Central extensions of Gauge Groups

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Abstract

This text deals with gauge groups and their central extensions. We are introducing the idea of gauge theories in the first chapter end explain why central extensions of these groups, as certain symmetry groups of physical theories, are important in the quantisation procedure of these theories.
In the second chapter we introduce principal fibre bundles $K \hookrightarrow P \twoheadrightarrow M$ as the mathematical framework for gauge theories and develop for compact base spaces an interpretation of their gauge groups $\mathrm{Gau}(\mathcal{P})$ as a subgroup of a finite product of mapping groups $C^{\infty}(\overline{V}_i, K)$ for finitely many trivial compact neighbourhoods $\overline{V}_i$. In view of the extendibility of the results presented in this text we are formulating these results for principal fibre bundles as well as for principal fibre bundles with boundary.
Following ideas from [Glö02] and [Nee01], this viewpoint enables us to topologise the gauge group and to obtain for locally exponential structure groups a Lie group structure on $\mathrm{Gau}(\mathcal{P})$. This will be done in the third chapter. After introducing the concept of central extensions for Lie groups and some results about them from [Nee02] in the fourth chapter, we extend some results for the mapping group $C^{\infty}(M, K)$ from [NM03] to gauge groups using their interpretation as mapping groups. This will in particular lead to the construction of a central extension

$$Z \hookrightarrow \widehat{\mathrm{Gau}(\mathcal{P})}_0 \twoheadrightarrow \mathrm{Gau}(\mathcal{P})_0$$

by the abelian group $Z \cong \mathfrak{z}/\Gamma$ where $\mathfrak{z}$ is a locally convex space and Γ is the image of the period map associated to the covariant cocycle constructed in the fifth chapter.

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Chapter 1

Motivation

1.1 Gauge Theories

Gauge theories occur in Physics in the description of elementary forces, namely the electromagnetic, the weak and the strong interaction. We will roughly line out a simple example to present the central idea of gauge theories, without going into physical details.

The integral parts of physical theories are their equations of motion which describe the behaviour of the considered physical systems. These equations are mostly differential equations and one of the most interesting properties of these equations are their symmetries. These symmetries are invariances under specific transformations Λ_φ where φ is element of a group G where these transformations may act on various elements of the given equation, e.g. functions or (differential-) operators. We say that Λ_φ is a *symmetry* of a given equation or theory and that G is their *symmetry group* , if the physical statements of the theory are invariant under the transformations Λ_φ for all $\varphi \in G$ (see below).

Symmetries give rise to conservation laws, forces and hence to physical statements about the validity of the given theory. Roughly speaking, one divides the occurring symmetries in two kinds, namely external symmetries and internal symmetries (cf. Section 2.4).

External symmetries are always related to a coordinate change of the configuration space, i.e. coordinate transformations. These symmetries are even present in classical mechanics, where the invariance of $F(t) = m_0\ddot{x}(t)$ under Gallilei transformations yields momentum conservation. Internal symmetries are more subtle. They are related to changes of the mathematical description, which do *not* come from a change of coordinates.

In Quantum Mechanics, the equation of motion for a quantum mechanical system, described by a function $\psi : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C}$ is the *Schrödinger equation*

$$i\frac{\partial}{\partial t}\psi(t,x) = \Delta\psi(t,x) \tag{1.1}$$

(where ψ is assumed to be of sufficiently high differentiable order). The physical relevant part of ψ is its modulus square

$$|\psi(t,x)|^2 = \psi^*(t,x)\psi(t,x) \tag{1.2}$$

which integrated over a space time region gives the probability for the particle being detected within this region.
Since both, (1.1) and (1.2) are invariant under transformations of the state of the form

$$\psi(t,x) \mapsto (\Lambda_\varphi.\psi)(t,x) := \varphi\ \psi(t,x)$$

for $\varphi \in U(1) = \{\varphi \in \mathbb{C} : |\varphi| = 1\}$, i.e.

$$\frac{\partial}{\partial t}\psi(t,x) = \Delta\psi(t,x) \Leftrightarrow \frac{\partial}{\partial t}(\Lambda_\varphi.\psi)(t,x) = \Delta(\Lambda_\varphi.\psi)(t,x) \tag{1.3}$$

$$\psi^*(t,x)\psi(t,x) = (\Lambda_\varphi.\psi)^*(t,x)\Lambda_\varphi.\psi(t,x) \tag{1.4}$$

the group $U(1)$ is a *symmetry group of Quantum Mechanics*. (1.4) is also valid, if φ depends on x, i.e. if φ is a function $\varphi : \mathbb{R}^3 \to U(1)$ and one of the next questions might be to ask for the invariance of (1.1) under transformations

$$\psi(t,x) \mapsto (\Lambda_\varphi.\psi)(t,x) = \varphi(x)\ \psi(t,x)$$

for $\varphi : \mathbb{R}^3 \to U(1)$. Of course, (1.3) does *not* hold, because the product rule produces additional terms. The problem here is, that the differentials are not affected by Λ_φ and that there are no counter terms cancelling the additional terms from the product rule. So we have to introduce these terms by the substitution

$$\frac{\partial}{\partial x^\mu} \mapsto D_\mu := \frac{\partial}{\partial x^\mu} + eA_\mu$$

where $e \in \mathbb{R}$ is a constant and $A_\mu : \mathbb{R} \to i\mathbb{R} = L(U(1))$ are functions, which change under Λ_φ according to

$$\Lambda_\varphi.A_\mu = A_\mu - \frac{1}{e}\ \varphi^{-1}\frac{\partial\varphi}{\partial x^\mu}\ .$$

The D_μ are called *covariant derivatives* and they transform as

$$\Lambda_\varphi.D_\mu = \frac{\partial}{\partial x^\mu} + A_\mu - \varphi^{-1}\frac{\partial\varphi}{\partial x^\mu}$$

and we obtain

$$(\Lambda_\varphi.D_\mu).(\Lambda_\varphi.\psi(t,x)) = \psi(t,x)\frac{\partial\varphi}{\partial x^\mu}(x) + \varphi(x)\frac{\partial\psi}{\partial x^\mu}(t,x)$$
$$+ eA_\mu(x)\varphi(x)\psi(t,x) - \varphi(x)^{-1}\frac{\partial\varphi}{\partial x^\mu}(x)\varphi(x)\ \psi(t,x) = \Lambda_\varphi.(D_\mu.\psi(t,x))$$

Now the substitution

$$\Delta = \sum_{i=1}^{3}\left(\frac{\partial}{\partial x^\mu}\right)^2 \mapsto \Delta_{cov} := \sum_{\mu=1}^{3} {D_\mu}^2$$

yields an operator, which is invariant under Λ_φ, i.e.

$$(\Lambda_\varphi.\Delta_{cov})\Lambda_\varphi.\psi(t,x) = \Lambda_\varphi.\big(\Delta_{cov}\psi(t,x)\big)$$

where $\Lambda_\varphi.\Delta_{cov} = \sum(\Lambda_\varphi.D_\mu)^2$. We now substitute Δ in (1.1) by the covariant operator Δ_{cov} and (1.3) now reads

$$\frac{\partial}{\partial t}\psi(t,x) = \Delta_{cov}\psi(t,x) \Leftrightarrow \frac{\partial}{\partial t}(\Lambda_\varphi.\psi)(t,x) = \Lambda_\varphi.\Delta_{cov}(\Lambda_\varphi.\psi)(t,x) \tag{1.5}$$

We thus have incorporated a new symmetry in the considered theory. Now one may ask the legitimate question, whether we only have made our theory more complicated or really got some new insights. The answer is quite deep and is the reason, why gauge theories are so important in our days physics. While the functions A_μ only seemed to be counter terms to force the equations to be invariant under Λ_φ, they have the *interpretation as the potential of a force* (cf. Section 2.4). This interpretation enables us to incorporate the description of forces into Quantum Mechanics and leads in the appropriate context to an understanding of the quanta of the three elementary forces mentioned above.

1.2 Central Extensions

Central extensions, especially by the circle group $U(1)$ occur whenever one studies representation of groups on Hilbert spaces which are induced by projective representations. An extension of G by Z is an exact sequence of groups

$$Z \hookrightarrow \hat{G} \twoheadrightarrow G,$$

where this notion means that the homomorphism on the left is injective, the one on the right is surjective and the image of the left hand equals the kernel of the right hand homomorphism. Then Z can be considered as a subgroup of $\hat{G}$ and G

is isomorphic to $\hat{G}/Z$. If Z is central in $\hat{G}$ the extension is called central and if, in addition, there exists a homomorphism $\sigma : G \to \hat{G}$ satisfying $q \circ \sigma = \mathrm{id}_G$, then the extension is said to split (cf. Section 4.1). If a central extension splits, then $\hat{G}$ is isomorphic to a direct product $Z \times G$ and in this sense central extensions form a context for decomposing groups into the parts they are built of which is more general than direct products with abelian groups. Central extensions occur in quantisation procedures, which will roughly be described in the remaining paragraph.
Having classified the symmetry groups of a given physical theory one wants to know what the representation of these groups are. The most important representations for quantised theories are the *unitary* ones, i.e. homomorphisms

$$\varphi : G \to U(\mathbb{H})$$

for a suitably chosen Hilbert space $\mathbb{H}$. Since the *unitary group* $U(\mathbb{H})$ is also a topological group [Sch95, Chapter III, Proposition 3.2], one requires the homomorphism φ to be continuous, whenever G is a topological group. A group, for which such a homomorphism exists is called *symmetry group of the quantum system* represented by the projective space

$$\mathbb{P} := \{\text{one-dimensional subspaces of } \mathbb{H}\}.$$

The projective space $\mathbb{P}$ inherits a projective product from the scalar product $\langle \cdot, \cdot \rangle_{\mathbb{H}}$, given for two states $[x]$ and $[y]$ by

$$\langle [x], [y] \rangle_{\mathbb{P}} := \frac{1}{\|x\|^2 \|y\|^2} \langle x, y \rangle_{\mathbb{H}} \langle y, x \rangle_{\mathbb{H}}.$$

As well as the *unitary operators*

$$U(\mathbb{H}) = \{U : \mathbb{H} \to \mathbb{H} : U \text{ is linear, bijective and } \langle x, y \rangle_{\mathbb{H}} = \langle U.x, U.y \rangle_{\mathbb{H}}\}$$

are the automorphisms of $\mathbb{H}$, one considers for $\mathbb{P}$ the mappings

$$\mathrm{Aut}(\mathbb{P}) = \Big\{ f : \mathbb{P} \to \mathbb{P} : \big\langle [x], [y] \big\rangle_{\mathbb{P}} = \big\langle f\big([x]\big), f\big([y]\big) \big\rangle_{\mathbb{P}} \Big\}$$

as automorphisms. Clearly each $U \in U(\mathbb{H})$ induces a $q(U) \in \mathrm{Aut}(\mathbb{P})$ by setting $q(U)([x]) := [U.x]$, and we denote by

$$PU(\mathbb{H}) = q\big(U(\mathbb{H})\big)$$

the subgroups of projective automorphisms coming from unitary operators. This leads to the short exact sequence

$$U(1) \hookrightarrow U(\mathbb{H}) \xrightarrow{q} PU(\mathbb{H})$$

and $PU(\mathbb{H})$ is a topological group with respect to the induced topology from $U(\mathbb{H})$. Now one axiom of quantum theory is that each classical symmetry group G admits a homomorphism $T : G \to PU(\mathbb{H})$ which is required to be continuous if G is a topological group. But this homomorphism does not lead in a natural way to a unitary representation of G. Consider

$$\hat{G} := \{(U, g) \in U(\mathbb{H}) \times G : q(U) = T(g)\}$$

as a closed subgroup of the topological group $U(\mathbb{H}) \times G$. Then the homomorphisms $q' : \hat{G} \to G$, $(U, g) \mapsto g$ and $S : \hat{G} \to U(\mathbb{H})$, $(U, g) \mapsto U$ are continuous, q' is surjective since q is so and the kernel of $\hat{q}$ is $\ker(q) \times \{e\}$. This leads to a central extension of G by $U(1)$, such that the diagram

$$\begin{array}{ccccc} U(1) & \xrightarrow{\text{incl}} & \hat{G} & \xrightarrow{q'} & G \\ \big\downarrow \text{id} & & \big\downarrow \text{S} & & \big\downarrow \text{T} \\ U(1) & \xrightarrow{\text{incl}} & U(\mathbb{H}) & \xrightarrow{q} & PU(\mathbb{H}) \end{array}$$

commutes. Since the central extension $U(1) \hookrightarrow \hat{G} \twoheadrightarrow G$ will in general *not* split, the classical symmetry group G will in general *not* be a symmetry group of the quantised system, but a central extension $\hat{G}$ of G by $U(1)$ with representation homomorphism $S : \hat{G} \to U(\mathbb{H})$.

Chapter 2

Principal Fibre Bundles and the Gauge Group

2.1 Manifolds with Boundary

Since we have to formulate the theory standing beyond the introductory example for manifolds with boundary (for some reasons that will become apparent later on) we first generalise the concept of a smooth manifold to manifolds with boundary.

Definition. Differentiable map, Differential, Smooth Map: If $U \subseteq \mathbb{R}^n$ is a set with $\mathrm{int}(U) \neq \varnothing$ and E a locally convex topological vector space, then a continuous map $f : U \to E$ is said to be a *differentiable map* or of *class* C^1 , if $f_{\mathrm{int}} := f|_{\mathrm{int}(U)}$ is of class C^1 as a map defined on open subset of $\mathbb{R}^n$ and the map

$$f_{\mathrm{int}} : \mathrm{int}(U) \times \mathbb{R}^n \to E, \quad (x, v) \mapsto df(x).v$$

extends continuously to $U \times \mathbb{R}^n$. This extension is called the *differential*

$$df : U \times \mathbb{R}^n \to E$$

and we inductively define f to be of *class* C^k if its differential df is of class C^{k-1}. Furthermore f is said to be *smooth map* if f is of class C^k for all $k \in \mathbb{N}$.

Remark 2.1.1 Since $\mathrm{int}(U \times \mathbb{R}^{2k-1}) = \mathrm{int}(U) \times \mathbb{R}^{2k-1}$ we have for $k = 1$ that $(df)_{\mathrm{int}} = d(f_{\mathrm{int}})$ and we inductively obtain $(d^k f)_{\mathrm{int}} = d^k(f_{\mathrm{int}})$. Hence *higher differentials* $d^k f$ are defined to be the continuous extensions of the differentials $d^k f_{\mathrm{int}}$.

Lemma 2.1.2 *If* $U \subseteq \mathbb{R}^n$ *and* $V \subseteq \mathbb{R}^m$ *such that* $\mathrm{int}(U) \neq \varnothing$, $\mathrm{int}(V) \neq \varnothing$ *and* $f : U \to \mathbb{R}^m$, $g : V \to \mathbb{R}^l$ *are mappings of class* C^k *such that* $f(U) \subseteq V$ *and* $f\big(\mathrm{int}(U)\big) \subseteq \mathrm{int}(V)$, *then* $g \circ f : U \to \mathbb{R}^l$ *is a map of class* C^k *and the differential is given by* $d(g \circ f)(x).v = dg\big(f(x)\big).df(x).v$.

Proof: Since $f\big(\mathrm{int}(U)\big) \subseteq \mathrm{int}(V)$ the map $(g \circ f)_{\mathrm{int}} = g \circ f_{\mathrm{int}}$ is of class C^1 as a map defined on an open subset of $\mathbb{R}^n$. Then we have for $v \in \mathbb{R}^n$ that the map

$$\mathrm{int}(U) \to \mathbb{R}^l, \quad x \mapsto d(g \circ f_{\mathrm{int}})(x).v = dg\big(f(x)\big).d\, f_{\mathrm{int}}(x).v$$

has the continuous extension

$$U \to \mathbb{R}^l, \quad x \mapsto dg\big(f(x)\big).df(x).v,$$

as the composition of continuous maps. Hence $g \circ f$ is of class C^1 and the same argument applies to mappings of class C^k for arbitrary k, hence also for smooth maps.

