Exercises for Algebra II, WS 12/13

Sheet 1

Exercise 1

Let S be the ring of $(n \times n)$-matrices with entries in a commutative ring R (recall that square matrices with entries in a commutative ring again form a ring with respect to matrix addition and multiplication). Is transposition the only ring isomorphism $S \to S^{\mathrm{op}}$?

Exercise 2

a) Find two polynomials over the field $\mathbb{F}_2$ with two elements that are different but have the same polynomial function.

b) Show that a polynomial algebra is unique up to unique isomorphisms, i.e. if (A, X) and (B, Y) are polynomial algebras over R, then there exists a unique isomorphism $A \to B$ mapping X to Y.

c) Let R be a commutative ring and let

$$R[x] := \{(r_0, r_1, \ldots) \mid r_i \in R \text{ for } i \in \mathbb{N}_0, r_i \neq 0 \text{ for only finitely many } i\}$$

be the finite sequences in R. Show that $(R[x], x)$ is a polynomial algebra over R if we set $x = (0, 1, 0, \ldots)$ and endow $R[x]$ with the multiplication

$$(r_0, r_1, \ldots) \cdot (s_0, s_1, \ldots) := \Big(\sum_{i+j=0} r_i \cdot s_j, \sum_{i+j=1} r_i \cdot s_j, \ldots\Big).$$

d) Which of the $\mathbb{C}$-algebras $\mathbb{C}[x]/(x^2)$, $\mathbb{C}[x, y]$, $\mathbb{C}[x, y]/(x)$, $\mathbb{C}[x, y]/(x - y)$ are polynomial algebras over $\mathbb{C}$ (recall $\mathbb{C}[x, y] := (\mathbb{C}[x])[y]$ and (f):= Ideal generated by f)?

Exercise 3

Let G, H be groups, $\varphi\colon G \to H$ a homomorphism and (V, ρ) a representation of H.

a) Show that $g \mapsto \rho(\varphi(g))$ defines a representation (V, ρ^φ) of G.

b) Suppose $G = H$. Show that if $\varphi\colon H \to H$ is a conjugation automorphism (i.e. given by $h \mapsto x \cdot h \cdot x^{-1}$ for some $x \in H$), then (V, ρ) and (V, ρ^φ) are isomorphic.

c) Is it also true for arbitrary automorphisms φ that (V, ρ) and (V, ρ^φ) are isomorphic?

Exercise 4

Let G be a finite group and R be a commutative ring. Denote by

$$Z(G) := \{g \in G \mid gx = xg \text{ for all } x \in G\}$$

the center of G (which always is a *normal* subgroup) and by

$$Z(R[G]) := \{y \in R[G] \mid xy = yx \text{ for all } x \in R[G]\}$$

be the center of the group algebra $R[G]$ over R.

a) Show that $Z(R[G])$ is a subalgebra of $R[G]$. Is it also an ideal?

b) Show that any $R[G]$-module is naturally a $Z(R[G])$-bimodule.

c) Show $R[Z(G)] \subseteq Z(R[G])$. Does here equality hold in general?