Infinite-dimensional Lie Theory for Gauge Groups

Christoph Wockel
Fachbereich Mathematik
Technische Universität Darmstadt

www.mathematik.tu-darmstadt.de/~wockel

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Overview

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1. Introduction and Motivation

Setting:

$$P \xrightarrow{\pi} M$$

smooth K-principal bundle (possibly infinite-dimensional), i.e.

- $P \times K \to P$ smooth free action
- $P/K \cong M$ and $\pi(x) = x.K$
- π admits smooth local sections

Examples:

- trivial bundles $M \times K \xrightarrow{\text{pr}} M$
- homogeneous spaces $G \xrightarrow{\text{q}} G/H$

Physical Motivation:

$$\begin{aligned} M &\leftrightarrow \text{space time} \\ \text{Diff}(M) &\leftrightarrow \text{symmetry group of } M \\ \text{Conn}(P) &\leftrightarrow \text{potentials of forces} \\ \text{Gau}(P) &\leftrightarrow \text{symmetry group of Conn}(P) \end{aligned}$$

Automorphisms:

$$\mathsf{Aut}(P) := \{f \in \mathsf{Diff}(P) : f(p.k) = f(p).k\}$$

$$\mathsf{Gau}(P) := \{f \in \mathsf{Aut}(P) : f(p) = p.\gamma_f(p)$$
$$\gamma_f \text{ smooth}, \gamma_f(p.k) = \gamma_f(p).k\}$$

$$\Rightarrow \mathsf{Gau}(P) \cong C^\infty(P,K)^K$$

Aim:

- Topologise $C^\infty(P,K)^K$
- Calculate $\pi_n(C^\infty(P,K)^K)$
- Construct central extensions of $C^\infty(P,K)^K$

Connection to mapping groups:

- P trivial (i.e. $\sigma : M \to P$ section)
  $\Rightarrow \sigma^* : C^\infty(P,K)^K \to C^\infty(M,K)$ iso

- K abelian & $f \in C^\infty(P,K)^K$
  $\Rightarrow f(p \cdot k) = k^{-1} \cdot f(p) \cdot k = f(p)$
  $\Rightarrow f^\# : P/K \cong M \to K$

2. The Topology of Gauge Groups

From now on : $\mathrm{Gau}(P) = C^\infty(P,K)^K$

Definition: $f \in C^\infty(P,K)$

$$\Rightarrow T^n f \in C(T^n P, T^n K)$$

$$\Rightarrow C^\infty(P,K)^K \hookrightarrow \prod_{n=0}^{\infty} C(T^n P, T^n K)_c$$

leads to group topology on $C^\infty(P,K)^K$.

Theorem (W.,'03): M compact,
$\exp : \mathfrak{k} \to K$ restricts to diffeomorphism

$\Rightarrow C^\infty(P,K)^K$ is ∞-dimensional Lie group,
$\exp_* : C^\infty(P,\mathfrak{k})^K \to C^\infty(P,K)^K$

Lie algebra: $C^\infty(P,\mathfrak{k})^K \cong \mathrm{sec}(P \times_{\mathrm{Ad}} \mathfrak{k})$

From now on assume M compact, K locally exponential!

Homotopy groups:

Aim: Calculation of $\pi_n(C^\infty(P,K)^K)$

Thorem (W.,'04):

$$\iota : C^\infty(P,K)^K \to C(P,K)^K$$

induces isomorphisms

$$\pi_n(\iota) : \pi_n(C^\infty(P,K)^K) \to \pi_n(C(P,K)^K)$$

$\Rightarrow$ suffices to consider $C(P,K)^K$

Consider bundles over orientable compact surfaces with connected K.

Fact: $\partial M \neq \emptyset \Rightarrow P \cong M \times K$

$$\Rightarrow C(P,K)^K \cong C(M,K)$$

Theorem (W.,'05): If $K = K_0$, $\partial M \neq \emptyset$, then

$$\pi_n(C(P,K)^K) \cong \pi_{n+1}(K)^{2g-m-1} \oplus \pi_n(K)$$

with $g = \mathrm{gen}(M)$ and $m = \#\mathrm{comp}(\partial M)$.

$\partial M = \emptyset \Rightarrow \mathrm{Bun}(K, M) \cong \pi_1(K)$

$\Rightarrow P$ determined by $\gamma \in C_*(\mathbb{S}^1, K)$

$$\Rightarrow_{\text{torus}} C(P,K)^K \cong \{f \in C([0,1]^2, K) : f(0,s) = f(1,s) \quad f(t,0) = \gamma(t)^{-1} \cdot f(t,1) \cdot \gamma(t)\}$$

$$\Rightarrow C(P,K)^K_* \cong \{f : f(0) = e\} \cong C_*(\mathbb{S}^2, K) \times C_*(\mathbb{S}^1, K)^{2g}$$

Theorem (W.,'05): If $K = K_0$, $\partial M = \emptyset$, $g = \mathrm{gen}(M)$, then

$$\pi_n(C(P,K)^K_*) \cong \pi_{n+2}(K) \oplus \pi_{n+1}(K)^{2g}.$$

Problem: $C(P,K)^K_* \to C(P,K)^K \to K$ does not split canonically!

