Exercises for Algebra II, WS 12/13

Sheet 2

Exercise 5
Prove or disprove the following assertions:

a) Any ring is faithful as a module over itself.

b) Any commutative ring is torsion free as a module over itself if and only if it is an integral domain.

c) Any free module over an integral domain is faithful as a module over itself.

d) Any free module is torsion free as a module over itself.

Exercise 6
If $(M_i)_{i\in I}$ is a family of R-modules, show that $((\prod_{i\in I} M_i), \pi_i)$ with $\pi_i((m_i)_{i\in I}) = m_i$ is a direct product (here $\prod_{i\in I} M_i$ denotes the product of the M_i as sets).
Show also that that $(\bigoplus_{i\in I} M_i, \iota_i)$ with $\iota_i(m) = m \cdot \delta_i$ (where δ_i is the usual δ-function at i) is a direct sum. Here $\bigoplus_{i\in I} M_i$ denotes

$$\{(m_i)_{i\in I} \in \prod_{i\in I} M_i \mid m_i \neq 0 \text{ for only finitely many } i\}.$$

Exercise 7
Let A be a commutative ring and I be an ideal of A.

a) Show that for each A-module M the map $A/I \otimes_A M \to M/IM$, $(a+I)\otimes x \mapsto ax + IM$ is an isomorphism of A-modules.

b) Each A/I-module is also an A-module. Show that for A/I-modules M, N we have $M \otimes_{A/I} N \cong M \otimes_A N$.

c) Compute $\mathbb{Z}_m \otimes_{\mathbb{Z}} \mathbb{Z}_n$ and $\mathbb{Z}_m \otimes_{\mathbb{Z}_m} \mathbb{Z}_n$ (the latter if m is a multiple of n).

Exercise 8
Show that $\mathbb{Q}/\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Q} = 0$ and $\mathbb{Q}/\mathbb{Z} \otimes \mathbb{Z}_n = 0$ for each $n \in \mathbb{N}^+$.

Exercise 9
Prove or disprove the following assertions:

a) In $\mathbb{Z}$, the sum of $15\mathbb{Z}$ and $6\mathbb{Z}$ is direct.

b) In $\mathbb{Z}$, the sum of $p\mathbb{Z}$ and $q\mathbb{Z}$ is direct if $\gcd(p,q) = 1$.

c) $\mathbb{Z}$ cannot be the direct sum of any two submodules.

d) $k[x]$ cannot be the direct sum of any two submodules.

e) A commutative ring A is a field if and only if each module over A is free.