Approximation Theorems for Locally Convex Manifolds, Lie Groups and Principal Bundles

Christoph Wockel

Universität Göttingen

Overview

Preliminaries

X: arbitrary topological space

M: finite-dimensional paracompact connected manifold (possibly with boundary or corners)

N: locally convex manifold

Topologies on function spaces

- $C(M, X)$ compact open topology
- $C^{\infty}(M, N)$ initial topology w.r.t.

$$C^{\infty}(M, N) \ni f \mapsto T^k f \in C(T^k M, T^k N)$$

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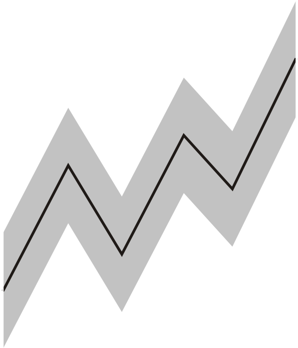
- take $f \in C(M, F)$ and a neighbourhood of f



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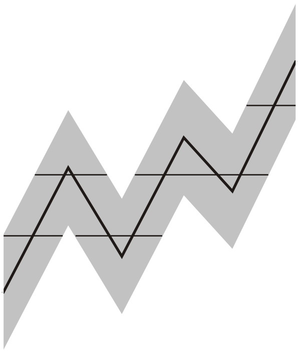

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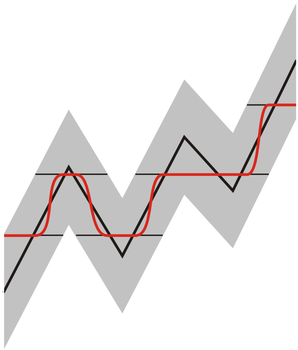
- use partition of unity to interpolate between these local approximations



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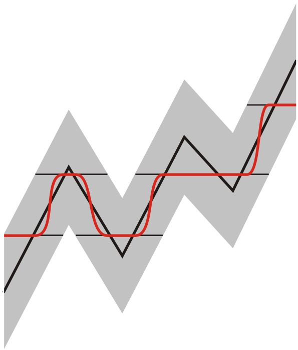

From locally convex spaces to locally convex manifolds

Problem

- construction uses convex combination to interpolate

$\Rightarrow$ not possible for functions with values in manifolds.

$\rightsquigarrow$ localised step-by-step smoothing process

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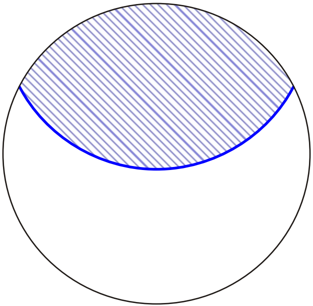

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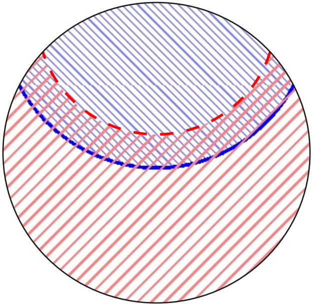

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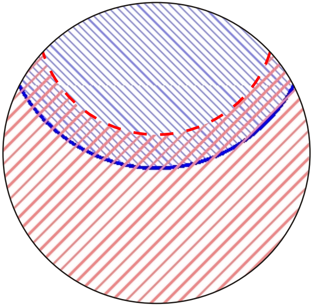

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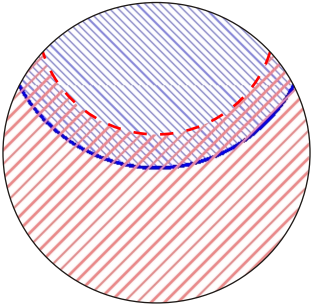

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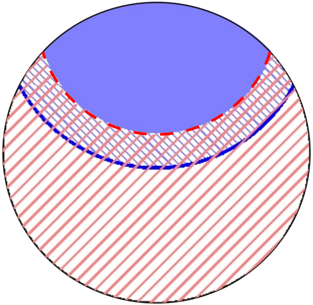

Smoothing sections in locally trivial bundles

Theorem (Generalised Steenrod Approximation)

- $\xi : E \to M$ *locally trivial smooth bundle with fibre a locally convex manifold*
- $C \subseteq M$ *closed,* $U \subseteq M$ *open*
- $\sigma : M \to E$ *section, smooth on* $C \setminus U$
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$\Rightarrow$ $\exists$ *section* $\tau : M \to O$*, smooth on* C*, equal to* σ *on* $M \setminus U$