□

Having this chain rule we can introduce the concept of manifolds with boundary.

Definition. Manifold with Boundary: If M is a paracompact Hausdorff space, then M is called a *manifold with boundary* if for each $x \in M$ there exists an open neighbourhood $U_x \subseteq M$ of x and a homeomorphism $\varphi_x : U_x \to V_x$ called a *chart* around x, such that

i) V_x is open in the half space $\mathbb{R}^n_+ := \{(x^1, \ldots, x^n) \in \mathbb{R}^n : x^1 \geq 0\}$,

ii) for all $x, y \in M$ with $U_x \cap U_y \neq \varnothing$ the coordinate change

$$\varphi_{xy} : \varphi_x(U_x \cap U_y) \to \varphi_y(U_x \cap U_y), \quad z \mapsto \varphi_y(\varphi_x^{-1}(z))$$

is of class C^∞.

The pairs $(U_i, \varphi_i)_{i \in I}$ are called a *differentiable structure* on M and a maximal differentiable structure is called an *atlas* .

Remark 2.1.3 Since the coordinate changes φ_{xy} are in particular homeomorphisms between open sets in $\mathbb{R}^n_+$ it follows that points of the hyperplane $H := \{(x^1, \ldots, x^n) \in \mathbb{R}^n : x^1 = 0\}$ are mapped by φ_{xy} into H again. Hence the set

$$\partial M := \{x \in M : \varphi_x(x) \in H\}$$

is well-defined and is called the *boundary* of M. We get the conventional notation of a *smooth manifold* (without boundary) if $\partial M = \varnothing$.

Remark 2.1.4 One has to take care when dealing with maps between manifolds with boundary because one wants the maps to preserve the distinction between boundary points and interior points. As seen in Lemma 2.1.2 we have to require a map $f : M \to N$ between manifolds with boundary to map $\mathrm{int}(M)$ into $\mathrm{int}(N)$ and in addition we require f to map ∂M into ∂N if ∂M and ∂N are non-empty. If one of the sets ∂M or ∂N is empty, there is no additional requirement. We will always point out these things in the corresponding statements, sometimes shortened by requiring that f has the *correct mapping property w.r.t. boundaries* .

Definition. Coordinate Map: If M is an n-dimensional manifold with boundary, N a smooth manifold modelled on the topological vector space E and $f : M \to N$ a continuous map, then f is said to be of class C^1 if for each point $x \in M$ and each pair of charts $\varphi_x : U_x \to \mathbb{R}^n$ and $\varphi_{f(x)} : U_{f(x)} \to E$ the corresponding *coordinate map*

$$\tilde{f} : \varphi_x(U_x \cap f^{-1}(U_{f(x)})) \to \varphi_{f(x)}(U_{f(x)}), \quad \varphi_x(y) \mapsto \varphi_{f(x)}\big(f(y)\big)$$

is of class C^1 in the sense defined above. In the same way f is said to be of class C^k if $\tilde{f}$ is of class C^k for all charts and f is said to be smooth if $\tilde{f}$ is of class C^k for all charts and for all $k \in \mathbb{N}$.

Remark 2.1.5 Since the differentials of the coordinate changes are forced to extend in the same way to the boundary as $\tilde{f}$ does it suffices to show that $\tilde{f}$ is of class C^k for an arbitrary pair of charts to verify that f is of class C^k. That is the reason why the dependence of $\tilde{f}$ on the charts is not made explicit in the definition.

A tangent vector to M is supposed to be something that looks like a vector in $\mathbb{R}^n$ attached to some point $x \in M$ which is in some sense invariant under changes of coordinates. Thus two triples (x, v, φ_x) and (x, v', φ'_x) with $x \in M$, $v, v' \in \mathbb{R}^n$ and φ_x, φ'_x charts around x are said to be equivalent if

$$v = d(\varphi_x^{-1} \circ \varphi'_x)(x).v'.$$

It is an easy calculation that for fixed x this relation actually defines an equivalence relation on the set $\mathbb{R}^n \times J_x$ where $J_x \subseteq I$ is the subset of the index set for a differentiable structure on M consisting of all $j \in I$ such that $x \in U_j$.

Definition. Tangent Space, Tangent Bundle, Tangent Map: If M is an n-dimensional manifold with boundary, then the *tangent space* at $x \in M$ is defined to be

$$T_xM := (\mathbb{R}^n \times J_x)/\sim,$$

where $\sim$ is the equivalence relation defined above. The *tangent bundle* TM is the disjoint union of all tangent spaces

$$TM := \bigcup_{x \in M} T_xM.$$

For a smooth map $f : M \to N$ having the correct mapping property w.r.t. boundaries the *tangent map* is defined to be

$$Tf : TM \to TN, \quad (x, [v, i]) \mapsto (f(x), [d(\varphi_j \circ f \circ \varphi_i^{-1})(\varphi_i(x)).v, j])$$

where (U_j, φ_j) is a chart around $f(x)$.

Remark 2.1.6 Although this definition seems to be quite technical, the reader should be reminded, that T_xM actually is a vector space isomorphic to $\mathbb{R}^n$ which is "attached to x". That Tf is well-defined simply is the chain rule. Compared with the ordinary differential df for maps between vector spaces, the tangent map Tf does not only yield the derivative $df(x,v)$, but also keeps track of the evaluation point x.

Lemma 2.1.7 *If M is a manifold with boundary, so is TM. The projection $\pi : TM \to M$, $(x,[v,i]) \mapsto x$ is a smooth map between manifolds with boundary having the correct mapping properties w.r.t. boundaries.*

Proof: Although the proof is straight forward, we will line it out to convince the reader that on this stage there is no big difference to the non-boundary case. First we have to define a topology on TM which turns it into a paracompact Hausdorff space. This can be done by endowing TM with the initial topology w.r.t. the mappings

- $\mathrm{pr}_1 : TM \to M$, $(x,[v,i]) \mapsto x$
- $\mathrm{pr}_i : TU_i \to \mathbb{R}^n$, $(x,v,i) \mapsto v$

where TU_i denotes the tangent bundle of the open sub-manifold with boundary U_i of M (note that the differentiable structure on TU_i consists of a single chart $\varphi_i \times d\varphi_i$) and i runs through I. This topology is Hausdorff since the topologies on M and $\mathbb{R}^n$ are, and the paracompactness also follows from the paracompactness of the topologies on M and $\mathbb{R}^n$.
If $(x,[v,i]) \in TM$ with $x \in U_i$, then TU_i is open in TM and the map

$$\varphi_i \times d\varphi_i : TU_i \to U_i \times \mathbb{R}^n, \;\; (y,[w,i]) \mapsto \big(\varphi_i(y), d\varphi_i(y).w\big)$$

defines a chart around $(x,[v,i])$. The requirement for the coordinate change to be smooth is exactly the requirement for the smoothness of the coordinate changes in M.
The smoothness of π holds trivially because the coordinate map of $\varphi_i \times d\varphi_i$ is simply the projection $\mathrm{pr}_1 : \varphi_i(U_i) \times \mathbb{R}^n$, $(x,v) \mapsto x$ which is smooth.
Since $\mathrm{int}(U_i \times \mathbb{R}^n) = \mathrm{int}(U_i) \times \mathbb{R}^n$ the definition of the charts for TM yields immediately that $(x,[v,i])$ is a a boundary point in TM if and only if x is a boundary point in M. Hence we have $\pi\big(\mathrm{int}(TM)\big) = \mathrm{int}(M)$ and $\pi(\partial TM) = \partial M$.

□

Corollary 2.1.8 *If M and N are manifolds with boundary and $f : M \to N$ is a smooth map having the correct mapping property w.r.t. boundaries, then*

$$Tf : TM \to TN$$

is a smooth map having the correct mapping properties w.r.t. boundaries.

For the next definition (and also the preceeding corollary) we first had to check, that *TM* inherits the structure of a manifold with boundary from M.

Definition. Higher Tangent Maps: If M and N are manifolds with boundary and $f : M \to N$ is smooth having the correct mapping property w.r.t. boundaries, then the *higher tangent maps* are inductively defined by

$$T^n f := T(T^{n-1} f) : T(T^{n-1} M) \to T(T^{n-1} N).$$

Lemma 2.1.9 *If M, N and O are a manifolds with boundary and $f : M \to N$, $g : N \to O$ are smooth maps required to have the correct mapping properties w.r.t boundaries, then for the tangent maps $T^n f : T^n M \to T^n N$ and $T^n g : T^n N \to T^n O$ we have $T^n(g \circ f) = T^n g \circ T^n f$.*

Proof: The identity $Tg \circ Tf = T(g \circ f)$ is

$$\begin{aligned}
&Tg(Tf(x, [v, i])) \\
=&(g(f(x)), [d(\varphi_k \circ g \circ \varphi_j^{-1})\Big(\varphi_j\big(f(x)\big)\Big).d(\varphi_j \circ f \circ \varphi_i^{-1})\big(\varphi_i(x)\big).v, k]) \\
=&(g(f(x)), [d(\varphi_k \circ g \circ f \circ \varphi_i^{-1})\big(\varphi_i(x)\big).v, k]) = T(g \circ f)(x, [v, i])
\end{aligned}$$

and induction on n yields the assertion.

□

Lemma 2.1.10 *If M and N are manifolds with boundary and $f : M \to N$ of class C^k, having the correct mapping properties w.r.t. boundaries, then the tangent map $T^n f : T^n M \to T^n N$ is of class C^{k-n} for all $n \leq k$.*

Proof: For $n = 1$ the tangent map $T^n f$ locally is the map $(x, v) \mapsto (f(x), df(x).v)$ which is of class C^{k-1}. The assertion now follows by induction on n.

□

Proposition 2.1.11 *If M is a manifold with boundary and $(U_i)_{i \in I}$ is an open cover of M, then there exists a partition of unity $(f_i)_{i \in I}$ subordinate to this open cover.*

Proof: The construction in the proof of [Hir76, Theorem 2.1] actually yields smooth functions $f_i : U_i \to \mathbb{R}$ also in the sense of manifolds with boundary.

□

2.2 Principal Fibre Bundles

The following objects give the natural context gauge theories are formulated in. The idea is that one wants to link the dependence on the phase factors to the coordinates of the manifold M modelling the space time. The easiest way to do this is to consider the space $P = M \times K$ for a Lie group K. Then K acts canonically on P by $\big((x,k),k'\big) \mapsto (x,kk')$. Principal fibre bundles are generalisations of these *trivial bundles*.

Definition. Equivariant Map, Invariant map: If K is a group acting on sets X and Y, then a map $f : X \to Y$ is said to be *equivariant* or *K-invariant* if $f(x \cdot k) = f(x) \cdot k$ for all $x \in X$ and $k \in K$. If X and Y have the structure of differentiable manifolds, K is a Lie group and the action on X and Y is smooth, then the set of K-invariant smooth functions from X to Y is denoted by $C^\infty(X,Y)^K$.

Definition. Principal Fibre Bundle: Assume that P and M are manifolds and K is a Lie group all modelled on locally convex spaces such that K acts smoothly on P from the right by $(p,k) \mapsto p \cdot k =: \rho_k(p)$ and $\pi : P \twoheadrightarrow M$ is a differentiable surjective map. Then $\mathcal{P} := (K,M,P,\pi)$ is said to be a *principal fibre bundle* if for each $x \in M$ there exists an open neighbourhood $U_x \subseteq M$ and an equivariant diffeomorphism $\Theta_x : \pi^{-1}(U_x) \to U_x \times K$ such that $\pi\big|_{U_x} = \mathrm{pr}_1 \circ \Theta_x$.

Remark 2.2.1 Note that we do not restrict to finite-dimensional bundles here. The reader not familiar with calculus in locally convex spaces and infinite-dimensional manifolds is referred to Chapter 3.

Definition. Principal Fibre Bundle with Boundary: Assume that P and M are finite-dimensional manifolds with boundary and K is a finite-dimensional Lie group, such that K acts smoothly on P from the right and $\pi : P \twoheadrightarrow M$ is a differentiable surjective map having the correct mapping property to boundaries. Then $\mathcal{P} := (K,M,P,\pi)$ is said to be a *principal fibre bundle with boundary* , if for each $x \in M$ there exists an open neighbourhood $U_x \subseteq M$ and an equivariant diffeomorphism $\Theta_x : \pi^{-1}(U_x) \to U_x \times K$ having the correct mapping properties with respect to boundaries.

Remark 2.2.2 The Lie group K is called the *structure group* , M the *base space* , P the *total space* and π the *bundle projection* of $\mathcal{P}$. Since we will not consider any other type of bundles in this text we will call a principal fibre bundle with structure group K, with or without boundary, shortly a *K-bundle* . If we need additional assumptions on the boundaries we will always point them out. A K-bundle is said to be finite-dimensional if the total space P is a finite-dimensional manifold (then K and M are automatically finite-dimensional). The open set U_x is called *trivialising* *neighbourhood* and the diffeomorphism Θ_x *local trivialisation*. For $x \in M$ the set $\pi^{-1}(x)$ is diffeomorphic to K and is called the *fibre* over x. Note that it does in general *not* admit a canonical group structure (cf. the following proposition).

Lemma 2.2.3 *If $\mathcal{P} = (K, M, P, \pi)$ is a K-bundle, then M is diffeomorphic to the orbit space P/K.*

Proof: We define $f : P/K \to M$, $[p] \mapsto \pi(p)$. This map is well-defined since $\pi(p \cdot k) = \mathrm{pr}_1\big(\Theta_{\pi(p)}(p \cdot k)\big) = \mathrm{pr}_1\big(\Theta_{\pi(p)}(p)\big) = \pi(p)$, surjective since π is so and injective since $\pi(p) = \pi(p') \Rightarrow p' = p \cdot k$ which can be seen in a local trivialisation. Since locally P is diffeomorphic to $U_x \times K$ and the diffeomorphism commutes with the action, f is locally the map $(U_x \times K)/K \to U_x$ and hence a diffeomorphism.

□

Definition. Homomorphism, Isomorphism, Automorphism:
If $\mathcal{P} = (K, M, P, \pi)$ and $\mathcal{P}' = (K, M', P', \pi')$ are K-bundles, then a smooth equivariant map $f : P \to P'$, having the correct mapping property with respect to boundaries, is called a *homomorphism of K-bundles* . It is called an *isomorphism* if it is also a diffeomorphism. If $\mathcal{P} = \mathcal{P}'$ then an isomorphism is called *automorphism* and the *group of automorphisms* of $\mathcal{P}$ is denoted by $\mathrm{Aut}(\mathcal{P})$.

Remark 2.2.4 The *automorphism group* $\mathrm{Aut}(\mathcal{P})$ is a group with respect to composition, since the inverse diffeomorphism to an equivariant diffeomorphism is again equivariant as an easy calculation shows.

Lemma 2.2.5 *A homomorphism $f : P \to P'$ of the K-bundles $\mathcal{P} = (K, M, P, \pi)$ and $\mathcal{P}' = (K, M', P', \pi')$ induces a smooth map*

$$f_M : M \cong P/K \to M' \cong P'/K, \quad [p] \mapsto [f(p)].$$

Proof: That f_M is well-defined follows from the property of f being a homomorphism. Locally f_M is the map $y \mapsto \pi'\Big(f\big(\Theta_x^{-1}(y, e)\big)\Big)$ and hence smooth.

□

Proposition 2.2.6 *If $\mathcal{P} = (K, M, P, \pi)$ is a K-bundle with a global section, i.e. a smooth map $\sigma : M \to P$ such that $\pi \circ \sigma = \mathrm{id}_M$, then $\mathcal{P}$ is isomorphic to the trivial bundle $M \times K$.*

Proof: The map $f : M \times K \to P$, $(x, k) \mapsto \sigma(x)k$ defines a global trivialisation.

