Exercises for Algebra II, WS 12/13

Sheet 3 – Solutions

Exercise 10

a) The ring structure is given by $(r,s)+(r',s') := (r+r', s+s')$ and $(r,s)\cdot(r',s') := (rr', ss')$. It is clear that π_i are homomorphisms of rings and thus $\pi_i \circ f$ is one. Thus $(\pi_i \circ f)_{i=1,2}$ is really an element of $\mathrm{Hom}(S, R_1) \times \mathrm{Hom}(S, R_2)$. An inverse map to $f \mapsto (\pi_i \circ f)$ is given by $(f_i)_{i=1,2} \mapsto (s \mapsto (f_1(s), f_2(s)))$.

b) ι_i is not a homomorphism of rings, since it does not map 1_{R_i} to $(1_{R_1}, 1_{R_2}) = 1_{R_1\times R_2}$.

c) The map $(r\otimes s), (r'\otimes s') \mapsto rr'\otimes ss'$ is well-defined on $R_1 \otimes_{\mathbb{Z}} R_2$ (why?) and turns $R_1 \otimes_{\mathbb{Z}} R_2$ into a ring (the identities like associativity and distributivity directly follow from the ones in R_1 and R_2 once the well-definedness is checked). Sicne $1\otimes 1$ is the unit in $R_1 \otimes R_2$ we have that ι_1 and ι_2 are morphisms of rings. Thus $f \circ \iota_i$ is a ring homomorphism.

An inverse to $f \mapsto (f\circ\iota_i)_{i=1,2}$ is constructed as follows. For $f_i\colon R_i \to S$ consider $f_1\cdot f_2\colon R_1\times R_2 \to S$. This is bi-additive and thus induces an additive map $\varphi\colon R_1 \otimes_Z R_2 \to S$. Since

$$\varphi((r\otimes s)\cdot(r'\otimes s')) = \varphi(rr'\otimes ss') = f(rr')f(ss') = f(r)f(s)f(r')f(s') = \varphi(r\otimes s) = \varphi(r'\otimes s')$$

by definition we have that φ is a morphism of rings. It satisfies $\varphi(r\otimes 1) = f_1(r)$ and $\varphi(1\otimes s) = f_2(s)$ and thus $(f_1, f_2) \mapsto \varphi$ is an inverse to $f \mapsto (f\circ\iota_i)_{i=1,2}$.

Exercise 11

If m has finite order, then $\langle m\rangle \cong \mathbb{Z}_n$ for some n and $\mathbb{Z}_n \otimes \mathbb{Q} = 0$ since $\mathbb{Z}_n$ has only torsion elements. Conversely, of m has infinite order then $\langle m\rangle \cong \mathbb{Z}$ and $\mathbb{Z}\otimes\mathbb{Q} = \mathbb{Q}$.
Since $\prod_{n\geq 1}\mathbb{Z}_n$ has the element $(1,1,...)$ which is of infinite order (why?) we have that $(\prod_{n\geq 1}\mathbb{Z}_n)\otimes\mathbb{Q}$ does not vanish. On the other hand, $\prod_{n\geq 1}(\mathbb{Z}_n\otimes\mathbb{Q}) = \prod_{n\geq 1} 0 = 0$.

Exercise 12

a) No: G is always bijective to the set with four elements.

b) Yes: $\mathbb{Z}_2 \to \mathbb{Z}_2\oplus\mathbb{Z}_2 \to \mathbb{Z}_2$ is short exact and $\mathbb{Z}_2 \xrightarrow{\cdot 2} \mathbb{Z}_4 \to \mathbb{Z}_2$ is also.

c) Yes: $\mathbb{Z} \to \mathbb{Z}\oplus\mathbb{Z}_2 \to \mathbb{Z}_2$ is short exact and $\mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \to \mathbb{Z}_2$ is also.

d) No: $\mathbb{Z}$ is free and from Proposition I.3.6 if follows that if $\mathbb{Z}_2 \to G \to \mathbb{Z}$ is short exact, then $G \cong \mathbb{Z}_2 \oplus \mathbb{Z}$.

Exercise 13

We show that φ_3 is injective: consider

$$\begin{tikzcd}
A_1 \arrow[r, "\alpha_1"] \arrow[d, "\varphi_1"] & A_2 \arrow[r, "\alpha_2"] \arrow[d, "\varphi_2"] & A_3 \arrow[r, "\alpha_3"] \arrow[d, "\varphi_3"] & A_4 \arrow[r] \arrow[d, "\varphi_4"] & A_5 \arrow[d, "\varphi_5"] \\
B_1 \arrow[r, "\beta_1"] & B_2 \arrow[r, "\beta_2"] & B_3 \arrow[r, "\beta_3"] & B_4 \arrow[r] & B_5
\end{tikzcd}$$

If $\varphi_3(a) = 0$, then $b := \beta_3(\varphi_3(a)) = 0$ and thus $\varphi_4^{-1}(b) = \alpha_3(a) = 0$. Thus $a = \alpha_2(c)$ for some $c \in A_2$ (why?). Then $\beta_2(\varphi_2(c)) = \varphi_3(\alpha_2(c)) = \varphi_3(a) = 0$ and thus $\varphi_2(c) = \beta_1(e)$ for some $e \in B_1$ (again why?). Then $f := \varphi_1^{-1}(e)$ satisfies $\alpha_1(f) = c$ and thus $a = \alpha_2(c) = \alpha_2(\alpha_1(f)) = 0$.
The surjectivity is shown similarly.

Exercise 14

a) If each M_i is projective, then there exist modules N_i such that $M_i \oplus N_i \cong R^{n_i}$ and thus
$$\bigoplus_i R^{n_i} \cong \bigoplus_i M \oplus \bigoplus_i N_i.$$
Since the left hand side is free it follows that $\bigoplus_i M_i$ is projective (Prop. I.4.7 b)).

Now let $\bigoplus_i M_i$ be projective. Each morphism $f\colon M_i \to M$ factors through the inclusion $M_i \to \bigoplus_i M_i$:
$$\begin{tikzcd}
& M_i \arrow[d, "\iota_i"] & \\
& \bigoplus_i M_i \arrow[d, "f"] \arrow[dl, dotted, "\alpha"'] & \\
N \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$
and since $\bigoplus_i M_i$ is projective there exists the dotted lift α. But $\alpha \circ \iota_i$ lifts $f \circ \iota_i$, and thus M_i is projective by Prop. I.4.7 a).

b) This is shown similarly.

Exercise 15

Assume $P' \oplus P \cong M$ is free. Then
$$F := \bigoplus_{n\in\mathbb{N}} (P' \oplus P) \cong \bigoplus_{n\in\mathbb{N}} M$$
is free. On the other hand, if we apply the isomorphism to
$$P \oplus F = P \bigoplus_{n\in\mathbb{N}} (P' \oplus P) = P \oplus P' \oplus P \oplus P' \oplus \cdots$$
that switches the $(2n+1)$- and $(2n+2)$-summands, then we see that
$$P \bigoplus_{n\in\mathbb{N}} (P' \oplus P) \cong \bigoplus_{n\in\mathbb{N}} (P' \oplus P) \cong \bigoplus_{n\in\mathbb{N}} M$$
is free. Thus F is free and $P \oplus F$ is also free.

Exercise 16

This is spelled out in Prop. VII.10.12 the book of Jantzen and Schwermer, cf. `http://www.springerlink.com/content/978-3-540-21380-2`