$\Rightarrow$ $\exists$ *homotopy* $R : [0,1] \times M \to O$ *with* $R_0 = \sigma$ *and* $R_1 = \tau$

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- *O open neighbourhood of $\sigma(M)$*

$\Rightarrow$ *$\exists$ section $\tau : M \to O$, smooth on C, equal to σ on $M \setminus U$*

$\Rightarrow$ *$\exists$ homotopy $R : [0,1] \times M \to O$ with $R_0 = \sigma$ and $R_1 = \tau$*

Corollary

- *$S^{\infty}(E)$ is dense in $S(E)$*
- *$C^{\infty}(M,N)$ is dense in $C(M,N)$ (even in graph topology)*
- *smooth and continuous homotopies agree (Kriegl, Michor)*

Application: Gauge groups

The gauge group

- G locally convex Lie group (locally exponential)
- $\pi : P \to M$ smooth principal G-bundle (M compact)
- $AD(P)$ associated G-bundle (by $AD(g).h = g \cdot h \cdot g^{-1}$)
- $\mathrm{Gau}^{\infty}(P) := S^{\infty}(AD(P))$, $\mathrm{Gau}(P) := S(AD(P))$

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- $\mathrm{Gau}^{\infty}(P)$ *and* $\mathrm{Gau}(P)$ *are infinite-dimensional Lie groups*
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Note: Reduces determination of $\pi_n(\mathrm{Gau}^{\infty}(P))$ to a purely topological setting, more appropriate for bundle theory

Application: Smoothing finite-dim. principal bundles

Facts

- Only need to consider compact G
- principal bundle given by a classifying map $f_P : M \to BG$
- bundle equivalences given by homotopies $f_P \simeq f_{P'}$

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$\rightsquigarrow$ smoothing procedure for cocycles

Principal bundles and cycles

M finite-dim. paracompact connected manifold

G locally convex Lie group (possibly $\dim(G) = \infty$)

Principal bundle

- $(U_i)_{i \in \mathbb{N}}$ locally finite open cover of M, $\overline{U_i}$ compact
- $g_{ij} : U_i \cap U_j \to G$ transition functions
  - $g_{ij}(x) \cdot g_{jk}(x) \cdot g_{ki}(x) = \mathbb{1}$ for $x \in U_i \cap U_j \cap U_k$
  - $g_{ii}(x) = \mathbb{1}$ for $x \in U_i$ ($\Leftrightarrow g_{ij}(x) = g_{ji}(x)^{-1}$ for $x \in U_i \cap U_j$)

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In the sequel:

- principal bundle given by $\mathcal{G} = (g_{ij} : U_i \cap U_j \to G)_{i,j \in \mathbb{N}}$
- $\mathcal{G}$ is continuous $\Leftrightarrow$ all g_{ij} are continuous
  $\mathcal{G}$ is smooth $\Leftrightarrow$ all g_{ij} are smooth

Bundle equivalences

Observation

Given:

- principal bundle $\mathcal{G} = (g_{ij} : U_i \cap U_j \to K)_{i,j \in \mathbb{N}}$
- mappings $(f_i : U_i \to G)_{i \in \mathbb{N}}$ (smooth or continuous, according to $\mathcal{G}$)

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$$\Rightarrow h_{ij}(x) := f_i^{-1}(x) \cdot g_{ij}(x) \cdot f_j(x)$$

defines a new principal bundle $\mathcal{H}$

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Equivalence of principal bundles

Two principal bundles

$$\mathcal{G} = (g_{ij} : U_i \cap U_j \to G)_{i,j \in \mathbb{N}}, \mathcal{H} = (h_{ij} : U_i \cap U_j \to G)_{i,j \in \mathbb{N}}$$

are *equivalent*, if there exists a bundle equivalence between them, i.e., $\mathcal{F} = (f_i : U_i \to G)_{i \in \mathbb{N}}$ with

$$h_{ij}(x) = f_i^{-1}(x) \cdot g_{ij}(x) \cdot f_j(x).$$

Examples of principal bundles

Example (Frame bundle)

$\varphi_i : U_i \to \varphi(U_i) \subseteq \mathbb{R}^n$ differential structure on M

$$g_{ij} := U_i \cap U_j \ni x \mapsto d(\varphi_i \circ \varphi_j^{-1})(\varphi_j(x)) \in \mathrm{GL}_n(\mathbb{R})$$