□

The following example is the situation we are dealing with in most of the parts of this text.

Example 2.2.7: Homogeneous Spaces If G is a Lie group and H a closed subgroup, then the homogeneous space G/H is an H-bundle with total space G and base space G/H. In particular we obtain for each sphere $\mathbb{S}^n$ a principal fibre bundle with base space $\mathbb{S}^n$ when considering $G = \mathbf{SO}(n)$ and $H = \mathbf{SO}(n-1)$. Since $\mathbf{SO}(n)/\mathbf{SO}(n-1)$ is diffeomorphic to the orbit of a point in $\mathbb{S}^n$ whose stabiliser subgroup is $\mathbf{SO}(n-1)$, e.g. $(0, \ldots, 0, 1)$, we have $\mathbf{SO}(n)/\mathbf{SO}(n-1) \cong \mathbb{S}^n$ since $\mathbf{SO}(n)$ acts transitively on $\mathbb{S}^n$.

Example 2.2.8: Frame Bundle If M is a finite-dimensional manifold, then

$$P := \bigcup_{x\in M} \{x\} \times \{(v_1, \ldots, v_n) \in (T_xM)^n : \\ (v_1, \ldots, v_n) \text{ is a basis of } T_xM\}$$

is the total space of the $\mathbf{GL}(n)$-bundle with base space M and bundle projection $\pi : P \to M$, $(x, v_1, \ldots, v_n) \mapsto x$. If $\varphi : U \to \varphi(U) \subseteq \mathbb{R}^n$ is a chart, then U is a trivialising neighbourhood with local trivialisation

$$\Theta(x, v_1, \ldots, v_n) = (\varphi(x), d\varphi_x.v_1, \ldots, d\varphi_x.v_n).$$

This bundle is called the *frame bundle* of the manifold M.

One also wants to describe K-bundles by glueing trivial bundles by locally given data, as describing a differentiable manifold by gluing open sets in $\mathbb{R}^n$. This alternative description of a K-bundle is given by the following proposition.

Proposition 2.2.9 *If $\mathcal{P} = (K, M, P, \pi)$ is a K-bundle and $(U_i)_{i\in I}$ an open cover of M consisting of trivialising neighbourhoods with local trivialisations Θ_i, then for each pair (i, j), such that $U_i \cap U_j \neq \varnothing$*

$$\Theta_i^{-1}(x, k_{ij}(x)) = \Theta_j^{-1}(x, e) \tag{2.1}$$

defines a smooth function $k_{ij} : U_i \cap U_j \to K$, such that

$$k_{ij}(x)k_{jl}(x) = k_{il}(x) \text{ for all } x \in U_i \cap U_j \cap U_l. \tag{2.2}$$

Conversely if M is a manifold with an open cover $(U_i)_{i\in I}$, K is a Lie group and for each pair (i, j), such that $U_i \cap U_j \neq \varnothing$ smooth functions $k_{ij} : U_i \cap U_j \to K$ are given, such that (2.2) is satisfied, then $\mathcal{P} := (K, M, P, \pi)$ is a K-bundle, where

$$P := \bigcup_{i\in I} \{i\} \times U_i \times K/\sim$$

with

$$(i, x, k) \sim (j, y, k') \Leftrightarrow x = y \in U_i \cap U_j \text{ and } k_{ij}(x) = k'k^{-1}.$$

Then the bundle projection is given by $\pi([i, x, k]) = x$ and the action of K on P by $[(i, x, k)] \cdot k' = [(i, x, kk')]$.

Proof: (cf.[KN63] for the non-boundary finite-dimensional case.) Defining the smooth function $k_i : \pi^{-1}(U_i) \to K$, $p \mapsto \mathrm{pr}_2\big(\Theta_i(p)\big)$, we see that

$$k_{ij}(x) = k_i\big(\Theta_j^{-1}(x, e)\big)$$

is also smooth and hence the first assertion holds. To verify the second assertion we introduce a differentiable structure on P by requiring the maps

$$U_i \times K \to P, \ \ (x, k) \mapsto \big([i, x, k]\big)$$

to be diffeomorphism onto their image. This equips P with the structure of a manifold, possibly with boundary. Clearly the K-action on P is smooth and π is smooth and surjective. The local trivialisations are given by the diffeomorphism introducing the differentiable structure on P. They are equivariant by the construction of the K-action on P.

□

Definition. Transition Functions: The functions k_{ij} from the preceding proposition are called the *transition functions* of the K-bundle $\mathcal{P}$. We will refer to a bundle determined by its transition functions by $\mathcal{P} := (K, M, (U_i)_{i\in I}, k_{ij})$.

Example 2.2.10 If $\mathcal{P}$ is the frame bundle of the manifold M, then the values of the transition functions $k_{ij}(x)$ are given by the matrix of the linear map

$$d(\varphi_j^{-1} \circ \varphi_i)(x) : T_xM \to T_xM.$$

Definition. Pull Back: If $\mathcal{P} = (K, M, P, \pi)$ is a principal fibre bundle, N is a manifold without boundary and $f : N \to M$ smooth such that $f(N) \subseteq \mathrm{int}(M)$, then the *pull back* $f^*(\mathcal{P})$ of $\mathcal{P}$ is defined to be the quadruple

$$f^*(\mathcal{P}) := (K, N, Q, \mathrm{pr}_2)$$

where $Q := \{(p, n) \in P \times N : \pi(p) = f(n)\}$.

Lemma 2.2.11 *If $\mathcal{P}$, N, f and $f^*(\mathcal{P})$ are chosen as in the previous definition, then $f^*(\mathcal{P})$ is a principal fibre bundle and the map*

$$f_{\mathcal{P}} : Q \to P, \ \ (p, m) \mapsto p$$

is a homomorphism of K-bundles.

Proof: Since $f(M') \subseteq \mathrm{int}(M)$, we can equivalently consider the restricted bundle $\mathcal{P}_{\mathrm{int}} := (K, \mathrm{int}(M), \mathrm{int}(P), \pi\big|_{\mathrm{int}(P)})$. If $\mathcal{P}$ is finite-dimensional the assertion follows from [KN63, Proposition 5.8]. Its proof also works for the locally convex case.

□

Remark 2.2.12 If $\mathcal{P} = (M, K, (U_i)_{i\in I}, k_{ij})$ is a K-bundle given by local data, then $f^*(\mathcal{P})$ is described by the open cover $f^{-1}(U_i)$ of N and the transition functions

$$k_{ij} : f^{-1}(U_i) \cap f^{-1}(U_j) \to K, \quad x \mapsto k_{ij}(f(x)),$$

i.e. $f^*(\mathcal{P}) = (K, N, f^{-1}(U_i), k_{ij} \circ f\big|_{f^{-1}(U_i \cap U_j)})$.

2.3 Connections on K-bundles

We want to introduce some geometry on K-bundles, which is supposed to extend the notion of connections in Riemannian manifolds to arbitrary structure groups. One ingredient is given canonically by the K-bundle structure.

Definition. Vertical Space: If $\mathcal{P} := (K, M, P, \pi)$ is a K-bundle possibly with boundary, then $V_p := \ker(d\pi_p) \subseteq T_pP$ is called the *vertical space* to the fibre.

If we consider the fibre $\pi^{-1}\big(\pi(p)\big)$ as a sub-manifold of P, then $V_p \cong T_p\pi^{-1}\big(\pi(p)\big)$. Each V_p is canonically isomorphic as a vector space to the Lie algebra $\mathfrak{k}$, the isomorphism given by

$$\tau_p : \mathfrak{k} \to T_pP, \quad \xi \mapsto d\eta_p(e).\xi,$$

where $\eta_p : K \to P$, $k \mapsto p{\cdot}k$ is the *orbit map* at $p \in P$. This map is an isomorphism of topological vector spaces since for a local trivialisation Θ_i around $\pi(p)$ it coincides with the map

$$\xi \mapsto d\Theta_i^{-1}\big(\pi(p), k_i(p)\big).d\lambda_{k_i(p)}\big(\pi(p), e\big).(0, \xi).$$

Hence the bundle structure admits a canonical subspace in each tangent space to P. Now one wants to decompose T_pP into the vertical subspace V_p and a suitable complement, but this complement cannot be chosen canonically any more and we have to introduce connections to determine these vector space complements.

Definition. Differential Form: If M is a smooth manifold, possibly with boundary, and Y a locally convex space, then a smooth Y-valued *k-form* or *differential form* on M is a function ω assigning to each $x \in M$ a k-linear alternating map $\omega(p) : T_x^kM \to Y$, such that in local coordinates the map

$$(p, v_1, \ldots, v_k) \mapsto \omega(p)(v_1, \ldots, v_k)$$

is smooth. We write $\Omega^k(M, Y)$ for the space of smooth k-forms on M with values in Y.

We consider the canonical K-action on K via conjugation $c_k : K \to K$, $k' \mapsto k^{-1}k'k$ and the derived action $\mathrm{Ad}(k) := dc_k(e)$.

Definition. Connection, Bundle with Connection: If $\mathcal{P} := (K, M, P, \pi)$ is a K-bundle, then a *connection* on $\mathcal{P}$ is a $\mathfrak{k}$-valued 1-form $\omega \in \Omega(P, \mathfrak{k})$, such that

$$\omega(p)\big(\tau_p(\xi)\big) = \xi \quad \text{for all} \quad \xi \in \mathfrak{k} \tag{2.3}$$
$$\rho_k^*(\omega) = \operatorname{Ad}(k) \circ \omega \quad \text{for all} \quad k \in K, \tag{2.4}$$

where $\rho_k^*(\omega)$ is the pull back of the 1-form ω by the diffeomorphism $\rho_k : P \to P$, $p \mapsto p \cdot k$. If ω is a connection on $\mathcal{P}$ the pair $(\mathcal{P}, \omega)$ is called a *K-bundle with connection* .

Remark 2.3.1 The *horizontal space* $H_p \subseteq T_pP$ is defined to be the kernel of $\omega(p)$. Since $\tau_p \circ \omega(p)\big|_{V_p} = \mathrm{id}_{V_p}$ and $\mathrm{im}\big(\tau_p \circ \omega(p)\big) = V_p$, each $\tau_p \circ \omega(p)$ defines a projection, and we have

$$T_pP = \ker\big(\tau_p \circ \omega(p)\big) \oplus \mathrm{im}\big(\tau_p \circ \omega(p)\big) = H_p \oplus V_p.$$

Since $T_pP = H_p \oplus V_p \cong T_{\pi(p)}M \oplus \mathfrak{k}$ as topological vector spaces, $V_p \cong \mathfrak{k}$ and all these spaces are closed, we have that H_p is isomorphic to $T_{\pi(p)M}$. In addition, (2.4) implies that $d\rho_k(H_p) = H_{p\cdot k}$ for all $k \in K$.

Proposition 2.3.2 *Each K-bundle $\mathcal{P} = (K, M, P, \pi)$ with smoothly paracompact base space M possesses a connection.*

Proof: We choose an open cover $(U_i)_{i\in I}$ of M consisting of trivialising neighbourhoods. Then on each trivial bundle $P_i := U_i \times K$ define

$$\omega_i(x, k) : T_{(x,k)}P \to T_{(x,k)}P, \quad v \mapsto \mathrm{pr}_{\mathfrak{k}}(v),$$

where $\mathrm{pr}_{\mathfrak{k}} : T_{(x,k)}P \cong T_xU_i \oplus \mathfrak{k} \to \mathfrak{k}$ is the projection to the $\mathfrak{k}$-component. Note, that in the case of a trivial bundle, each tangent space T_pP is canonically isomorphic to the direct sum $T_{\pi(p)}U_i \oplus \mathfrak{k}$.
Clearly ω_i defines a connection on each K-bundle $U_i \times K$. Now define a $\mathfrak{k}$-valued 1-form on $\mathcal{P}$ by setting

$$\omega(p) := \sum f_i(p)\omega_i(p),$$

where $(f_i)_{i\in I}$ is a partition of unity subordinate to $(U_i)_{i\in I}$ and the sum ranges over all $j \in I$, such that $\pi(p) \in U_j$. Since the sum is a convex combination, (2.3) and (2.4) are satisfied, and hence ω is a connection form.

□

There are many different ways in describing a connection. An alternative one is given by a unique lift of vector fields on M to K-invariant horizontal vector fields on P.

Definition. Invariant Vector Field: If $\mathcal{P} = (K, M, P, \pi)$ is a K-bundle we consider TP as a K-module, where K acts via the differentials of the action on P, i.e. $T\rho_k : TP \to TP$. Then a *K-invariant vector field* is defined to be a K-invariant smooth function $X : P \to TP$, such that $X(p) \in T_pP$. The set of K-invariant vector fields are denoted by $\mathcal{V}(P)^K$.

Lemma 2.3.3 *If $(\mathcal{P}, \omega)$ is a K-bundle with connection and connection form ω, then each vector field $X \in \mathcal{V}(M)$ has a unique lift to a horizontal K-invariant vector field $\sigma(X)$ on P, i.e. $d\pi(p).\sigma(X)(p) = X\big(\pi(p)\big)$, $\omega(p)\big(\sigma(X)(p)\big) = 0$, and $\sigma(X) \in \mathcal{V}(P)^K$.*

Proof: The value of $\sigma(X)$ in $p \in P$ is uniquely determined by the requirement $d\pi_p\big(\sigma(X)(p)\big) = X(p)$ and $\omega(p)\big(\sigma(X)(p)\big) = 0$, since $d\pi(p)\big|_{H_p} : H_p \to T_{\pi(p)}M$ and $\omega(p)\big|_{V_p} : V_p \to \mathfrak{k}$ are isomorphisms and $T_pP = H_p \oplus V_p$. Hence it remains to verify the smoothness and K-invariance of $\sigma(X)$. First we define X' to have in $p \in P$ the value

$$X'(p) := d\Theta_i^{-1}\big(\pi(p), k_i(p)\big).\Big(X\big(\pi(p)\big), 0\Big) \in T_pP$$

in T_pP for a local trivialisation Θ_i, such that $\pi(p) \in U_i$, and we observe

$$d\pi(p).X'(p) = \mathrm{pr}_1 \circ d\Theta_i(p).\big(X'(p)\big) = X\big(\pi(p)\big).$$

Then $\sigma(X)(p) := X'(p) - \tau_p\Big(\omega(p)\big(X'(p)\big)\Big)$ satisfies $\omega(p)\big(\sigma(X)(p)\big) = 0$, hence $\sigma(X)(p)$ is independent of the chosen trivialisation and thus defines a smooth vector field. The first summand is invariant due to

$$\begin{aligned} &d\Theta_i^{-1}\big(\pi(p\cdot k), k_i(p\cdot k)\big).\Big(X\big(\pi(p\cdot k), 0\big)\Big) \\ =&d\Big(\Theta_i^{-1}\big(\pi(p), k_i(p)\big) \circ \rho_k\Big).\Big(X\big(\pi(p), 0\big)\Big) \\ =&d\rho_k(p) \circ d\Theta_i^{-1}\big(\pi(p), k_i(p)\big).\Big(X\big(\pi(p), 0\big)\Big). \end{aligned}$$

For the second summand we note that

$$\rho_k^*(\omega) = \mathrm{Ad}(k) \circ \omega \Rightarrow \omega(p).X'(p) = \mathrm{Ad}(k^{-1}).\omega(p\cdot k)\big(d\rho_k.X'(p)\big)$$

Hence we get with $\eta_p \circ \rho_k = \rho_k \circ \eta_p$ and $\eta_p \circ \lambda_k = \eta_{p\cdot k}$

$$\begin{aligned} d\rho_k(p).\tau_p\Big(\omega(p)\big(X'(p)\big)\Big) &= d(\rho_k \circ \eta_p)(e).\Big(\omega(p)\big(X'(p)\big)\Big) \\ &= d\eta_p(k).d\rho_k(e).\mathrm{Ad}(k^{-1}).\Big(\omega(p\cdot k)\big(X'(p\cdot k)\big)\Big) \\ &= d\eta_p(k).d\lambda_k(e).\Big(\omega(p\cdot k)\big(X'(p\cdot k)\big)\Big) \\ &= d\eta_{p\cdot k}.\Big(\omega(p\cdot k)\big(X'(p\cdot k)\big)\Big) = \tau_{p\cdot k}\Big(\omega(p\cdot k)\big(X'(p\cdot k)\big)\Big), \end{aligned}$$

such that the second summand is also invariant.

