Exercises for Algebra II, WS 12/13

Sheet 5 – Solutions

Exercise 21

a) Since M is Artinian and Noetherian, N and M/N are so and thus of finite length. If $N = P_0 \supset \cdots \supset P_r = 0$ is a composition series of N, $M/N = Q_0 \supset \cdots \supset Q_s = 0$ is a composition series of M/N and $\pi\colon M \to M/N$ is the canonical projection, then

$$M = \pi_{-1}(Q_0) \supset \pi^{-1}(Q_1) \supset \cdots \supset \pi_{-1}(Q_s) = P_0 \supset P_1 \supset \cdots \supset P_r$$

is a composition series of M (why?) of length $r + s$.

b) This follows from the last point an the short exact sequence $P \cap Q \to P \oplus Q \to P + Q$.

Exercise 22

Since $\mathbb{Z}$ is a pid a module M of finite length has to be Noetherian (implies finitely generated) and Artinian (implies no elements of infinite order), thus a finite abelian group. By the classification of finite abelian groups we have that there exist primes p_i for $i = 1, ..., n$ and $\nu_i \in \mathbb{N}^+$ such that

$$M \cong \bigoplus_{i=1}^{n} \mathbb{Z}_{p_i^{\nu_i}}.$$

Since the simple abelian groups are $\mathbb{Z}_p$ for p a prime we have that that $l(\mathbb{Z}_{p^n}) = n$ and thus by exercise 21 that $l(M) = \sum \nu_i$. From this is is also clear that a finite abelian group M has a unique composition series if $M = \mathbb{Z}_{p^i}$ for some prime p and $i \in \mathbb{N}$.

Exercise 23

a) n is not divided by any square if and only if in its prime factorisation there only occur primes to the power 1. By the Chinese Remainder Theorem, this is the case if and only if $\mathbb{Z}_n$ is isomorphic to a direct sum of $\mathbb{Z}_p$'s. Since $\mathbb{Z}_q$ is simple if and only if q is prime, the latter is equivalent to semi-simplicity.

b) $R = \mathbb{R}[\mathbb{Z}]$, $M = \mathbb{R}^2$, module strucutre determined by $n \mapsto \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}$ and $N = \mathbb{R} \cdot e_1$. Then N is simple (one-dimensional), as well as M/N, but M has exactly one proper submodule, thus cannot be simple nor the sum of simple submodules.

Exercise 24

a) R is an k-algebra with $\dim_k(R) = \frac{n(n+1)}{2} < \infty$, thus Artinian (and also Noetherian), and thus $R/\operatorname{rad}(R)$ is semi-simple.

b) If $\mathfrak{m}$ is a maximal ideal in S, then $S/\mathfrak{m}$ is a field. Since the only nilpotent elements in a field are 0 we have that $s + \mathfrak{m} = 0$, and thus $s \in \mathfrak{m}$. Hence s is contained in each maximal ideal of S and thus in $\operatorname{rad}(S)$.

c) Each matrix with only off-diagonal entries is nilpotent, thus contained in $\operatorname{rad}(R)$. If at least one diagonal entry (say m_{kk}) of a matrix in R is not zero, then the surjective homomorphism $\lambda_k \colon R \to k$ $(m_{ij}) \mapsto m_{kk}$ does not vanish on this matrix, which thus is not contained in the maximal ideal $\ker(\lambda_k)$ and thus is not contained in $\operatorname{rad}(R)$. Thus $\operatorname{rad}(R)$ are exactly the matrices with zeros on the diagonal.

d) From the latter it is clear that $R/\operatorname{rad}(R) \cong k^n$ as vector spaces. It is not the case that the induced action $R/\operatorname{rad}(R) \cong k^n$ is the natural one, it is rather given by the action of the diagonal entries only $(m_{ij}).(k_i) = (m_{ii}k_i)$. Since $\operatorname{rad}(R)$ does not act trivially in the natural module structure on k^n, these modules are non-isomorphic.

e) $\mathbb{Z}$ is a non-Artinian $\mathbb{Z}$-module. Since each maximal ideal in $\mathbb{Z}$ is of the form $p\mathbb{Z}$ for p a prime we have that $\operatorname{rad}(\mathbb{Z}) = \cap_p p\mathbb{Z} = \{0\}$ and thus $\mathbb{Z}/\operatorname{rad}(\mathbb{Z}) = \mathbb{Z}$. Since the simple abelian groups are finite this is also true for the semi-simple ones, thus $\mathbb{Z}/\operatorname{rad}(\mathbb{Z})$ cannot be semi-simple.