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- $d(\varphi_i \circ \varphi_j^{-1}) \cdot d(\varphi_j \circ \varphi_k^{-1}) \cdot d(\varphi_k \circ \varphi_i^{-1}) \equiv \mathbb{1}$ (Chain Rule)
- $d(\varphi_i \circ \varphi_i^{-1}) = d(\mathrm{id}_{\varphi(U_i)}) \equiv \mathbb{1}$

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Example (Equivalence classes)

- (continuous) bundles over $\mathbb{S}^n$, classified by $\pi_{n-1}(G)$

  (bundles over hemispheres U_N, U_S are trivial,
  $g_{NS} : U_N \cap U_S \cong \mathbb{S}^{n-1} \to G$ is transition function)
- (continuous) line bundles, classified by $H^2(M; \mathbb{Z})$
- (continuous) $\mathrm{PU}(\mathcal{H})$-bundles, classified by $H^3(M; \mathbb{Z})$

Smoothing arbitrary principal bundles and bundle equivalences

Question

What is the relation between smooth and continuous principal bundles and bundle equivalences?

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Theorem (Müller, W.)

Each continuous principal bundle is continuously equivalent to a smooth principal bundle. Moreover, two smooth principal bundles are smoothly equivalent if and only if they are continuously equivalent.

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Proof

Uses method to smooth cocycles and avoid classifying spaces.

Smoothing Lie group valued mappings

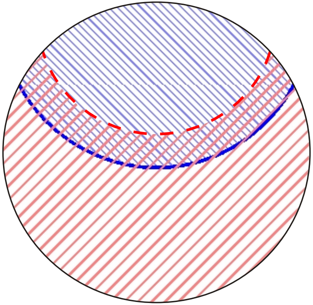
M finite-dimensional paracompact manifold

G locally convex Lie group

Tool

$C \subseteq M$ closed, $U \subseteq M$ open, $f \in C(M, G)$ smooth on $C \backslash U$

$\Rightarrow$ $\exists \widetilde{f} \in C(M, G)$, smooth on C arbitrarily close to f



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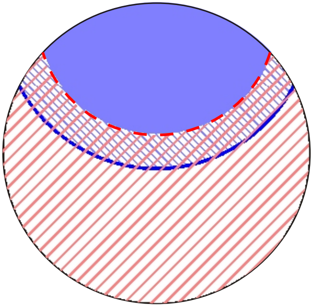

Smoothing bundle equivalences

Given

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$$\mathcal{H} = (h_{ij} : U_i \cap U_j \to G)_{i,j \in \mathbb{N}}$$

(i.e., all g_{ij}, h_{ij} smooth) and a *continuous* equivalence

$$\mathcal{F} = (f_i : U_i \to G)_{i \in \mathbb{N}}$$

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Problem

ensure $h_{ij} = \widetilde{f}_i^{-1} \cdot g_{ij} \cdot \widetilde{f}_j$ during the smoothing procedure!

The idea of induction

Idea

On $U_i \cap U_j$: $\quad h_{ij} = \widetilde{f}_i^{-1} \cdot g_{ij} \cdot \widetilde{f}_j \quad \Leftrightarrow \quad \widetilde{f}_i = g_{ij} \cdot \widetilde{f}_j \cdot h_{ji}$

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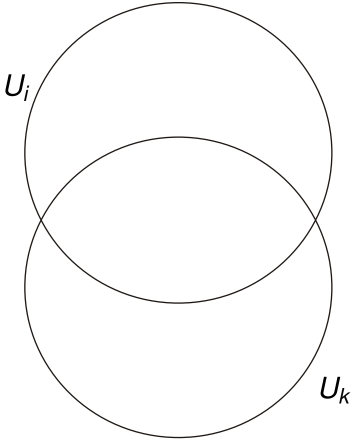
$\Rightarrow$ Use $\widetilde{f}_i = \underbrace{g_{ij} \cdot \widetilde{f}_j \cdot h_{ji}}_{\text{smooth}}$, to define $\widetilde{f}_i$ inductively for $i = 1, 2, \ldots$.