□

We now turn to the algebraic properties of the spaces $C^\infty(P,\mathfrak{k})^K$, $\mathcal{V}(P)^K$ and $\mathcal{V}(M)$ and their relationship.

Lemma 2.3.4 *The spaces $C^\infty(P,\mathfrak{k})^K$, $\mathcal{V}(P)^K$ and $\mathcal{V}(M)$ are real Lie algebras.*

Proof: Clearly $\mathcal{V}(M)$ is a real Lie algebra. We check that $C^\infty(P,\mathfrak{k})^K$ is a subalgebra of $C^\infty(P,\mathfrak{k})$ and that $\mathcal{V}(P)^K$ is a subalgebra of $\mathcal{V}(P)$.
For $\xi,\eta\in C^\infty(P,\mathfrak{k})^K$, we have that $[\xi,\eta]=\Big(p\mapsto\big[\xi(p),\eta(p)\big]\Big)\in C^\infty(P,\mathfrak{k})^K$, since

$$\begin{aligned}\big(\mathrm{Ad}(k).[\xi,\eta]\big)(p) :&= \mathrm{Ad}(k).\big[\xi(p),\eta(p)\big]=\big[\mathrm{Ad}(k).\xi(p),\mathrm{Ad}(k).\eta(p)\big]\\ &=\big[\xi(p\cdot k),\eta(p\cdot k)\big]=[\xi,\eta](p\cdot k).\end{aligned}$$

For the vector fields $X,Y\in\mathcal{V}(P)^K$, the Lie bracket is defined to be the map

$$p\mapsto[X,Y](p):=d\varphi^{-1}(\tilde{p}).\big(d\tilde{Y}(\tilde{p}).\tilde{X}(\tilde{p})-d\tilde{X}(\tilde{p}).\tilde{Y}(\tilde{p})\big),$$

where $\varphi:U_p\to\varphi(U_p)\subseteq E$ is a chart around p, $\tilde{p}:=\varphi(p)$ and $\tilde{X}(\tilde{p}'):=d\varphi(p').X(p')$ for $\tilde{p}'\in\varphi(U_p)$ and respectively for $\tilde{Y}$ are the coordinate representations of the vector fields X and Y. First we observe that for any vector field Z on P we have

$$d\rho_k(p).Z(p)=Z(p\cdot k)\Leftrightarrow\tilde{Z}(\widetilde{p\cdot k})=d\tilde{\rho_k}(\tilde{p})\tilde{X}(\tilde{p}),$$

such that it suffices to perform a local calculation. Hence we may assume w.l.o.g. that U_p is an open subset of a locally convex space E. Then

$$\begin{aligned}d\rho_k.[X,Y](p)&=d\rho_k(p).\big(dY(p).X(p)-dX(p).Y(p)\big)\\ &=d\rho_k(p).dY(p).X(p)-d\rho_k(p).dX(p).Y(p)\\ &=d\rho_k(p).dY(p).d\rho_{k^{-1}}(p\cdot k).X(p\cdot k)-\quad\dots\\ &=d\rho_k(p).d(Y\circ\rho_{k^{-1}})(p\cdot k).X(p\cdot k)-\quad\dots\\ &\overset{i)}{=}d\rho_k(p).d\rho_{k^{-1}}(p\cdot k).dY(p\cdot k).X(p\cdot k)-\quad\dots\\ &=dY(p\cdot k).X(p\cdot k)-dX(p\cdot k).Y(p\cdot k)=[X,Y](p\cdot k),\end{aligned}$$

where $i)$ holds since the second derivatives $\partial^2\rho_k(p)\big(Y(p),X(p)\big)$ cancel out. Hence the Lie bracket of K-invariant vector fields is again left invariant and thus $\mathcal{V}(P)^K$ is a Lie subalgebra of the Lie algebra $\mathcal{V}(M)$.

□

The vector fields on M have canonically the structure of a $C^\infty(M,\mathbb{R})$-module (where $C^\infty(M,\mathbb{R})$ is considered as commutative algebra), by means of $(f.X)(x) = f(x)X(x)$. This is also valid for $C^\infty(P,\mathfrak{k})^K$ and $\mathcal{V}(P)^K$.

Lemma 2.3.5 *The spaces $C^\infty(P,\mathfrak{k})^K$ and $\mathcal{V}(P)^K$ possess the structure of a $C^\infty(M,\mathbb{R})$-module defined by*

$$(f.\xi)(p) = f\big(\pi(p)\big)\xi(p) \quad \text{for } \xi \in C^\infty(P,\mathfrak{k})^K$$
$$(f.X)(p) = f\big(\pi(p)\big).X(p) \text{ for } X \in \mathcal{V}(P)^K.$$

Proof: Since $\pi(p) = \pi(p\cdot k)$ and $\mathrm{Ad}(k)$ and $d\rho_k(p)$ are linear for all $k \in K$ and $p \in P$ this actually defines a module structure on $C^\infty(P,\mathfrak{k})^K$ and $\mathcal{V}(P)^K$.

□

Lemma 2.3.6 *If $\mathcal{P} := (K,M,P,\pi)$ is a K-bundle, then the sequence of $C^\infty(M,\mathbb{R})$ modules and Lie algebras*

$$C^\infty(P,\mathfrak{k})^K \overset{\tau}{\hookrightarrow} \mathcal{V}(P)^K \overset{d\pi}{\twoheadrightarrow} \mathcal{V}(M),$$

where $\big(d\pi(X)\big)(x) := d\pi(p).X(p)$ for some $p \in \pi^{-1}(\{x\})$, is exact.

Proof: Since $\xi \in C^\infty(P,\mathfrak{k})^K$ is K-invariant, so is

$$\tau_{p\cdot k}\big(\xi(p\cdot k)\big) = d\eta_{p\cdot k}(e).\xi(p\cdot k) = d\eta_p(k).d\lambda_k(e).\mathrm{Ad}(k).\xi(p)$$
$$= d(\eta_p \circ \rho_k)(e).\xi(p) = d\rho_k(p).\eta_p(e).\xi(p) = d\rho_k(p).\tau_p\big(\xi(p)\big).$$

Since $\tau_p(\xi)(p) \in V_p$ we have that $\mathrm{im}(\tau) \subseteq \ker(d\pi)$. If $X \in \ker(d\pi)$, then $X(p) \in V_p \cong \mathfrak{k}$, such that $p \mapsto X(p)$ actually defines a $\mathfrak{k}$-valued K-invariant function on P.
Since $d\pi(p\cdot k) = d\pi(p) \circ T\rho_{k^{-1}}(p\cdot k)$ we see that $d\pi(X)$ is well-defined for K-invariant vector fields on P. Since $d\eta_p(e)$ is linear τ is a module homomorphism. Also by definition of the module structure on $\mathcal{V}(P)^K$ we get that $d\pi$ is a module homomorphism.
What remains to show is that the mappings τ and $d\pi$ are also Lie algebra homomorphisms. For the vector field $\tau(\xi) \in \mathcal{V}(P)^K$ we have the coordinate representation $\widetilde{\tau(X)}\big(\varphi(p)\big) = \xi(p)$ for a suitably chosen chart around p, such that τ is a Lie algebra homomorphism. Since passing to the coordinate representation of a vector field on P commutes with $d\pi$, i.e. $\big(\widetilde{d\pi(X)}\big)(\tilde{p}) = d\tilde{\pi}(\tilde{p})(\tilde{X})\widetilde{\pi(p)}$ it is also immediate that $d\pi$ is a Lie algebra homomorphism.

□

Lemma 2.3.7 *The mapping $\sigma : \mathcal{V}(M) \to \mathcal{V}(P)^K$ from Lemma 2.3.3 is $C^\infty(M,\mathbb{R})$-linear and for $X, Y \in \mathcal{V}(M)$ the horizontal component of $[\sigma(X), \sigma(Y)]$ is $\sigma([X,Y])$, i.e. $\big[\sigma(x), \sigma(Y)\big](p) - \sigma\big([X,Y]\big)(p) \in V_p$ for all $p \in P$.*

Proof: The $C^\infty(M,\mathbb{R})$-linearity follows from the construction of $\sigma(X)$. For $X, Y \in \mathcal{V}(M)$, we compute

$$d\pi(p).\Big(\big[\sigma(X), \sigma(Y)\big](p) - \sigma\big([X,Y]\big)\Big)(p)$$
$$= \Big[d\pi.\sigma(X), d\pi.\sigma(Y)\Big](p) - d\pi(p).\sigma\big([X,Y]\big)(p)$$
$$= [X,Y](p) - [X,Y](p) = 0,$$

since $d\pi$ is a Lie algebra homomorphism. This shows that $\sigma\big([X,Y]\big)$ is the horizontal component of $\big[\sigma(X), \sigma(Y)\big]$.

□

Summarising the preceding lemmas, we get the following proposition.

Proposition 2.3.8 *If $\mathcal{P} = (K, M, P, \pi)$ is a K-bundle with connection given by the horizontal lift of vector fields $\sigma : \mathcal{V}(M) \to \mathcal{V}(P)^K$, then the exact sequence of Lie algebras*

$$C^\infty(P, \mathfrak{k})^K \hookrightarrow \mathcal{V}(P)^K \twoheadrightarrow \mathcal{V}(M)$$

has a $C^\infty(M,\mathbb{R})$-linear splitting given by the connection $\sigma : \mathcal{V}(M) \to \mathcal{V}(P)^K$.

Remark 2.3.9 The splitting can also be described by the connection form ω, considered as a map $\omega : \mathcal{V}(P)^K \to C^\infty(P, \mathfrak{k})^K$. We thus obtain the split exact sequence of $C^\infty(M,\mathbb{R})$-modules

$$C^\infty(P, \mathfrak{k})^K \hookrightarrow \mathcal{V}(P)^K \twoheadrightarrow \mathcal{V}(M).$$

In general σ will not be a Lie algebra homomorphism. The failure of σ being a Lie algebra homomorphism is measured by the curvature of the connection σ.

Definition. Curvature Form: If $(\mathcal{P}, \omega)$ is a K-bundle with connection, then the $\mathfrak{k}$-valued 2-form

$$\Omega : \mathcal{V}(P) \times \mathcal{V}(P) \to C^\infty(P, \mathfrak{k}), \quad (X, Y) \mapsto d\omega(X_H, Y_H),$$

where X_H, Y_H denote the horizontal component of X and Y, is called the *curvature form* of the connection ω.

Lemma 2.3.10 *For a K-bundle with finite-dimensional structure group K, connection form ω and curvature form Ω, the identity*

$$\tau_p\Big(\Omega\big(\sigma(X),\sigma(Y)\big)(p)\Big) = \sigma\big([X,Y]\big)(p) - \big[\sigma(X),\sigma(Y)\big](p)$$

holds for all $p \in P$, where $\sigma : \mathcal{V}(M) \to \mathcal{V}(P)^K$ is the unique lift of vector fields determined by ω.

Proof: Since $\sigma(X)$ and $\sigma(Y)$ are horizontal we have

$$\begin{aligned}\Omega\big(\sigma(X),\sigma(Y)\big)(p) &= (d\omega)\big(\sigma(X),\sigma(Y)\big)(p)\\ &= \underbrace{\sigma(X).\omega\big(\sigma(Y)\big)(p) - \sigma(Y).\omega\big(\sigma(X)\big)(p)}_{=0} - \omega\big([\sigma(X),\sigma(Y)]\big)(p).\end{aligned}$$

With Lemma 2.3.7 this yields the assertion since $\tau_p\Big(\omega\big(X(p)\big)\Big)$ is the vertical component of any vector field $X \in \mathcal{V}(P)$.

□

2.4 The Gauge Group Gau($\mathcal{P}$)

We now turn to the analysis of inner symmetries for a given K-bundle. Inner symmetries are supposed to be transformations of the total space which are compatible with the group operation and which do not induce a transformation on the base space.

Definition. Gauge Transformations, Gauge Group: An automorphism $f : P \to P$ of the K-bundle $\mathcal{P} = (K, M, P, \pi)$ is called *gauge transformation* , if the induced diffeomorphism $f_M : M \to M$, $[p] \mapsto [f(p)]$ is the identity on M and the *group of gauge transformations*

$$\mathrm{Gau}(\mathcal{P}) := \{f \in \mathrm{Aut}(\mathcal{P}) : [p] = [f(p)] \ \text{ for all } \ p \in P\}$$

is called the *gauge group* .

Remark 2.4.1 The gauge group Gau($\mathcal{P}$) is a group with respect to composition, since the inverse automorphism of a gauge transformation also induces the identity on M.

Since a gauge transformation, as a fibre preserving map, maps each fibre in a equivariant fashion into itself, it can be represented by considering for each element in the fibre the group element k such that $f(p) = p \cdot k$.

Lemma 2.4.2 *Each gauge transformation* $f \in \mathrm{Gau}(\mathcal{P})$ *induces a K-invariant smooth map* $\gamma_f \in C^\infty(P,K)^K$ *uniquely determined by requiring* $p \cdot \gamma_f(p) = f(p)$ *for all* $p \in P$*, and the map* $\mathrm{Gau}(\mathcal{P}) \to C^\infty(P,K)^K$*,* $f \mapsto \gamma_f$ *is an isomorphism of groups.*

Proof: For $p \in P$ the value $\gamma_f(p)$ is uniquely determined since the action of K on P is free. Hence we see $k^{-1}\gamma_f(p)k = \gamma_f(p \cdot k)$ since

$$p \cdot k\, \gamma_f(p \cdot k) = f(p \cdot k) \Rightarrow p \cdot \underbrace{k\, \gamma_f(p \cdot k)\, k^{-1}}_{=\gamma_f(p)} = f(p).$$

That γ_f is smooth can be seen in a local trivialisation $\Theta : \pi^{-1}(U) \to U \times K$ for a trivialising neighbourhood U of $\pi(p)$; defining the smooth function $k : \pi^{-1}(U) \to K$, $p \mapsto \mathrm{pr}_2\big(\Theta(p)\big)$ we observe

$$\begin{aligned} p \cdot \gamma_f(p) = f(p) = \Theta^{-1}\big(\pi(p), k(p)\big) \cdot \gamma_f(p) &\Rightarrow \Theta\big(f(p)\big) = \big(\pi(p), k(p)\gamma_f(p)\big) \\ &\Rightarrow k(p)^{-1}k\big(f(p)\big) = \gamma_f(p). \end{aligned}$$

This shows that γ_f is smooth. To see that the map $f \mapsto \gamma_f$ is a homomorphism we observe for $f, g \in \mathrm{Gau}(\mathcal{P})$

$$\begin{aligned} & f(p) = p \cdot \gamma_f(p) \quad \text{for all} \quad p \in P \\ \Rightarrow\ & f\big(g(p)\big) = g(p) \cdot \gamma_f\big(g(p)\big) \quad \text{for all} \quad p \in P \\ \Rightarrow\ & f\big(g(p)\big) = p \cdot \gamma_g(p)\gamma_f\big(p \cdot \gamma_g(p)\big) \overset{i)}{=} p \cdot \gamma_f(p)\gamma_g(p) \quad \text{for all} \quad p \in P \\ \Rightarrow\ & \gamma_{(f \circ g)}(p) = \gamma_f(p)\gamma_g(p) \quad \text{for all} \quad p \in P, \end{aligned}$$

where $i)$ holds due to the K-invariance of γ_f. Clearly the map $f \mapsto \gamma_f$ is injective and since each map $\gamma \in C^\infty(P,K)^K$ induces a gauge transformation via $p \mapsto p \cdot \gamma(p)$ it is also surjective.

