\to} C_{n-2})$, choosing a pre-image — in this case of the form $(c,x) \in C_{n-1} \oplus D_n$ with some arbitrary $x \in D_n$ —, applying the middle differential to this — yielding $(-dc, f(c) + dx) \in C_{n-2} \oplus D_{n-1}$ —, choosing a pre-image of this — can be taken to be $f(c) + dx \in D_{n-1}$ and then taking the class $[f(c) + dx] = [f(x)] \in \operatorname{coker}(d)$. This shows the claim.

f) $f_\bullet$ is a quasi-isomorphism
$\Leftrightarrow H_i(f_\bullet)$ iso $\forall\ i$
$\Leftrightarrow \delta\colon H_i(C[-1]_\bullet) \to H_i(D_\bullet)$ iso $\forall\ i$ (by part d))
$\Leftrightarrow \delta$ always injective and surjective
$\Leftrightarrow H_i(E(f)_\bullet) = 0\ \forall\ i$ (by exactness of the sequence (2)).

Exercise 45

a) If $n = 1$, then we have the short exact sequence $K \to P_0 \to M$, inducing the long exact sequence

$$\cdots \to L_iF(P_0) \to L_iF(M) \to L_{i-1}F(K) \to L_{i-1}F(P_0) \to \cdots .$$

Since $L_iF(P)$ vanish for $i \geq 1$ and P projective, this shows $L_2F(M) \cong L_1(K)$. Iterating this argument (one can also do a formal induction) then shows the claim.

b) If $P_\bullet \to M$ is a projective resolution, then the sequence

$$0 \to K := \ker(P_n \to P_{n-1}) \to P_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0$$

is exact. Thus $LF_n(M) \cong L_1(K)$ which vanishes by assumption.