Exercises for Algebra II, WS 12/13

Sheet 3

Exercise 10

Let R_1, R_2 and S be commutative rings.

a) Show that the product $R_1 \times R_2$ has a ring structure such that the projections $\pi_i \colon R_1 \times R_2 \to R_i$ are morphisms of rings turning $(R_1 \times R_2, (\pi_i)_{i=1,2})$ into a direct product of rings, i.e.

$$\mathrm{Hom}(S, R_1 \times R_2) \to \mathrm{Hom}(S, R_1) \times \mathrm{Hom}(S, R_2), \quad f \mapsto (\pi_i \circ f)_{i=1,2}$$

into an isomorphism of sets (in particular, the product on the right hand side denotes the Cartesian product of sets).

b) Why don't the inclusions $\iota_i \colon R_i \to R_1 \times R_2$ turn $(R_1 \times R_2, (\iota_i)_{i=1,2})$ into a direct sum?

c) Show that the tensor product $R_1 \otimes_{\mathbb{Z}} R_2$ has a ring structure such that the maps $\iota_1 \colon R_1 \to R_1 \otimes_{\mathbb{Z}} R_2$, $r \mapsto r \otimes 1$ and $\iota_2 \colon R_2 \to R_1 \otimes_{\mathbb{Z}} R_2$, $s \mapsto 1 \otimes s$ are morphisms of rings an that $(R_1 \otimes_{\mathbb{Z}} R_2, (\iota_i)_{i=1,2})$ is a direct sum, i.e.,

$$\mathrm{Hom}(R_1 \otimes_{\mathbb{Z}} R_2, S) \to \mathrm{Hom}(R_1, S) \times \mathrm{Hom}(R_2, S), \quad f \mapsto (f \circ \iota_i)_{i=1,2}$$

is an isomorphism of sets.

Exercise 11

Show that

$$\left(\prod_{n \geq 1} \mathbb{Z}_n \right) \otimes_{\mathbb{Z}} \mathbb{Q} \not\cong 0 \quad \text{and} \quad \prod_{n \geq 1} (\mathbb{Z}_n \otimes_{\mathbb{Z}} \mathbb{Q}) \cong 0$$

Hint: Show first that for an arbitrary $\mathbb{Z}$-module M and $m \in M$ we have $\langle m \rangle \otimes \mathbb{Q} = 0$ if and only if m has finite order.

Exercise 12

a) Is it possible to find an infinite abelian group G such that $\mathbb{Z}_2 \to G \to \mathbb{Z}_2$ is short exact?

b) Is it possible to find finite abelian groups G_1, G_2 such that $\mathbb{Z}_2 \to G_i \to \mathbb{Z}_2$ is short exact but $G_1 \not\cong G_2$?

c) Is it possible to find abelian groups G_1, G_2 such that $\mathbb{Z} \to G_i \to \mathbb{Z}_2$ is short exact but $G_1 \not\cong G_2$?

d) Is it possible to find abelian groups G_1, G_2 such that $\mathbb{Z}_2 \to G_i \to \mathbb{Z}$ is short exact but $G_1 \not\cong G_2$?

Exercise 13

Show the **5-Lemma:** Consider the commutative diagram

$$\begin{tikzcd}
A_1 \arrow[r] \arrow[d, "\varphi_1"] & A_2 \arrow[r] \arrow[d, "\varphi_2"] & A_3 \arrow[r] \arrow[d, "\varphi_3"] & A_4 \arrow[r] \arrow[d, "\varphi_4"] & A_5 \arrow[d, "\varphi_5"] \\
B_1 \arrow[r] & B_2 \arrow[r] & B_3 \arrow[r] & B_4 \arrow[r] & B_5
\end{tikzcd}$$

of R-modules with exact sequences as top and bottom rows. If $\varphi_1, \varphi_2, \varphi_4$ and φ_5 are isomorphisms, then φ_3 is also an isomorphism.

Hint: An element in the kernel of φ_3 must come from an element of A_2 (why?). This then comes in turn from an element of A_1 (why?), thus its image in A_3 vanishes (why?).

Exercise 14

Let $(M_i)_{i\in I}$ be a family of R-modules. Show

a) The direct sum $\oplus_{i\in I} M_i$ is a projective R-modules if and only if all modules M_i are projective.

b) The direct product $\prod_{i\in I} M_i$ is an injective R-module if and only if all modules M_i are injective.

Exercise 15

Show the **Eilenberg-swindle**: Let P be a projective R-module. Show that there always exist a free R-module F such that $P \oplus F$ is free.

Hint: Find P' such that $P \oplus P'$ is free and consider

$$P' \bigoplus_{n\in\mathbb{N}} (P \oplus P').$$

Exercise 16

Show **Lemma I.2.9**: If A is a commutative ring and $\varphi\colon M \otimes_{\mathbb{Z}} N \to M \otimes_A N$ is the morphism of abelian groups, induced by the $\mathbb{Z}$-bilinear (aka biadditive) map $\otimes_A\colon M \times N \to M \otimes_A N$, then $\ker(\varphi) = \langle T\rangle$ is the subgroup generated by

$$T := \{ax \otimes_{\mathbb{Z}} u - x \otimes_{\mathbb{Z}} au \mid x \in M, u \in N, a \in A\}$$

by the following steps:

a) Show that $\ker(\varphi) \subseteq \langle T\rangle$.

b) Show that $a \cdot (x \otimes_{\mathbb{Z}} u) := ax \otimes_{\mathbb{Z}} u$ defines an A-module structure on $M \otimes_{\mathbb{Z}} N$.

c) Show that $\langle T\rangle$ is an A-submodule and that $(M \otimes_{\mathbb{Z}} N)/\langle T\rangle \cong M \otimes_A N$.

d) Conclude that $\ker(\varphi) = \langle T\rangle$.