□

Remark 2.4.3 Note that in the preceding lemma we passed from a subgroup of the diffeomorphism group $\mathrm{Diff}(P)$ to a mapping group. This will enable us later on to use existing results for mapping groups, respectively to modify them slightly to the more general situation of gauge groups. In general the mappings in $C^\infty(P,K)^K$ do *not* factorise to mappings in $C^\infty(M,K)$. This is the reason why we cannot topologise $\mathrm{Gau}(\mathcal{P})$ directly, not even for compact base space M. However, under certain restrictions we can identify $\mathrm{Gau}(\mathcal{P})$ with the well-known mapping group $C^\infty(M,K)$.

Lemma 2.4.4 *If the K-bundle* $\mathcal{P} = (K, M, P, \pi)$ *is trivial or the structure group K is abelian, then* $\mathrm{Gau}(\mathcal{P})$ *is isomorphic to the mapping group* $C^\infty(M,K)$*.*

Proof: a) If $\mathcal{P}$ is trivial, i.e. $\Theta : P \to M \times K$ is an equivariant diffeomorphism, and $f \in \mathrm{Gau}(\mathcal{P})$, then $(\gamma_f)_M : M \to K$, $x \mapsto \gamma_f\big(\Theta^{-1}(x,e)\big)$ defines a smooth K-valued function on M.
b) If K is abelian the K-invariance of γ_f implies that γ_f is constant on each fibre and hence factors through a smooth map $(\gamma_f)_M : M \to K$.
It is easily checked that the map $\mathrm{Gau}(\mathcal{P}) \to C^\infty(M,K)$, $f \mapsto (\gamma_f)_M$ actually defines an isomorphism of groups.

□

This result is the first hint how to topologise Gau($\mathcal{P}$), since locally restricted to a trivial bundle the gauge group coincides with a mapping group. Since the sole obstruction for Gau($\mathcal{P}$) to be a mapping group is the absence of a global trivialisation, we expect Gau($\mathcal{P}$) to look like a product of mapping groups, twisted somehow. We will turn to this subject in Section 3.3.
Having clarified the mathematical framework, we now turn again to the physical interpretation of gauge theories for finite-dimensional K-bundles. A connection form ω is supposed to represent the potential of a force. This connection form can in local coordinates, i.e. in a trivialising open subset $U \subseteq M$, be expressed as a tuple of functions $A_\mu : U \to \mathfrak{k}$,

$$x \mapsto \omega(p)\big(\frac{\partial}{\partial x^\mu}(p)\big)$$

where $p \in \pi^{-1}(x)$, $1 \leq \mu \leq n$ and $n = \dim(M)$. These functions are the same functions as in Section 1.1 and they are called in the physical literature *local gauge potentials*. The exterior derivative of this 1-form is the curvature 2-form, having in local coordinates the coefficient functions

$$F_{\mu\nu}(x) = \frac{\partial}{\partial x^\mu}A_\nu(x) - \frac{\partial}{\partial x^\nu}A_\mu(x)$$

(cf. Lemma 2.3.10). This tensor is called in the physical literature the *energy-momentum tensor*. If this tensor vanishes, i.e. if the connection ω is flat, the force vanishes. Hence the force, described by a connection in a K-bundle, has the interpretation of the curvature of this bundle. Since gauge transformations are supposed to map connections describing the same force into another, one is interested in the space of connections modulo gauge transformations. To understand this statement we have to clarify which structure the space of all connections for a given K-bundle has and how the gauge transformations act on this space.

Lemma 2.4.5 *The space* conn($\mathcal{P}$) *of connections for a fixed K-bundle $\mathcal{P}$ is an affine space with translational vector space*

$$\Omega^1_0(P,\mathfrak{k})^K := \{\omega \in \Omega^1(P,\mathfrak{k})^K : \omega(p)\big|_{V_p} = 0\}$$

of $\mathfrak{k}$-valued 1-forms on P, vanishing on the vertical subspaces, where the action of K on $\Omega^1(P,\mathfrak{k})$ is given by the pull back $(k,\omega) \mapsto \rho_k^*(\omega)$. Hence

$$\mathrm{conn}(\mathcal{P}) = \omega_0 + \Omega_0^1(P,\mathfrak{k})^K$$

for an arbitrary connection form ω_0, such that $\mathrm{conn}(\mathcal{P})$ is an affine subspace of the vector space $\Omega^1(P,\mathfrak{k})^K$.

Proof: This is an easy calculation.

□

Gauge transformations act on connections via the pull backs of the diffeomorphism they describe, i.e.

$$\mathrm{Gau}(\mathcal{P}) \times \mathrm{conn}(\mathcal{P}) \to \mathrm{conn}(\mathcal{P}),\ (f,\omega) \mapsto f^*(\omega).$$

Hence the configuration space of a gauge theory described by a K-bundle $\mathcal{P}$ is the space $\mathrm{conn}(\mathcal{P})/\mathrm{Gau}(\mathcal{P})$.

2.5 K-bundles over $\mathbb{S}^1$

Since the bundles over $\mathbb{S}^1$ will play a crucial rule in the following text we derive here a characterisation result for them.

Proposition 2.5.1 *Let $\mathcal{P} = (K, M, P, \pi)$ be a finite-dimensional K-bundle without boundary and ω be a connection with vanishing curvature. If M is simply connected, then $\mathcal{P}$ is isomorphic to the trivial bundle $(K, M, M \times K, \mathrm{pr}_1)$.*

Proof: [KN63, Corollary 9.2]

□

Proposition 2.5.2 *Each K-bundle $\mathcal{P} = (K,\mathbb{S}^1,P,\pi)$ over $\mathbb{S}^1$ with finite-dimensional structure group K is isomorphic to a bundle of the form*

$$\mathcal{P}_k := (K, \mathbb{S}^1, P_k, \pi)$$

with $k \in K_0$, where

$$P_k := \mathbb{R} \times_k K := \mathbb{R} \times K/\sim \tag{2.5}$$

with $(x,k') \sim (x+n, k^{-n}k')$.

Proof: Consider the open cover (U_1, U_2) of $\mathbb{S}^1$ consisting of $U_1 := \exp\big(i(-\varepsilon, 1+\varepsilon)\big)$ and $U_1 := \exp\big(i(1-\varepsilon, 2+\varepsilon)\big)$ for $\varepsilon < \frac{1}{4}$. Then each restriction of $\mathcal{P}$ to the K-bundle over U_1, U_2 is flat since every 2-form on the circle vanishes. Hence the restrictions to the bundles over $U_{1,2}$ are isomorphic to $U_i \times K$, i.e. $\Theta_i : P_i := \pi^{-1}(U_i) \to U_i \times K$ are equivariant diffeomorphisms, due to the preceding proposition. We can also assume the pull back $d(\Theta_i^{-1})^*(\omega)$ of the connection on $\mathcal{P}$ to connections on $U_i \times K$ to be the canonical connection [KN63, Theorem 9.1], and hence the description of the connection via horizontal lifts is

$$\sigma : \mathcal{V}(\mathbb{S}^1) \to \mathcal{V}(U_i \times K)^K, \;\; X \mapsto \Big(p \mapsto \big(X(p), 0\big)\Big).$$

Now consider the two curves $\gamma_1 : I_1 := (-\varepsilon, 1+\varepsilon) \to \mathbb{S}^1$ and $\gamma_2 : I_2 := (1-\varepsilon, 2+\varepsilon) \to \mathbb{S}^1$, $t \mapsto e^{2\pi i t}$. For each $p_0 \in \pi^{-1}(t_0)$ these curves can be lifted to unique horizontal curves $\gamma_{\mathrm{hor}\, i} : I_i \to P$, i.e. $\gamma_{\mathrm{hor}\, i}(t_0) = p_0$, $\dot{\gamma}_{\mathrm{hor}\, i}(t) \in H_{\gamma_{\mathrm{hor}\, i}(t)}$ and $d\pi_{\gamma_{\mathrm{hor}\, i}(t)} \cdot \dot{\gamma}_{\mathrm{hot}\, i}(t) = \dot{\gamma}(t)$, for $i = 1, 2$.
Since the local trivialisation were assumed to preserve the connection we have the description of the horizontal lifts $\gamma_{\mathrm{hor}\, i}(t) = \Theta_i^{-1}\Big(\big(\gamma_i(t), k_i(p_0)\big)\Big)$ since the connection on $U_i \times K$ is the canonical one. Hence the uniqueness of the horizontal lifts and the definition of the transition functions $k_{12} : U_1 \cap U_2 \to K$ implies that

$$\gamma_{\mathrm{hor}1}(t) \cdot k_{12}\big(\gamma_1(t)\big) = \gamma_{\mathrm{hor}2}(t) \Rightarrow \Big(k_{12}\big(\gamma_1(t)\big), \gamma_1(t)\Big) = \big(e, \gamma_2(t)\big)$$

for all $t \in U_1 \cap U_2$. Hence $\frac{d}{dt}\Big(k_{12}\big(\gamma_1(t)\big)\Big) \equiv 0$ and thus k_{12} is constant on every connected component of $U_1 \cap U_2$.
Denoting by k_1 and k_2 the values of k_{12} on the two connected components of $U_1 \cap U_2$ we consider the isomorphism $P_2 \to P_2$, $(x, k') \mapsto (x, k') \cdot k_2^{-1}$. This isomorphism changes the value of the transition functions to $\widetilde{k}_{12}(x) = k_{12}(x) k_2^{-1}$ and we see that we can assume k_2 to be e, whereby k_1 changes its value to $k = k_1 k_2^{-1}$. Hence

$$t \mapsto k_i\big(\gamma_{\mathrm{hor}i}(t)\big) \text{ if } t \in U_i$$

is a well-defined continuous curve from e to k. Finally the construction procedure of Proposition 2.2.9 yields that the bundle determined by k can be described by 2.5.

□

Corollary 2.5.3 *If $\mathcal{P} = (K, M, P, \pi)$ is a finite-dimensional K-bundle with connection ω and $\alpha : \mathbb{S}^1 \to M$ smooth then the pull back $\alpha^*(\mathcal{P})$ is described by $k \in K$ determined by $\alpha_{\mathrm{hor}}(0) \cdot k = \alpha_{\mathrm{hor}}(1)$. In particular for a given bundle $\mathcal{P} = (K, \mathbb{S}^1, P, \pi)$ we obtain the describing element $k \in K$ by lifting the identity on $\mathbb{S}^1$ and considering the difference of start- and ending point.*

Remark 2.5.4 The preceding Proposition also follows from Proposition 2.5.1 when one considers the universal covering map $f : \mathbb{R} \to \mathbb{S}^1$ and the pull back $f^*(\mathcal{P})$ to a bundle over $\mathbb{R}$ which is flat and hence trivial. In general the element $k \in K$ obtained in the preceding corollary depends on the choice of a connection on $\mathcal{P}$, but different connections lead to isomorphic bundles.

Chapter 3

Differential Calculus in Locally Convex Spaces

3.1 The Gâteaux Derivative

In this text we follow the notion of differentiability on locally convex topological vector spaces dating from Hamilton and first applied to Lie groups by Milnor in [Mil83].

Definition. Gâteaux Derivative, Differentiable Map: If E, F are topological vector spaces, F is locally convex and $U \subseteq E$ is open, then for a continuous map $f : U \to E$, $x \in U$ and $v \in E$ we consider the *Gâteaux derivative*

$$df(x).v := \lim_{h \to 0} \frac{1}{h}\big(f(x + hv) - f(x)\big)$$

whenever the limit exists. We say, that $f : U \to F$ is of *class* C^1 , if $df(x).v$ exists for all $x \in U$, $v \in E$ and the map

$$df : U \times E \to F, \quad (x, v) \mapsto df(x, v)$$

is continuous. Inductively f is defined to be of *class* C^k , if $df : U \times E \to F$ is of class C^{k-1} and $d^k f := d^{k-1}\big(df\big)$. Furthermore f is said to be of class C^∞ or *smooth* , if it is of class C^k for all $k \in \mathbb{N}$.

Remark 3.1.1 In the following we will sometimes write $df(x, v)$ instead of $df(x).v$ when pointing out that we consider df as a map of two variables and not as linear maps dependent on a variable x.

The link to the directional derivatives and their continuity is given by the following lemma.

Lemma 3.1.2 (Directional Derivatives) *If E, F are topological vector spaces, F is locally convex and $U \subseteq E$ is an open subset and $f : U \to E$ is of class C^k, then the directional derivative , inductively defined by*

$$\partial^k f(x).(v_1, \dots, v_k) \\ = \lim_{h \to 0} \frac{1}{h}\big(\partial^{k-1} f(x + hv_k).(v_1, \dots, v_{k-1}) - \partial^{k-1} f(x).(v_1, \dots, v_{k-1})\big),$$

with $\partial^0 f := f$ exists for all $x \in U$, $(v_1, \dots, v_k) \in E^k$. The map $\partial^k f : U \times E^k \to F$ is continuous and satisfies

$$d^{k-1}\big(df\big)(x, v_1).\big((v_2, 0) \dots, (v_k, 0)\big) = \partial^k f(x).(v_1, \dots, v_k) \tag{3.1}$$

for all $k \geq 2$.

Proof: Since the fact that f is of class C^k implies that $d^{k-1}\big(df\big)$ is continuous, we only have to verify (3.1) by induction on k. The case $k = 2$ reads

$$\begin{aligned} d\big(df\big)(x, v_1)(v_2, 0) &= \lim_{h \to 0} \frac{1}{h}\Big(df\big((x, v_1) + h(v_2, 0)\big) - df(x, v_1)\Big) \\ &= \lim_{h \to 0} \frac{1}{h}\Big(df\big((x + hv_2, v_1)\big) - df(x, v_1)\Big) = \partial^2 f(x).(v_1, v_2) \end{aligned}$$

because $\partial^1 f(x, v) = df(x.v)$. Now assume, that (3.1) holds for $k \in \mathbb{N}$ and calculate

$$\begin{aligned} & d^k\big(df\big)\big((x, v_1), (v_2, 0), \dots, (v_{k+1}, 0)\big) \\ = & \lim_{h \to 0} \frac{1}{h}\Big(d^{k-1}\big(df\big)\big((x, v_1) + h(v_{k+1}, 0), (v_2, 0), \dots, (v_k, 0)\big) \\ & \qquad - d^{k-1}\big(df\big)\big((x, v_1), (v_2, 0), \dots, (v_k, 0)\big)\Big) \\ = & \lim_{h \to 0} \frac{1}{h}\Big(\partial^k f\big((x + hv_{k+1}), v_1, \dots, v_k\big) - \partial^k f(x, v_1, \dots, v_k)\Big) \\ = & \, \partial^{k+1} f(x, v_1, \dots, v_{k+1}) \end{aligned}$$

Thus (3.1) also holds for $k + 1$.

□

The reason why we assume Y to be locally convex is that for locally convex spaces one has the notion of a weak integral. This provides a powerful tool in many calculations, namely the generalisation of the *Fundamental Theorem of Calculus*

$$\gamma(b) - \gamma(a) = \int_a^b \gamma'(t) dt.$$

Definition. Weak Integral: Assume that F is a locally convex vector space, $I \subseteq \mathbb{R}$ is an open nonempty interval, $a, b \in I$ and $\gamma : I \to F$ is a continuous curve. If there exists a $z \in F$ such that

$$\lambda(z) = \int_a^b \lambda(\gamma(t))dt \tag{3.2}$$

holds for all $\lambda \in F'$, where F' denotes the space of continuous linear functionals on F, then $z =: \int_a^b \gamma(t)dt$ is called the *weak integral* of γ from a to b. The element z is uniquely determined by (3.2) since F is assumed to be locally convex and hence the elements of F' separate the points of F by the Theorem of Hahn-Banach.

