Sheet 9 – Solutions

Exercise 39

If $U \to X$ and $U \to Y$ are given, then they determine a unique morphism into $X \times Y$ that gives the original morphism after composition with the respective projection morphisms. Since the compositions of $U \to X$ and of $U \to Y$ to a Z-valued morphisms coincide, it follows that the morphism $U \to X \times Y$ actually takes values in $\{(x, y) \mid f(x) = g(x)\}$.

Exercise 40

Spelling out the definitions one sees that



is a pull-back if and only if $A \to B$ is a kernel of $B \to C$. Similarly, the diagram is a push-out if and only if $B \to C$ is a cokernel of $A \to B$. Thus $A \to B$ is mono (kernels are always monomorphisms), ker $(B \to C) \cong$ coker $(A \to B)$, and $B \to C$ is epi (likewise).

Exercise 41

Let $f: A \to U$ and $g: B \to U$ be group homomorphisms that agree on $A \cap B$. Then we may define $\tilde{f}: \langle A \cup B \rangle \to U$ by setting it to f(a) if $a \in A$, g(b) if $b \in B$ and extend it as a group homomorphism to $\langle A \cup B \rangle$. This is well-defined, e.g.

$$\begin{aligned} a_1 \cdot b_1 &= a_2 \cdot b_2 \Rightarrow a_2^{-1} \cdot a_1 = b_2 \cdot b_1^{-1} \in A \cap B \\ &\Rightarrow f(a_2)^{-1} \cdot f(a_1) = g(b_2) \cdot g(b_1)^{-1} \\ &\Rightarrow \widetilde{f}(a_1 \cdot b_1) = f(a_1) \cdot g(b_1) = f(a_2) \cdot g(b_2) = \widetilde{f}(a_2 \cdot b_2). \end{aligned}$$

Since \tilde{f} is uniquely determined by its restriction to A and B this shows the claim since $G = \langle A \cup B \rangle$. This also shows that in general a push-out of two subgroups A and B is given by the subgroup generated by A and B.

Exercise 42

a) An inverse to the morphism

$$\operatorname{Hom}_{\operatorname{Ab}}(N, A) \to \operatorname{Hom}_{\operatorname{Mod-R}}(N, \operatorname{Hom}_{\operatorname{Ab}}(R, A)), \quad \varphi \mapsto (n \mapsto \varphi(n \cdot r))$$

is given by

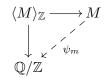
$$\operatorname{Hom}_{\operatorname{\mathbf{Mod-R}}}(N,\operatorname{Hom}_{\operatorname{\mathbf{Ab}}}(R,A)) \to \operatorname{Hom}_{\operatorname{\mathbf{Ab}}}(N,A), \quad \varphi \mapsto (a \mapsto \varphi(a,1)) \,.$$

b) Since the forgetful functor $\operatorname{Mod-R} \to \operatorname{Ab}$ is exact (check this), it follows from Proposition ... that the right adjoint preserves injectives. Since \mathbb{Q}/\mathbb{Z} is divisible and thus is injective, $\operatorname{Hom}_{\operatorname{Ab}}(R, \mathbb{Q}/\mathbb{Z})$ is injective.

- c) This also follows from Proposition ..., since the product functor is right adjoint to the diagonal functor (which is of course exact).
- d) We define

$$\Phi \colon M \to \prod_{\operatorname{Hom}_{\operatorname{\mathbf{Mod-R}}}(M,I_0)} \operatorname{Hom}_{\operatorname{\mathbf{Ab}}}(R, \mathbb{Q}/\mathbb{Z}), \quad (\Phi(m)_{\psi})(r) := \psi(m)(r).$$

This is injective, since for each $m \in M$ we find a homomorphism $\psi_m \colon M \to \mathbb{Q}/\mathbb{Z}$ (of abelian groups) making the diagram



commute. If we identify ψ_m with the associated homomorphism $M \to \operatorname{Hom}_{Ab}(R, \mathbb{Q}/\mathbb{Z})$, then $(\Phi(m)_{\psi_m})(1) \neq 0$ and thus Φ is injective.

By b) we know that I_0 is injective, by c) we know that I(M) is injective. Since M injects into I(M) this shows that **R-Mod** \cong **Mod-R** has enough injectives.