1 Higher dimensional covers

(Joint work with Sven Porst & Chenchang Zhu)

Motivation:
Categorification of structure groups for gauge theories.

Notion of cover: \( X (n-1) \)-connected space

\[ n \text{-cover: } a: Y \to X \text{ fibration with } \pi_k(a) \text{iso for } k \neq n, \pi_n(Y) = 0 \]

Construction:

For characteristic map \( X \xrightarrow{f} K(\pi_n, n) \) iso on \( \pi_n \),

\[ Y = f^*(PK(\pi_n, n)) \]

the pullback of the path loop fibration. This is an \( n \)-cover of \( X \).

This is unsatisfactory from a group perspective.

Example:

\[
\begin{array}{ccc}
(n = 1) & Spin & \to & SO & 1 \text{-cover (simply connected)} \\
(n = 2) & ? & \to & \Omega Spin & (\infty \text{-dimensional Lie group with } \pi_2 \neq 0) \\
(n = 3) & ? & \to & Spin & (\? \cong \text{“string group”})
\end{array}
\]

2 A simple but instructive example: \( n = 1 \)

For \( G \) a connected Lie group (or also a topological group), consider the simply connected cover

\[ \pi_1 \hookrightarrow \tilde{G} \twoheadrightarrow G \]

This is

- a \( \pi_1 \)-principal bundle
• a central extension of $G$ by $\pi_1$

Now we know from group cohomology that $\tilde{G}$ is equivalent to $\pi_1 \times \theta_1 G$, which is the set $\pi_1 \times G$, endowed with the group multiplication

$$(a, g) \cdot (b, h) = (a + b + \theta_1(g, h), g \cdot h)$$

for some function $\theta_1 : G \times G \rightarrow \pi_1$

• associativity requires: $\theta_1(g, h) + \theta_1(gh, k) = \theta_1(g, hk) + \theta_1(h, k)$

• $\theta_1(g, e) = \theta_1(e, g) = 0$ implies that $(0, e)$ is a unit

This defines the group structure $\pi_1 \times \theta_1 G$, but how about the smooth structure?

Assume that $\theta_1 |_{U \times U}$ is smooth on a unit neighborhood $U \subset G$, then $\theta_1$ gives rise to a Čech cohomology class

$$[\tau \theta_1] \in \tilde{H}^1(G, \pi_1).$$

Endowing $\pi_1 \times \theta_1 G$ with the topology making $\pi_1 \times \theta_1 G \rightarrow G$ a $\pi_1$-principal bundle with the characteristic class $[\tau \theta_1]$ yields a Lie group topology on $\pi_1 \times \theta_1 G$ such that

$$\pi_1 \hookrightarrow \pi_1 \times \theta_1 G \rightarrow G$$

is equivalent to $\tilde{G}$ as a central extension.

### 3 Construction of $\theta_1$

For each $g \in G$, choose a smooth path $\alpha_g$, connecting the identity $e$ with $g$, i.e., a section $\alpha : G \rightarrow PG$ of the evaluation map $ev : PG \rightarrow G$, where $PG$ is the smooth path space of $G$ (w.l.o.g. we can assume $\alpha$ to be smooth on a unit neighbourhood). Then we can interpret $\alpha$ as a map from $G$ to the group $C_1$ of singular 1-chains on $G$, and thus we may take its group differential $d_{\text{gr}} \alpha$.

The crucial observation is that $d_{\text{gr}} \alpha$ takes values in the subgroup of 1-cycles $Z_1$, instead of $C_1$ (cf. Figure [1]). With this we set $\theta_1 := q \circ (d_{\text{gr}} \alpha) : G \times G \rightarrow \pi_1$, where $q : Z_1 \rightarrow H_1 \cong \pi_1$ is the canonical quotient map. From this it is obvious that $\theta_1$ is a cocycle.