Proposition 3.1.3 *If F is a locally convex vector space, $I \subseteq \mathbb{R}$ is an open non empty interval, $a, b \in I$ and $\gamma : I \to F$ is a curve of class C^1, then*

$$\gamma(b) - \gamma(a) = \int_a^b \gamma'(t)dt$$

where $\gamma'(t) := d\gamma(t).1$.

Proof: For $\lambda \in F'$ we have

$$\begin{aligned}(\lambda \circ \gamma)'(t) &= \lim_{h \to 0} \frac{1}{h}\Big(\lambda\big(\gamma(t+h)\big) - \lambda\big(\gamma(t)\big)\Big) \\ &= \lim_{h \to 0} \lambda\Big(\frac{1}{h}\big(\gamma(t+h) - \gamma(t)\big)\Big) = \lambda\Big(\lim_{h \to 0} \frac{1}{h}\big(\gamma(t+h) - \gamma(t)\big)\Big) = \lambda\big(\gamma'(t)\big)\end{aligned}$$

due to the linearity and the continuity of λ. Thus $\lambda \circ \gamma : I \to \mathbb{R}$ is a continuously differentiable curve with $(\lambda \circ \gamma)'(t) = \lambda\big(\gamma(t)'\big)$. Hence the (standard) Fundamental Theorem of Calculus yields

$$\lambda\big(\gamma(b) - \gamma(a)\big) = (\lambda \circ \gamma)(b) - (\lambda \circ \gamma)(a) = \int_a^b (\lambda \circ \gamma)'(t)dt = \int_a^b \lambda\big(\gamma'(t)\big)dt$$

Since $\lambda \in F'$ was arbitrary, $\gamma(b) - \gamma(a)$ satisfies the defining properties of the weak integral $\int_a^b \gamma'(t)dt$.

□

3.2 Groups of Mappings

Since we will mostly be interested in mapping groups, such as $C^\infty(M, K)$ for a compact manifold M, possibly with boundary and K a Lie group, we are now aiming at giving these groups a Lie group structure. First we recall a few results from

[Glö03]. If X is a topological space and F a topological vector space, then $C(X,F)$ endowed with pointwise operations and the topology of uniform convergence on compact subsets is a topological vector space. The topology on $C(X,G)$ coincides with the compact open topology and hence the evaluation maps

$$\mathrm{ev}_x : C(X,F) \to F, \quad f \mapsto f(x)$$

are continuous for all $x \in F$. If, in addition, X is Hausdorff and F is locally convex the same holds for $C(X,F)$.

Definition. Topology on $C^\infty(M,F)$: Assume that M is a finite-dimensional manifold with boundary, modelled on the finite-dimensional vector space E and F is locally convex space. There exists a natural embedding

$$C^\infty(M,F) \hookrightarrow \prod_{k=0}^{\infty} C(T^k M, F), \quad f \mapsto (d^k f)_{k\in\mathbb{N}},$$

where $d^k f := \mathrm{pr}_{2^k} \circ T^k f$. We endow $C^\infty(M,F)$ with the topology making this map a topological embedding, i.e. with the coarsest topology making all the maps $C^\infty(M,F) \to C(TM,F)$, $f \mapsto d^k f$ continuous.

Remark 3.2.1 Since the product topology is the final topology w.r.t projections a map $f : X \to C^\infty(M,F)$ for a topological space X is continuous if and only if the maps $X \to C(T^k M, F)$, $x \mapsto d^k\big(f(x)\big)$ are continuous for all $k \in \mathbb{N}$.

Lemma 3.2.2 *If M is a finite-dimensional manifold with boundary possessing a countable differentiable structure $(\varphi_i, U_i)_{i\in\mathbb{N}}$ and F a Frécht space, then $C^\infty(M,F)$ is a Fréchet space too.*

Proof: If $U \subseteq \mathbb{R}^n$ is a manifold with boundary and F is a Fréchet space then the topology on $C(U,F)$ can be described by the seminorms

$$\sup_{x\in V_i}\big(p_j(\cdot)\big) : C(U,F) \to \mathbb{R}$$

where p_i are a countable family of seminorms describing the topology on F and $V_j \subseteq U$ are a countable family of compact subsets covering U. Hence $C(U,F)$ is a Fréchet space. Since the topology on $C^\infty(M,F)$ coincides with the topology induced by the maps

$$C^\infty(M,F) \mapsto C\Big(T^k\big(\varphi_i(U_i)\big), F\Big), \quad f \mapsto d^k(f \circ \varphi_i^{-1}),$$

which are countably many, it can be described by countably many seminorms and hence is a Fréchet space.

□

The aim of this section is to show, that for a smooth map $f : M \times U \to F$ the map $f_\sharp : C^\infty(M, U) \to C^\infty(M, F)$, $\gamma \mapsto f \circ (\mathrm{id}_M, \gamma)$ is smooth. We first observe

Lemma 3.2.3 *If E is a locally convex spaces, $U \subseteq E$ is open and M is a compact manifold with boundary, then the set*

$$C^\infty(M, U) := C^\infty(M, E) \cap U^M = \{f \in C^\infty(M, E) : f(M) \subseteq U\}$$

is open in $C^\infty(M, E)$.

Proof: Since $C^\infty(M, U) = C(M, U) \cap C^\infty(M, E)$ and

$$C(M, U) = \{f \in C(M, E) : f(M) \subseteq U\}$$

is open in $C(M, E)$, $C^\infty(M, U)$ is open as well.

□

For the rest of this section we will follow closely the way described in [Glö02, p.366-375] and [Nee01], where the results are proved for manifolds without boundary. The results carry over in exactly the same way for the case of manifolds with boundary and the main point are the correct definitions and observations made in Section 2.1.

Lemma 3.2.4 *If M and N are finite-dimensional manifolds with boundary, F is locally convex and $f : N \to M$ is a smooth map having the correct mapping property w.r.t boundaries, then the pull back*

$$f^* : C^\infty(M, F) \to C^\infty(N, F), \quad \gamma \mapsto \gamma \circ f$$

is continuous.

Proof: Since f is smooth we know that $T^k f$ is continuous for all $k \in \mathbb{N}$. Hence

$$d^k \circ f^* : C^\infty(M, F) \to C(T^k N, F), \quad \gamma \mapsto d^k(\gamma \circ f) = d^k\gamma \circ T^k f$$

is continuous.

□

Lemma 3.2.5 *If M is a finite-dimensional manifold with boundary and E a locally convex space, then the map*

$$C^\infty(M, E) \to C^\infty(TM, TE), \quad \gamma \mapsto T\gamma,$$

where $T\gamma([x, i, v]) = (\gamma(x), d(\gamma \circ \varphi_i^{-1})(\varphi_i(x), v))$ is continuous. In addition, the maps inductively defined by

$$C^\infty(M, E) \to C^\infty(T^k M, E^{2k}), \quad \gamma \mapsto T^k\gamma = T(T^{k-1}\gamma)$$

are continuous.

Proof: The map $C^\infty(M,E) \to C^\infty(TM,E)$, $\gamma \mapsto d\gamma = \mathrm{pr}_2 \circ T\gamma$ is obviously continuous. In addition, the map $C^\infty(M,E) \to C^\infty(TM,E)$, $\gamma \mapsto \gamma \circ \pi$ is continuous by Lemma 3.2.4, where $\pi : TM \to M$ is the bundle projection. Identifying $C^\infty(TM,TE)$ canonically with $C^\infty(TM,E) \times C^\infty(TM,E)$, we thus observe that the map $\gamma \mapsto T\gamma$ is continuous.
The continuity of the maps $\gamma \mapsto T^k\gamma$ follows directly from the definition as a composition of k continuous maps.

□

Lemma 3.2.6 *If M is a topological space, E and F are topological vector spaces, $U \subseteq E$ is open and $f : M \times U :\to F$ is continuous, then the mapping*

$$f_\sharp : C(M,U) \to C(M,F), \;\; \gamma \mapsto f \circ (\mathrm{id}_M, \gamma)$$

is continuous.

Proof: [Nee01, Lemma III.6]

□

Lemma 3.2.7 *If M is a finite-dimensional manifold with boundary, E and F are locally convex spaces, $U \subseteq E$ is open and $f : M \times U \to F$ is smooth, then the mapping*

$$f_\sharp : C^\infty(M,U) \to C^\infty(M,F), \;\; \gamma \mapsto f \circ (\mathrm{id}_M, \gamma)$$

is continuous.

Proof: For $\gamma \in C^\infty(M,U)$ we have

$$T(f_\sharp\gamma) = T(f \circ (\mathrm{id}_M, \gamma)) = Tf \circ T(\mathrm{id}_M, \gamma) = Tf \circ (\mathrm{id}_{TM}, T\gamma) = (Tf)_\sharp(T\gamma)$$

and thus inductively

$$\begin{aligned} T^n(f_\sharp\gamma) &= T\big(T^{n-1}(f_\sharp\gamma)\big) = T\big((T^{n-1}f)_\sharp T^{n-1}\gamma\big) \\ &= T\big(T^{n-1}f \circ (\mathrm{id}_{T^{n-1}M}, T^{n-1}\gamma)\big) = T^n f \circ (\mathrm{id}_{T^nM}, T^n\gamma) = \big(T^n f\big)_\sharp T^n\gamma. \end{aligned}$$

We now can write the mapping $\gamma \mapsto T^n(f_\sharp\gamma)$ as the composition of the two maps $\gamma \mapsto (\mathrm{id}_{T^nM}, T^n\gamma)$ and $(\mathrm{id}_{T^nM}, T^n\gamma) \mapsto (T^nf)_\sharp T^n\gamma$ which are continuous by the two preceeding lemmas. Hence the map

$$f_\sharp : C^\infty(M,U) \to C^\infty(M,F), \;\; \gamma \mapsto f \circ (\mathrm{id}_M, \gamma)$$

is continuous because a map from any topological space to $C^\infty(M,F)$ is continuous if all composition with $d^n = \mathrm{pr}_{2^n} \circ T^n$ are continuous (cf. Remark 3.2.1).

□

Proposition 3.2.8 *If M is a finite-dimensional compact manifold with boundary, E and F are locally convex spaces, $U \subseteq E$ is open and $f : M \times U \to F$ is smooth, then the mapping*

$$f_\sharp : C^\infty(M,U) \to C^\infty(M,F), \;\; \gamma \mapsto f \circ (\mathrm{id}_M, \gamma)$$

is smooth.

Proof: (cf. [Nee01, Proposition III.7]) We claim that

$$d^k(f_\sharp) = (d_2^k f)_\sharp \tag{3.3}$$

holds for all $k \in \mathbb{N}$, where $d_2^k f(x,y).v := d^k f(x,y).(0,v)$. This claim immediately proves the assertion due to Lemma 3.2.7.
To verify (3.3) we perform an induction on k. The case $k = 0$ is trivial, hence assume that (3.3) holds for $k \in \mathbb{N}_0$,

$$\gamma \in C^\infty(M, U \times E^{2k-1}) \cong C^\infty(M,U) \times C^\infty(M,E)^{2k-1}$$

and

$$\eta \in C^\infty(M, E^{2k}) \cong C^\infty(M,E)^{2k}.$$

Then $\mathrm{im}(\gamma) \subseteq U \times E^{2k-1}$ and $\mathrm{im}(\eta) \subseteq E^{2k}$ are compact and there exists an $\varepsilon > 0$ such that

$$\mathrm{im}(\gamma) + (-\varepsilon,\varepsilon)\mathrm{im}(\eta) \subseteq U \times E^{2k-1}.$$

Hence $\gamma + h\eta \in C^\infty(M, U \times E^{2k-1})$ for all $h \in (-\varepsilon,\varepsilon)$ and we calculate

$$\begin{aligned}
\big(d(d^k f_\sharp)(\gamma,\eta)\big)(x) &= \lim_{h\to 0} \frac{1}{h}\Big(\big(d^k f_\sharp(\gamma + h\eta) - d^k f_\sharp(\gamma)\big)(x)\Big) \\
&\overset{i)}{=} \lim_{h\to 0} \frac{1}{h}\Big(d_2^k f\big(x, \gamma(x) + h\eta(x)\big) - d_2^k f\big(x,\gamma(x)\big)\Big) \\
&\overset{ii)}{=} \lim_{h\to 0} \int_0^1 d_2\bigg(\Big(d_2^k f\big(x, \gamma(x) + th\,\eta(x)\big)\Big), \eta(x)\bigg) dt \\
&\overset{iii)}{=} \int_0^1 \lim_{h\to 0} d_2\bigg(\Big(d_2^k f\big(x, \gamma(x) + th\,\eta(x)\big)\Big), \eta(x)\bigg) dt \\
&= d_2^{k+1} f\big(x, \gamma(x), \eta(x)\big) = \big(d_2^{k+1} f\big)_\sharp(\gamma,\eta)(x),
\end{aligned}$$

where $i)$ holds by the induction hypothesis, $ii)$ holds by the Fundamental Theorem of Calculus (Proposition 3.1.3) and is the clue in this proof. Finally $iii)$ holds because the integrand is uniformly bounded as a continuous function on the compact set $[0,1] \times [-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}]$. This establishes our claim and hence completes the proof.

□

Corollary 3.2.9 *If M is a manifold with boundary, E and F are locally convex spaces, $U \subseteq E$ are open and $f : U \to F$ smooth, then the push forward $f_* : C^\infty(M,U) \to C^\infty(M,F)$, $\gamma \mapsto f \circ \gamma$ is a smooth map.*

Proof: Define $\tilde{f} : M \times U \to F$, $(x,v) \mapsto f(x)$ and apply Proposition 3.2.8. Then f_* is given by $\tilde{f}_\sharp$ and hence smooth.

□

3.3 Gau($\mathcal{P}$) as Lie group

First we will construct a smooth Lie group structure on the space $C^\infty(M,K)$ for an arbitrary Lie Group K by applying the following proposition.

Proposition 3.3.1 *Let G be a group with a smooth manifold structure on $U \subseteq G$ modelled on the locally convex space E. Furthermore assume that there exists $V \subseteq U$ open such that $e \in V$, $VV \subseteq U$, $V = V^{-1}$ and*

i) $V \times V \to U$, $(g,h) \mapsto gh$ is smooth,

ii) $V \to V$, $g \mapsto g^{-1}$ is smooth,

iii) for all $g \in G$ there exists an open unit neighbourhood $W \subseteq U$ such that $g^{-1}Wg \subseteq U$ and the map $W \to U$, $h \mapsto g^{-1}hg$ is smooth.

Then there exists an unique manifold structure on G, such that V is an open sub-manifold of G which turns G into a Lie-group.

