# Sheet 4

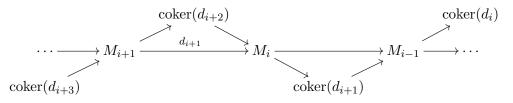
#### Exercise 17

Let

$$M_{\bullet} := \left( \cdots \xrightarrow{d_2} M_1 \xrightarrow{d_1} M_0 \xrightarrow{d_0} M_{-1} \xrightarrow{d_{-1}} \cdots \right)$$

be a chain complex of R-modules. Show that the following are equivalent:

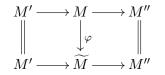
- a)  $M_{\bullet}$  is an exact sequence.
- b) In the diagram



the diagonal sequences are short exact. It is part of the exercise to determine all the maps in the previous diagram (recall that  $\operatorname{coker}(f) := N/\operatorname{im}(f)$  for  $f \colon M \to N$ ).

## Exercise 18

Let  $M' \to M \to M''$  and  $M' \to \widetilde{M} \to M''$  be short exact sequences of *R*-modules and  $\varphi \colon M \to \widetilde{M}$  be a morphism of *R*-modules such that



commutes. Show that  $\varphi$  is then automatically an isomorphism.

#### Exercise 19

- a) Let M be an R-module. Show that M is projective and finitely generated if and only if there exist  $x_1, ..., x_n \in M$  and  $f \in \operatorname{Hom}_R(M, R)$  such that  $x = \sum_{i=1}^n f_i(x) \cdot x_i$ .
- b) Show that if I is a two-sided Ideal in R and M is a projective R-module, then M/IM is a projective R/I-module.

## Exercise 20

Let R, S be commutative rings. Show that

- a) R is flat as a module over itself.
- b) If  $S \subseteq R$  is multiplicative, then the localization  $S^{-1}R$  is flat over R.
- c) Show that if M is flat over R and that  $R \to S$  is a morphism of rings, then  $M \otimes_R S$  is flat over S.