Concrete description of the induction

Let $\widetilde{f_i}$ be defined for $i < k$.
Then $\widetilde{f_k}$ is



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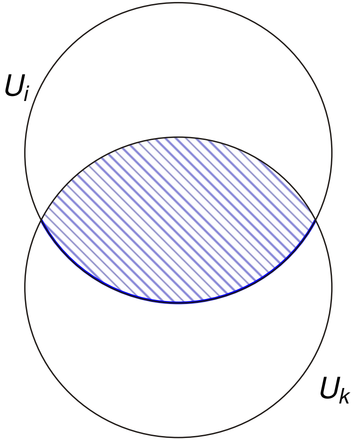

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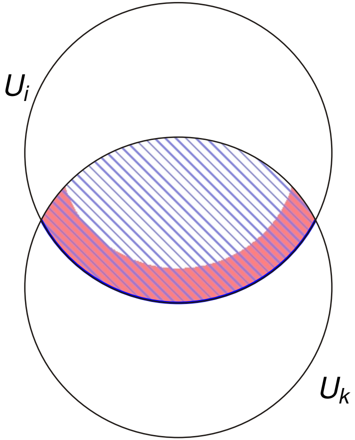

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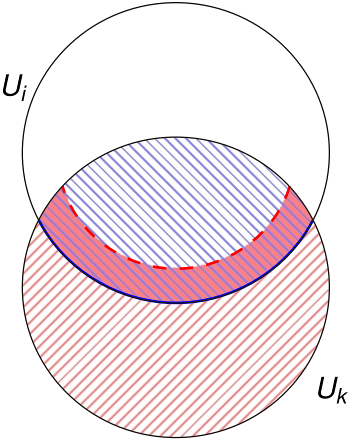

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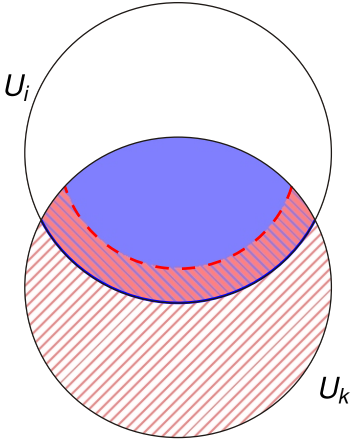
- extended by f_k on $U_k \setminus \bigcup_{i<k} U_i$
- and, eventually, smoothed out.



Concrete description of the induction

Let $\widetilde{f}_i$ be defined for $i < k$.
Then $\widetilde{f}_k$ is

- defined on $U_k \cap \bigcup_{i<k} U_i$ by
$$\widetilde{f}_k := g_{ki} \cdot \widetilde{f}_i \cdot h_{ik}$$
- "near" ∂U_i modified to f_k,
- extended by f_k on $U_k \setminus \bigcup_{i<k} U_i$
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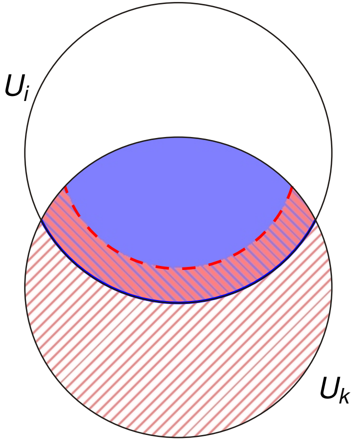

Problem

We violate $\widetilde{f}_k = g_{ki} \cdot \widetilde{f}_i \cdot h_{ik}$ *"near"* ∂U_i*!*

The safety margin

Remedy

“Capture” region, on which $\widetilde{f}_k = g_{ki} \cdot \widetilde{f}_i \cdot h_{ik}$ may be violated in a “safety margin”, which is “near” ∂U_i.

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Observation

$(V_i)_{i \in \mathbb{N}}$ open cover of M, $V_i \subseteq U_i$ and $\widetilde{f}'_i : V_i \to G$ smooth with $\widetilde{f}'_i = g_{ij} \cdot \widetilde{f}'_j \cdot h_{ji}$ on $V_i \cap V_j$

$$\Rightarrow \widetilde{f}_k(x) := g_{ki}(x) \cdot \widetilde{f}'_i(x) \cdot h_{ik}(x) \text{ for } x \in U_k \cap V_i$$

defines a *smooth* bundle equivalence.

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Upshot

Bundle equivalence is determined by its values on a finer cover!