Proof: [Glö03, Chapter 15]

□

Now let M be a compact manifold with boundary and assume that K is an arbitrary Lie group modelled on the locally convex space E and that $\varphi : U \to \tilde{U} := \varphi(U)$ is a chart around e. We consider the push forward

$$\varphi_* : C^\infty(M,U) \to C^\infty(M,\tilde{U}), \quad \gamma \mapsto \varphi \circ \gamma$$

which is bijective and equip $C^\infty(M,U)$ with the manifold structure making φ_* a diffeomorphism. Note that $C^\infty(M,\tilde{U})$ is an open subset of the locally convex space $C^\infty(M,E)$ due to Lemma 3.2.3.
Since K is a topological group there exists an open unit neighbourhood $V \subseteq U$ such that $VV \subseteq U$ and $V^{-1} = V$. Then $C^\infty(M,V)$ is an open sub-manifold of $C^\infty(M,U)$ since $C^\infty(M,V) = \varphi_*\big(C^\infty(M,V)\big)$ is open. Furthermore the maps

- $\tilde{m} : \tilde{V} \times \tilde{V} \to \tilde{U}$, $(x, y) \mapsto \varphi\big(\varphi^{-1}(x)\varphi^{-1}(y)\big)$
- $\tilde{i} : \tilde{V} \to \tilde{V}$, $x \mapsto \varphi\big(\varphi(x)^{-1}\big)$

are smooth and so are their push forwards

- $\tilde{m}_* : C^\infty(M, \tilde{V} \times \tilde{V}) \to C^\infty(M, \tilde{U})$, $(\gamma, \eta) \mapsto \tilde{m} \circ (\gamma, \eta)$
- $\tilde{i}_* : C^\infty(M, \tilde{V}) \to C^\infty(M, \tilde{V})$, $\gamma \mapsto \tilde{i} \circ \gamma$

due to Corollary 3.2.9. We thus have made the first step towards the following proposition. First we need a technical lemma.

Lemma 3.3.2 *If X is a compact topological space, K is a topological group, $U \subseteq K$ is an open unit neighbourhood and $f : X \to K$ is continuous, then there exists an open unit neighbourhood $W \subseteq K$ and an open neighbourhood P of $f(X)$ such that $pWp^{-1} \subseteq U$ for all $p \in P$.*

Proof: For all $x \in X$ there exists an open unit neighbourhood $\tilde{W}_x$ such that $f(x)\tilde{W}_x f(x)^{-1} \subseteq U$. Then choose an open unit neighbourhood $W_x \subseteq \tilde{W}_x$ such that $W_x^{-1} = W_x$ and $W_x W_x W_x \subseteq \tilde{W}_x$. Since $f(X)$ is compact it is covered by finitely many sets $f(x_1)W_{x_i}, \ldots, f(x_n)W_{x_n}$. Set $W = \cap_{i=1}^n W_{x_i}$ and $P = f(X)W$. If $p \in P$ then we have $p \in f(x_i)W_{x_i}$ for at least one $1 \leq i \leq n$. Hence

$$pwp^{-1} \in f(x_i)U_{x_i}U_{x_i}U_{x_i}^{-1}f(x_i)^{-1} \subseteq f(x_i)\tilde{U}_{x_i}f(x_i)^{-1} \subseteq U$$

holds for all $p \in P$ and $w \in W$.

□

Proposition 3.3.3 *If M is a compact manifold with boundary and K is a Lie-group modelled on the locally convex space F, then $C^\infty(M, K)$ is a Lie group w.r.t. pointwise group operation and the topology induced from the push forwards*

$$\varphi_* : C^\infty(M, U) \to C^\infty(M, \tilde{U}), \;\; \gamma \mapsto \varphi \circ \gamma$$

where $\varphi : U \to \tilde{U}$ is a chart around e.

Proof: We continue with the notation introduced above. What remains to check is *iii)* from Proposition 3.3.1, so assume $\gamma \in C^\infty(M, K)$. The preceding lemma yields an open neighbourhood P of $\gamma(M)$ such that $p^{-1}Wp \subseteq V$ for all $p \in P$. Set $\tilde{W} := \varphi(W)$. Since conjugation $c_k : K \to K$ is smooth, so are the maps

$$\tilde{f} : P \times \tilde{W} \to \tilde{V}, \;\; (p, y) \mapsto \varphi\Big(c_p\big(\varphi^{-1}(y)\big)\Big),$$
$$f : M \times \tilde{W} \to \tilde{V}, \;\; (x, y) \mapsto \tilde{f}\big(\gamma(x), y\big).$$

Proposition 3.2.8 now implies, that

$$f_\sharp : C^\infty(M, \tilde{W}) \to C^\infty(M, \tilde{V}), \quad \eta \mapsto f \circ (\mathrm{id}_M, \eta)$$

is smooth. For $x \in M$ we see that

$$f_\sharp(\eta)(x) = f(x, \eta(x)) = \tilde{f}(\gamma(x), \eta(x)) = \varphi\Big(c_{\gamma(x)}(\varphi^{-1}\big(\eta(x)\big)\Big)$$

is the coordinate map of conjugation with γ. This completes the proof.

□

Corollary 3.3.4 *If M is a compact manifold with boundary and K a Fréchet-Lie group modelled on the Fréchet space F, then $C^\infty(M, K)$ is a Fréchet-Lie group i.e. $C^\infty(M, F)$ is a Fréchet space.*

Proof: These are Proposition 3.3.3 and Lemma 3.2.2.

□

Remark 3.3.5 The topology on $C^\infty(M, K)$ can alternatively be described as the topology making the natural map

$$C^\infty(M, K) \hookrightarrow \prod_{k=1}^{\infty} C(T^k M, T^k K), \quad \gamma \mapsto T^n \gamma$$

a topological embedding. This can directly be seen from the construction of the topology via the push forward of the charts $\varphi : U \to \tilde{U}$.

We now turn to the problem of topologising the gauge group Gau($\mathcal{P}$). Since we already know that for trivial bundles $\mathcal{P}$ the group Gau($\mathcal{P}$) is isomorphic to $C^\infty(M, K)$ (cf. Lemma 2.4.4), we see that restricted to a trivialising neighbourhood U, Gau($\mathcal{P}$) looks like $C^\infty(U, K)$. But globally the locally trivial pieces of Gau($\mathcal{P}$) might not fit together in a trivial way. For this aim we have to restrict to a special class of Lie groups as structure groups, so called *locally exponential* Lie groups as well as to compact base spaces M.

Definition. Logarithmic derivative: If M is a manifold, K a Lie group and $f \in C^\infty(M, K)$, then we denote by $\delta^l(f)$ the $\mathfrak{k}$-valued 1-form on M

$$\delta^l(f) : \mathcal{V}(M) \to C^\infty(M, \mathfrak{k}), \quad \big(\delta^l(f).X\big)(x) = d\lambda_{f(x)^{-1}}\big(f(x)\big).df(x).X(x),$$

called the *left logarithmic derivative* of f. It is also denoted by the shorthand notation $\delta^l(f) := f^{-1}.df$ and we define the *right logarithmic derivative* $\delta^r(f) := \delta^l(f^{-1}) = f.df^{-1}$.

Lemma 3.3.6 *For smooth functions $\gamma_{1,2} : M \to K$, we have*

$$\delta^r(\gamma_1\gamma_2) = \delta^r(\gamma_1) + \mathrm{Ad}(\gamma_1) \circ \delta^r(\gamma_2)$$

and

$$\delta^l(\gamma_1\gamma_2) = \delta^l(\gamma_2) + \mathrm{Ad}(\gamma_2) \circ \delta^l(\gamma_1).$$

If $\delta^l(\gamma_1) = \delta^l(\gamma_2)$ and M is connected, then there exists $k \in K$ such that $\gamma_1 = \lambda_k \circ \gamma_2$.

Proof: The first two identities follow directly from the definition of the logarithmic derivatives and

$$dm(k_1,k_2)(X_1,X_2) = dm(k_1,k_2)(X_1,0) + dm(k_1,k_2)(0,X_2) = d\rho_{k_2}(k_1).X_1 + d\lambda_{k_1}(k_2).X_2.$$

Hence we get for $\delta^l(\gamma_1) = \delta^l(\gamma_2)$

$$\delta^l(\gamma_1\gamma_2^{-1}) = \delta^l(\gamma_2^{-1}) + \mathrm{Ad}(\gamma_2^{-1})\delta^l(\gamma_1) = \delta^l(\gamma_2^{-1}) + \mathrm{Ad}(\gamma_2^{-1})\delta^l(\gamma_2) = \delta^l(\gamma_2\gamma_2^{-1}) = 0.$$

Thus $d(\gamma_1\gamma_2^{-1}) = 0$, such that $\gamma_1\gamma_2^{-1}$ is locally constant.

□

Definition. Locally Exponential Lie group: For a Lie group K denote for $X \in \mathfrak{k}$ by γ_X the constant function $t \mapsto X$ in $C^\infty([0,1],\mathfrak{k})$. Then K is called *locally exponential* if for each $X \in \mathfrak{k}$ the initial value problem

$$\gamma(0) = e, \quad \delta^l(\gamma) = \xi_X$$

has a solution $\gamma_X \in C^\infty(I,K)$ and the exponential function

$$\exp_K : \mathfrak{k} \to K, \ \ \gamma \mapsto \gamma_X(1)$$

is smooth and restricts to a local diffemorphism. Note that the preceding lemma implies that for each Lie group there exists at most one exponential function.

Lemma 3.3.7 *If K and K' are regular Lie groups, then for each Lie group homomorphism $\alpha : K \to K'$ and the induced Lie algebra homomorphism $d\alpha(e) : \mathfrak{k} \to \mathfrak{k}'$ the diagram*

$$\begin{array}{ccc} U & \xrightarrow{\alpha} & K' \\ \uparrow{\scriptstyle \exp_K} & & \uparrow{\scriptstyle \exp_{K'}} \\ \mathfrak{k} & \xrightarrow{d\alpha(e)} & \mathfrak{k}' \end{array}$$

commutes.

Proof: For $X \in \mathfrak{k}$ consider the curve

$$\tau : [0,1] \to K, \ \ t \mapsto \exp_K(tX).$$

Then $\gamma := \alpha \circ \tau$ is a curve such that $\gamma(0) = e$ and $\gamma(1) = \alpha\big(\exp_K(X)\big)$ with

$$\gamma'(t) = d\alpha\big(\exp_K(tX)\big).d\exp_K(tX).X\ .$$

For its left logarithmic derivate we compute

$$\begin{aligned}\delta^l(\gamma)(t) &= d\lambda_{\gamma(t)^{-1}}\big(\gamma(t)\big).\gamma'(t) = d\lambda_{\gamma(t)^{-1}}\big(\gamma(t)\big).d\alpha\big(\exp_K(tX)\big).d\exp_K(tX).X \\ &\overset{i)}{=} d\alpha(e).d\lambda_{\exp_K(tX)^{-1}}\big(\exp_K(tX)\big).d\exp_K(tX).X = d\alpha(e).\delta^l(\tau)(t) = d\alpha(e).X,\end{aligned}$$

where i) holds since $\alpha \circ \lambda_k = \lambda_{\alpha(k)} \circ \alpha$ holds for the homomorphism α. Hence $\gamma(1) = \exp_{K'}\big(d\alpha(e).X\big)$ holds and thus $\exp_{K'}\big(d\alpha(e).X\big) = \alpha\big(\exp_K(X)\big)$ for all $X \in \mathfrak{k}$.

□

Corollary 3.3.8 *If K is a locally exponential Lie group, then for each $k \in K$ there exists an open unit neighbourhood $\widetilde{U_k} \subseteq K$ diffeomorphic to an open zero neighbourhood $U_k := \exp_K^{-1}(\widetilde{U_k}) \subseteq \mathfrak{k}$ an an open neighbourhood V_k of k such that*

$$c_k \circ \exp_K|_{U_k} = \exp_K \circ \operatorname{Ad}(k)|_{U_k}$$

holds for all $k \in K$.

Proof: This follows directly from the continuity of the adjoint action and the preceeding lemma.

□

Lemma 3.3.9 *If K is a Banach-Lie group, then K is locally exponential.*

Proof: First we note that the existence of solutions to ordinary differential equations on open domains of Banach spaces and their smooth dependence on initial values implies that every Banach-Lie group is regular. It follows directly from the definition of $\exp_K$ that $\exp_K(tX) = \gamma_X(t)$ for $\gamma \in C^\infty(\mathbb{R}, K)$ such that $\delta^l(\gamma) = X$. Hence $d\exp_K(e) = \mathrm{id}_{\mathfrak{k}}$ and the Inverse Function Theorem yields that $\exp_K$ restricts to a diffeomorphism on some open zero neighbourhood of $\mathfrak{k}$.

□

Lemma 3.3.10 *If $\mathcal{P} = (K, M, P, \pi)$ is a K-bundle with compact base space M, then there exists a finite open cover $(V_i)_{i \in I_n}$, $I_n = \{1, \ldots, n\}$, such that each $\overline{V}_i$ is a manifold with boundary contained in some trivialising neighbourhood and contained in the domain of some chart.*

Proof: For each $x \in M$ we consider a trivialising neighbourhood $U_x \subseteq M$ for which there exists a chart φ_x with domain U_x, i.e. $\varphi_x : U_x \to \varphi_x(U_x) \subseteq \mathbb{R}^m$. Note that the compactness of M implies that M is finite-dimensional. Then there exists a closed ball $W_i \subseteq \mathbb{R}^n$ of radius small enough such that $\overline{W}_i \subseteq \varphi_x(U_x)$ and set $V_x := \varphi^{-1}(W_x)$. Clearly $\overline{V}_x = \varphi_x(\overline{W}_x)$ is a manifold with boundary and $(V_x)_{x\in M}$ is an open cover of M. Hence there exists a finite subcover $V_1, \dots, V_n$ with the desired properties.

□

The cover $V_1, \dots, V_n$, respectively the closed cover $\overline{V}_1, \dots, \overline{V_n}$, is the key for the topologisation of the gauge group.

Proposition 3.3.11 *If $\mathcal{P} = (K, M, P, \pi)$ is a K-bundle with compact base space M and locally exponential structure group K, then*

$$G(\mathcal{P}) := \left\{ (\gamma_i)_{i\in I_n} \in \prod_{i=1}^{n} C^\infty(\overline{V}_i, K) : \gamma_i(x) = k_{ij}(x)\gamma_j(x)k_{ji}(x) \text{ for all } x \in \overline{V_i} \cap \overline{V}_j \right\}$$

is a Lie group, where $(V_i)_{i\in I_n}$ is a finite open cover of M consisting of trivialising neighbourhoods, such that the $\overline{V}_i$'s are manifolds with boundary. It is modelled on the locally convex space

$$\mathfrak{g}(\mathcal{P}) := \left\{ (\xi_i)_{i\in I_n} \in \bigoplus_{i=1}^{n} C^\infty(\overline{V}_i, \mathfrak{k}) : \xi_i(x) = \mathrm{Ad}\big(k_{ji}(x)\big).\xi_j(x) \text{ for all } x \in \overline{V}_i \cap \overline{V}_j \right\}$$

endowed with the subspace-topology from $\bigoplus_{i=1}^{n} C^\infty(\overline{V_i}, \mathfrak{k})$.

Corollary 3.3.12 *If $\mathcal{P} = (K, M, P, \pi)$ is a K-bundle with compact base space and locally exponential structure group K, then* Gau($\mathcal{P}$) *and $C^\infty(P, K)^K$ are Lie groups isomorphic to $G(\mathcal{P})$. In addition $C^\infty(P, \mathfrak{k})^K$ is isomorphic to $\mathfrak{g}(\mathcal{P})$.*

Proof: For each $\gamma \in C^\infty(P, K)^K$ we consider the tuple of functions $(\gamma_i)_{i\in I_n}$ defined by $\gamma_i : \overline{V}_i \to K$, $x \mapsto \gamma\big(\Theta_i^{-1}(x, e)\big)$, where $\Theta_i : \pi^{-1}(\overline{V}_i) \to \overline{V}_i \times K$ is a local trivialisation. Due to the definition of the transition functions $\Theta_i^{-1}\big(x, k_{ij}(x)\big) = \Theta_j^{-1}(x, e)$ we have

$$\begin{aligned}\gamma_i(x) = \gamma\big(\Theta_i^{-1}(x, e)\big) = \gamma_i\big(\Theta_j^{-1}(x, e)k_{ji}(x)\big) = k_{ji}^{-1}(x)\gamma\big(\Theta_j^{-1}(x, e)\big)k_{ji}(x)\\ = k_{ij}(x)\gamma_j(x)k_{ji}(x).\end{aligned}$$

Clearly the map $C^\infty(P, K)^K \to G(\mathcal{P})$, $\gamma \mapsto (\gamma_i)_{i\in I_n}$ is a homomorphism. On the other hand each tuple $(\eta_i)_{i\in I_n} \in G(\mathcal{P})$ defines via

$$\eta(p) = \eta\big(\Theta_i^{-1}(x, e)k_i(p)\big) = k_i(p)\eta_i(x)k_i^{-1}(p) \text{ for } \pi(p) \in \overline{V}_i,$$

where $k_i : \pi^{-1}(\overline{V_i}) \to K$, $p \mapsto \mathrm{pr}_2\big(\Theta_i(p)\big)$, a map in $C^\infty(P,K)^k$. This mapping is inverse to the constructed map from $C^\infty(P,K)^K$ to $G(\mathcal{P})$ and thus these groups are isomorphic as groups.
Now we endow $\mathrm{Gau}(\mathcal{P}) \cong C^\infty(P,K)^K$ and $C^\infty(P,K)^K$ with the topology induced from $G(\mathcal{P})$ and obtain a Lie group structure on them. The same construction leads to the isomorphism $C^\infty(P,\mathfrak{k})^K \cong L\big(G(\mathcal{P})\big)$.