**Theorem 1.** $[\theta_1]$ is universal for 2-cocycles $f$ which vanishes on some unit neighborhood, i.e.,

$$\text{Hom}(\pi_1, A) \rightarrow H^2_{\text{gr}}(G, A), \quad \varphi \mapsto [\varphi \circ \theta_1]$$

is bijective for each discrete abelian group $A$.

• Use standard covering theory for proof. In particular, the path lifting property (or parallel transport).
\[(d_{gp} \alpha)(g, h) = \alpha_g - \alpha_{gh} + g.\alpha_h =\]

\[\text{Figure 1: } (d_{gp} \alpha)(g, h) \text{ is a closed 1-cycle in } G\]

- \(H^n_{gr}(G, A)\): locally smooth group cohomology

**Upshot:** The universal locally constant 2-cocycle \(\theta_1\) describes simply connected covers! \(\rightsquigarrow\) We shall take this as the fundamental property for a generalisation to higher dimensions.

### 4 Construction of \(\theta_2\)

Now assume that \(G\) is simply connected. Then we find for each \(g, h \in G\) a (smooth) map \(\beta_{g, h} : \Delta^2 \to G\) such that \(\partial \beta_{g, h} = (d_{gp} \alpha)(g, h)\) (cf. Figure 2).

\[\text{Figure 2: } \partial \beta_{g, h} = (d_{gp} \alpha)(g, h)\]

As before, we observe that \((d_{gp} \beta)(g, h, k)\) is a 2-cycle in \(G\) (cf. Figure 3) and we set \(\theta_2 := g \circ (d_{gp} \beta) : G^3 \to \pi_2\). Again, it is obviously true that \(\theta_2\) defines a group 3-cocycle. Assuming w.l.o.g. that \(\beta_{g, h}\) depends smoothly on \(g\) and \(h\) on some unit neighbourhood and thus that \(\theta_2\) is constant on some unit neighbourhood.

**Theorem 2.** \([\theta_2] \in H^3_{gp}(G, \pi_2(G))\) is universal for locally constant 3-cocycles.

- Proof use path lifting (parallel transport) in 2-bundles.
- Question: To what extend describes \(\theta_2\) a 2-connected covering of \(G\)?
- Algebraically: \(\theta_2\) defines and extension of 2-groups

\[B\pi_2 \to G_{\theta_2} \to G\]
Theorem 3. Principal $\mathcal{G}$-2-bundles (for $\mathcal{G}$ a strict Lie 2-group) over $G$ are classified by $\tilde{\mathcal{H}}(G, \mathcal{G})$.

In particular, if $\mathcal{G}$ is $\mathcal{B}\pi_2$, then $\tilde{H}(G, \mathcal{G}) \cong \tilde{H}^2(G, \pi_2)$ and $[\tau \theta_2] \in \tilde{H}^2(G, \pi_2)$ gives rise to a principal $\mathcal{B}\pi_2$-2-bundle $P_{\tau \theta_2} \to G$. What would be nice is Lie 2-group structure on $P_{\tau \theta_2}$, but that is too much to ask for! Remedy: invert Morita morphisms of bundles obtain a weak group object in the category of smooth stacks, i.e., a stacky Lie group.

5 What is this good for?

- $\mathcal{G}_{\theta_2}$ provides a generalisation of Lie’s Third Theorem to Banach–Lie algebras, (which fails in general when trying to integrate Banach–Lie algebras to Banach–Lie groups).

- Generalisation to higher dimensions in possible, giving a cohomology class $[\theta_n] \in H^{n+1}_{\text{gp}}(G, \pi_n)$, which is universal for locally constant $(n+1)$-group cocycles $\mapsto$ relation to string group models!

- For $G$ compact, simple and simply connected, the transgression map $\tau : H^3_{\text{gp}}(G, \pi_3) \to \tilde{H}^3(G, \pi_3)$ may be understood in terms of the Dijkgraaf-Witten correspondence.

Figure 3: $(d_{\text{gp}}(\beta))(g, h, k)$ is a closed 2-cycle in $G$