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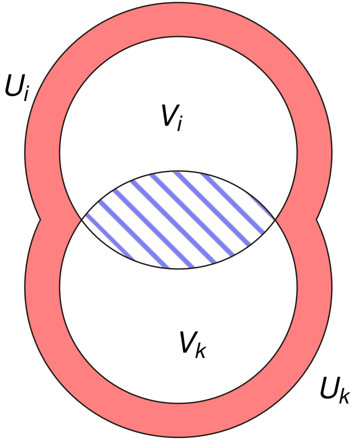
- choose $(V_i)_{i\in\mathbb{N}}$ with $\overline{V_i} \subseteq U_i$



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- choose $(V_i)_{i\in\mathbb{N}}$ with $\overline{V_i} \subseteq U_i$
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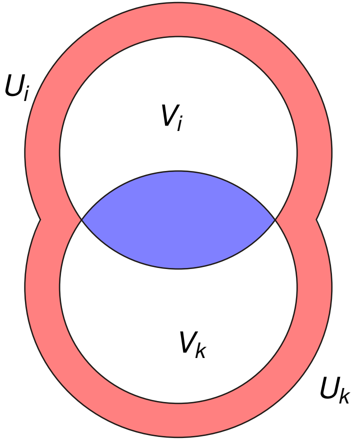

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- when all $\widetilde{f}_i$ are constructed, define
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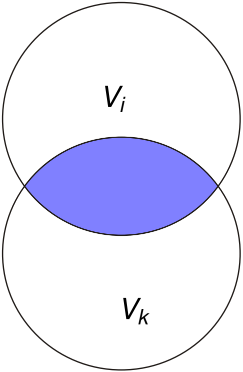

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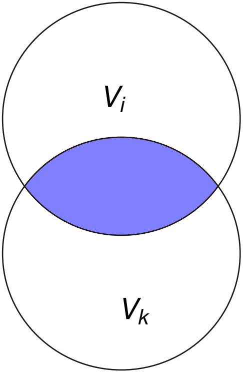
$\rightsquigarrow$ yields *smooth* bundle equivalence, uniquely determined by $\widetilde{f}_i' : V_i \to G$



Technical difficulties

Where are the difficulties?

- satisfy $\widetilde{f}_i = g_{ij} \cdot \widetilde{f}_j \cdot h_{ji}$
- extension from $U_k \cap \bigcup_{i<k} U_i$ to U_k

Solutions

- safety margin
- fade $\widetilde{f}_k$ (on $U_k \cap \bigcup_{i<k} U_i$) out to f_k (on $U_k \setminus \bigcup_{i<k} U_i$)
  - by convex combination from $\widetilde{f}_k \cdot f_k^{-1}$ to $\mathbb{1}$ (G is locally convex)
  - control smoothing procedure in the compact-open topology $\rightsquigarrow \widetilde{f}_k \cdot f_k^{-1}$ has values in a fixed "convex" $\mathbb{1}$-neighbourhood

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Note: Only methods from elementary topology are used!

Smoothing principal bundles

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continuous principal bundle

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(i.e., $h_{ij} \cdot h_{jk} \cdot h_{ki} = \mathbb{1}$) and continuous bundle equivalence

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$$\Rightarrow h_{ki} = h_{kj} \cdot h_{ji}$$

$\rightsquigarrow$ same inductive construction with appropriate order on tupels $(k, i) \in \mathbb{N} \times \mathbb{N}$

Result

G locally convex Lie group, M finite-dim. paracompact

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- proof uses only elementary topological concepts
- proof avoids classifying spaces
- relative version "for free"

Applications

Non-abelian Čech cohomology

Reformulation of the theorem

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Twisted K-theory

Element $c \in H^3(M;\mathbb{Z})$ describes "twisting" or ordinary K-theory (homotopy classes of sections in associated Fred($\mathcal{H}$)-bundle)

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Reformulation of the theorem

Twisted K-theory

Element $c \in H^3(M; \mathbb{Z})$ describes "twisting" or ordinary K-theory (homotopy classes of sections in associated $\mathrm{Fred}(\mathcal{H})$-bundle)

- c-twisted K-theory may be formulated "smoothly"
- construction of the twisted Chern Character

$$\mathrm{ch} : K_c(M) \to H^*(M, c),$$

needs existence of smooth structures on a principal $\mathrm{PU}(\mathcal{H})$-bundle corresponding to c.

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- "smooth and continuous homotopies agree" $\rightsquigarrow$ smoothing procedure for finite-dimensional principal bundles
- if $\dim(G) = \infty$, smooth structures on classifying spaces need not exist
- smoothing procedure for arbitrary principal bundles and bundle equivalences

Crucial points

- local compactness of M to "control" smoothing procedure
- local convexity of G to make "interpolation" work

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