$\Rightarrow$ Obtain an exact homotopy sequence

$$\cdots \to \pi_{n+1}(K) \to \pi_n(C(P,K)^K_*) \to \pi_n(C(P,K)^K) \to \pi_n(K) \to \ldots$$

3. Central Extensions of Gauge Groups (according to Losev et.al.)

- $G := C^\infty(P, K)_0^K$
- $\mathfrak{g} := C^\infty(P, \mathfrak{k})^K \cong \mathrm{sec}(P \times_{\mathrm{Ad}} \mathfrak{k})$

Aim: Construct central extensions

$$Z \hookrightarrow \widehat{G} \twoheadrightarrow G$$

Philosophy:

- Central extensions are parametrised by cohomology classes $H^2(\mathfrak{g}, \mathfrak{z})$, $H^2(G, Z)$.

- Construct cocycle in $H^2(\mathfrak{g}, \mathfrak{z})$, then try to 'integrate'.

Mapping Groups (Pressley&Segal,86; Maier&Neeb,03): Construction of central extensions for $C^\infty(M, K)_0$ for $\dim(K) < \infty$.

Central extensions of $\mathfrak{g}$: Construct cocycle

$$\omega_{\kappa,\nabla} : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{z}$$

- $\kappa : \mathfrak{k} \times \mathfrak{k} \to V$ invariant, symmetric, bilinear
- $\nabla : \mathfrak{g} \times \mathcal{V}(M) \to \mathfrak{g}$, $(\xi, X) \mapsto \nabla_\xi.X$ connection (or covariant derivative on $\mathfrak{g} \cong \mathrm{sec}(P \times_{\mathrm{Ad}} \mathfrak{k})$)
- $\mathfrak{z} := \Omega^1(M,V)/dC^\infty(M,V) \supseteq H^1_{dR}(M,V)$

$$\omega_{\kappa,\nabla}(\xi,\eta) := [\kappa(\xi,\nabla\eta)] \in \mathfrak{z}$$

∇, ∇' connect. $\Rightarrow \omega_{\kappa,\nabla} - \omega_{\kappa,\nabla'}$ coboundary $\Rightarrow [\omega_\kappa] := [\omega_{\kappa,\nabla}] \in H^2(\mathfrak{g},\mathfrak{z})$ independent of choice of ∇, yields central extension

$$\mathfrak{z} \hookrightarrow \underbrace{\mathfrak{z} \oplus_{\omega_\kappa} \mathfrak{g}}_{\widehat{\mathfrak{g}}} \twoheadrightarrow \mathfrak{g}.$$

Central extension of G: Period map

$$\mathrm{per}_\omega : \pi_2(G) \times \pi_1(G) \to \mathfrak{z} \times \mathrm{Lin}(\mathfrak{z}, \mathfrak{g})$$

$\mathrm{im}(\mathrm{per}_\omega) = \Gamma \times 0 \Rightarrow$ ex. corresponding central extension

$$Z \hookrightarrow \widehat{G} \twoheadrightarrow G$$

Reduction Theorem (W.,'03): If $K = K_0$, then

$$\mathrm{im}(\mathrm{per}_\omega) = \Gamma \times 0 \Leftrightarrow \mathrm{im}(\mathrm{per}_{\omega,\mathbb{S}^1}) = \Gamma_{\mathbb{S}^1} \times 0$$

Is in particular satisfied if $\dim(K) < \infty$.

Universality? For mapping groups, finite-dim. s.-s. $\mathfrak{k}$: relies on $\mathfrak{k} \leq C^\infty(M, \mathfrak{k})$, i.e.

$$C^\infty_*(M, \mathfrak{k}) \hookrightarrow C^\infty(M, \mathfrak{k}) \twoheadrightarrow \mathfrak{k}$$

splits. For gauge algebras not true, same problem as before!

Conclusion

- $C^\infty(P,K)^K$ is ∞-dim. Lie group
- topology on $C^\infty(P,K)^K$ accessible, $\pi_n(C^\infty(P,K)^K)$ computable
- can construct central extensions
- problems:
  - K no canonical subgroup of $C(P,K)^K$ (resp. $C^\infty(P,K)^K$)
  - $\mathfrak{k}$ no canonical subalgebra of $C^\infty(P,\mathfrak{k})^K$ (resp. $C(P,\mathfrak{k})^K$)