□

Corollary 3.3.13 *If $\mathcal{P}_k$ is a K-bundle over $\mathbb{S}^1$ determined by $k \in K$ (cf. Proposition 2.5.2), then* $\mathrm{Gau}(\mathcal{P})$ *is isomorphic tohe twisted loop group*

$$C^\infty_k(\mathbb{R},K) := \{\gamma \in C^\infty(\mathbb{R},K) : \gamma(x) = k^n\gamma(x+n)k^{-n}\}.$$

Definition. Gauge Algebra: The Lie algebra $\mathfrak{gau}(\mathcal{P}) := \mathfrak{g}(\mathcal{P})$ is said to be the *gauge algebra* of the K-bundle $\mathcal{P} = (K,M,P,\pi)$ with compact base space M and locally exponential structure group K.

Remark 3.3.14 In the following text we will always identify the gauge group $\mathrm{Gau}(\mathcal{P})$ with $C^\infty(P,K)^K$ or $G(\mathcal{P})$. The viewpoint identifying $\mathrm{Gau}(\mathcal{P})$ with $C^\infty(P,K)^K$ is more useful when performing calculations in the global picture, where $\mathcal{P}$ is given as the quadruple (K,M,P,π). For calculations in local coordinates, i.e. when $\mathcal{P}$ is given by transition functions $(K,M,(U_i)_{i\in I},k_{ij})$ and for topological considerations, the viewpoint $\mathrm{Gau}(\mathcal{P}) \cong G(\mathcal{P})$ is more convenient. We will always point this out by stating either $\mathrm{Gau}(\mathcal{P}) \cong C^\infty(P,K)^K$ or $\mathrm{Gau}(\mathcal{P}) \cong G(\mathcal{P})$.
The same distinction we make on the level of the gauge algebra, where the two viewpoints are $\mathfrak{gau}(\mathcal{P}) \cong C^\infty(P,\mathfrak{k})^K$ and $\mathfrak{gau}(\mathcal{P}) \cong \mathfrak{g}(\mathcal{P})$.

3.4 An Approximation Theorem

Since the natural constructions in many proofs yield only continuous maps, but we are always interested in smooth ones, we will often need approximation arguments. This section will provide the relevant changes to the statements given in [Nee02, Section A.3] and is closely related the second chapter in [Hir76].

Definition. Continuous Automorphism and Gauge Transformation: If $\mathcal{P} = (K,M,P,\pi)$ is a K-bundle then a *continuous automorphism* is an equivariant homeomorphism $f : P \to P$ and the *group of continuous automorphisms* is denoted by $\mathrm{Aut}_{\mathrm{cont}}(\mathcal{P})$. A *continuous gauge transformation* is a continuous automorphism $f : P \to P$ satisfying $\pi \circ f = \pi$ and the *group of continuous gauge transformations* is denoted by

$$\mathrm{Gau}_{\mathrm{cont}}(\mathcal{P}) := \{f \in \mathrm{Aut}_{\mathrm{cont}}(\mathcal{P}) : \pi \circ f = \pi \ \text{ for all } \ p \in P\}.$$

Remark 3.4.1 The same consideration as in the precesding section lead to

$$\mathrm{Gau}_{\mathrm{cont}}(\mathcal{P}) \cong \{(\gamma_i)_{i\in I_n} \in \prod_{i=1}^{n} C(\overline{V}_i, K) : \gamma_i(x) = k_{ij}(x)\gamma_j(x)k_{ji}(x) \text{ for all } x \in \overline{V}_i \cap \overline{V}_j\}.$$

Each $C(\overline{V_i}, K)$ is a topological group with respect to the compact-open topology. Since the evaluation maps are continuous we conclude that $\mathrm{Gau}_{\mathrm{cont}}(\mathcal{P})$ is a closed subgroup of the topological group $\prod_{i=1}^{n} C(\overline{V_i}, K)$

In the following $C(X,Y)_c$ denotes the set of continuous functions from X to Y equipped with the compact open topology.

Lemma 3.4.2 *If M is a finite-dimensional σ-compact manifold with boundary, then for each locally convex space F the space $C^\infty(M,F)$ is dense in $C(M,F)_c$. If $f \in C(M,F)$ has compact support and U is an open neighbourhood of $\mathrm{supp}(f)$, then each neighbourhood of f in $C(M,F)$ contains a smooth function whose support is contained in U.*

Proof: The proof of [Nee02, Theorem A.3.1] can be taken over in exactly the same way.

□

Corollary 3.4.3 *If M is a finite-dimensional σ-compact manifold with boundary and V is an open subset of the locally convex space F, then $C^\infty(M,V)$ is dense in $C(M,V)_c$.*

Lemma 3.4.4 *Let M be a σ-compact manifold with boundary, F be a locally convex space, $W \subseteq F$ be open and $f : M \to W$ be continuous. If $L \subseteq M$ is closed and $U \subseteq M$ is open such that f is smooth on a neighbourhood of $L\backslash U$, then each neighbourhood of f in $C(M,F)_c$ contains a continuous map h which is smooth on a neighbourhood of L and which equals f on $M\backslash U$.*

Proof: (cf. [Hir76, Theorem 2.5]) We may w.l.o.g. assume that W is convex. Let $A \subseteq M$ be an open set containing $L\backslash U$ such that $f\big|_A$ is smooth. Then $L\backslash A \subseteq U$ is closed in M such that there exists $V \subseteq U$ open with

$$L\backslash A \subseteq V \subseteq \overline{V} \subseteq U.$$

Then $\{U, M\backslash\overline{V}\}$ is an open cover of M, and there exists a smooth partition of unity $\{f_1, f_2\}$ subordinated to this cover. Then

$$G_f : C(M,W)_c \to C(M,F)_c, \ \ G(\gamma)(x) = f_1(x)\gamma(x) + f_2(x)f(x)$$

is continuous since $\gamma \mapsto f_1\gamma$ is continuous (for $f_1(x) \in [0,1]$ for all $x \in U$ and the compact open topology on $C(U,F)$ coincides with the topology of uniform convergence on compact subsets) and $f_1\gamma \mapsto f_1\gamma + f_2 f$ is continuous since addition in $C(M,F)_c$ is so.

Then $G_f(\gamma)$ is smooth on $A \cup V \supseteq A \cup (L \backslash A) \supseteq L$ if γ is so since f_1 is smooth, f is smooth on A and $f_2\big|_V \equiv 0$. Furthermore we have $G_f(\gamma) = \gamma$ on V and $G_f(\gamma) = f$ on $M \backslash U$. Since $G(f) = f$ there is for each open neighbourhood O of f an open neighbourhood O' of f such that $G_f(O') \subseteq O$. By the preceding lemma there is a smooth function $h \in O'$ such that $G_f(h)$ has the desired properties.

$\square$

Corollary 3.4.5 *Let K be a Lie group and M be a manifold with boundary. If $\widetilde{W} \subseteq K$ is diffeomorphic to an open subset of $\mathfrak{k}$, $L \subseteq M$ is closed, $U \subseteq M$ is open and $f \in C(M,\widetilde{W})$ is smooth on a neighbourhood of $L \backslash U$, then each neighbourhood of f in $C(M,\widetilde{W})_{c.o.}$ contains a map g which is smooth on a neighbourhood of L and which equals f on $M \backslash U$.*

Proposition 3.4.6 *Let K be a Lie group and M be a σ-compact manifold with boundary. If $L \subseteq M$ is closed, $V \subseteq M$ is open and $f \in C(M,K)$ is smooth on a neighbourhood of $L \backslash V$, then each open neighbourhood O of f in $C(M,K)_c$ contains a $g \in C(M,K)$ which is smooth on a neighbourhood of L and equals f on $M \backslash V$.*

Proposition 3.4.7 *If M is compact and K is locally exponential, then the group* $\mathrm{Gau}_{\mathrm{cont}}(\mathcal{P})$ *of continuous gauge transformation is dense in* $\mathrm{Gau}(\mathcal{P})$.

Lemma 3.4.8 *If $(\gamma_i)_{i \in I_n} \in \mathrm{Gau}(\mathcal{P})$ is a smooth gauge transformation which is close to identity, in the sense that $\gamma_i(\overline{V}_i) \subseteq U$ for an open unit neighbourhood $U \subseteq K$ diffeomorphic to an open convex zero neighbourhood, then $(\gamma_i)_{i \in I_n}$ is homotopic to e.*

Proof: We may assume that $U \subseteq K$ is the open unit neighbourhood from Corollary 3.3.8 and that $\varphi := \exp_K^{-1} : U \to \tilde{U}$ be the inverse diffeomorphism of the exponential function on U. Since $\gamma_i(\overline{V}_i) \subseteq U$, U is convex and $\mathrm{Ad}(k_{ij})(x)$ is linear

$$\tilde{F}_i : [0,1] \to C^\infty\big(\overline{V}_i, \varphi(U)\big), \quad \tilde{F}_i(t)(x) = t\,\varphi\big(\gamma_i(x)\big)$$

defines a homotopy in $\mathfrak{gau}(\mathcal{P})$ and hence

$$F_i : [0,1] \to C^\infty\big(\overline{V}_i, U\big), \quad F_i(t)(x) = \exp_K\big(\tilde{F}_i(t)(x)\big)$$

defines a homotopy in $\mathrm{Gau}(\mathcal{P})$ from e to $(\gamma_i)_{i \in I_n}$.

$\square$

Corollary 3.4.9 *If $\mathcal{P} = (K, M, P, \pi)$ is a K-bundle with compact base space then*

$$\mathrm{Gau}(\mathcal{P}) \cap \mathrm{Gau}_{\mathrm{cont}}(\mathcal{P})_0 = \mathrm{Gau}(\mathcal{P})_0.$$

Proof: With the preceding lemma the argument from [Nee02, Lemma A.3.6] carries over in exactly the same way.

□

Lemma 3.4.10 *If K is a topological group and (X, x_0) an arcwise connected pointed topological space, then*

$$\pi_0\big(C_*(X,K)\big) \cong \pi_0\big(C(X,K_0)\big)$$

holds, where $C_*(X,K)$ denotes the space of base point preserving continuous maps from X to K.

Proof: We have that $C(X,K_0) \cong C_*(X,K_0) \rtimes_\alpha K_0$ for

$$\alpha : K_0 \to \mathrm{Aut}\big(C_*(X,K_0)\big), \;\; \alpha(k)(f) = kfk^{-1},$$

where we identify k with the constant function $x \mapsto k$. Thus

$$\pi_0\big(C(X,K_0)\big) \cong \pi\big(C_*(X,K_0) \rtimes K_0\big) = \pi_0\big(C_*(X,K_0)\big)$$

and since X is assumed to be arcwise connected we have $C_*(X,K) = C_*(X,K_0)$.

□

Corollary 3.4.11 *For a topological group K we have the isomorphism*

$$\pi_k(K) := \pi_0\big(C_*(\mathbb{S}^k, K)\big) \cong \pi_0\big(C(\mathbb{S}^k, K_0)\big).$$

Theorem 3.4.12 *For a K-bundle $\mathcal{P} = (M, K, P, \pi)$ the natural inclusion*

$$\mathrm{incl} : \mathrm{Gau}(\mathcal{P}) \hookrightarrow \mathrm{Gau}_{\mathrm{cont}}(\mathcal{P})$$

is a weak homotopy equivalence, i.e. the induced mappings

$$\pi_k(\mathrm{incl}) : \pi_k(\mathrm{Gau}(\mathcal{P})) \to \pi_k\big(\mathrm{Gau}_{\mathrm{cont}}(\mathcal{P})\big)$$

are isomorphisms of groups.

Proof: To verify surjectivity, consider the local description by trivialising neighbourhoods $(U_i)_{i\in I}$ and transition functions $k_{ij} : U_i \cap U_j \to K$ of $\mathcal{P}$ and the manifold $\mathbb{S}^k \times M$. Note that if M is a manifold with boundary, so is $\mathbb{S}^k \times M$ since $\partial(\mathbb{S}^k) = \varnothing$. Then $(\mathbb{S}^k \times U_i)_{i\in I}$ is an open cover of $\mathbb{S}^k \times M$ and

$$\tilde{k}_{ij} : (\mathbb{S}^k \times U_i) \cap (\mathbb{S}^k \times U_j) = \mathbb{S}^k \times (U_i \cap U_j) \to K, \quad (t,x) \mapsto k_{ij}(x)$$

are the transition functions of a K-bundle over $\mathbb{S}^k \times M$ corresponding to the pull back $\mathrm{pr}_2^*(\mathcal{P})$ for the projection $\mathrm{pr}_2 : \mathbb{S}^k \times M \to M$. With the local description of this bundle we derive

$$\begin{aligned}\mathrm{Gau}_{\mathrm{cont}}(\mathrm{pr}_2^*(\mathcal{P})) \cong \{(\gamma_i)_{i\in I} \in \prod_{i=1}^{n} C(\mathbb{S}^k \times \overline{V}_i) \to K :& \\ \gamma_i(t,x) = k_{ij}(x)\gamma_j(t,x)k_{ji}(x) \text{ for all } t \in \mathbb{S}^k, x \in \overline{V}_i \cap \overline{V}_j\}&\end{aligned}$$

Since $C(\mathbb{S}^k \times \overline{V}_i, K) \cong C\big(\mathbb{S}^k, C(\overline{V}_i, K)\big)$ [Glö03, Chapter 23] this yields

$$\mathrm{Gau}_{\mathrm{cont}}(\mathrm{pr}_2^*(\mathcal{P})) \cong C\big(\mathbb{S}^k, \mathrm{Gau}_{\mathrm{cont}}(\mathcal{P})\big).$$

Hence we get with Corollary 3.4.11 for $k \geq 1$

$$\begin{aligned}\pi_k\big(\mathrm{Gau}(\mathcal{P})\big) := \pi_0\Big(C_*\big(\mathbb{S}^k, \mathrm{Gau}(\mathcal{P})\big)\Big) \cong \pi_0\Big(C\big(\mathbb{S}^k, \mathrm{Gau}(\mathcal{P})_0\big)\Big) \xrightarrow{\iota}& \\ \pi_0\Big(C\big(\mathbb{S}^k, \mathrm{Gau}_{\mathrm{cont}}(\mathcal{P})_0\big)\Big) \to \pi_0\Big(C_*\big(\mathbb{S}^k, \mathrm{Gau}_{\mathrm{cont}}(\mathcal{P})\big)\Big) := \pi_0\Big(\mathrm{Gau}_{\mathrm{cont}}\big(\mathrm{pr}_2^*(\mathcal{P})\big)\Big),&\end{aligned}$$

where ι is induced by the inclusion $\mathrm{Gau}(\mathcal{P}) \to \mathrm{Gau}_{\mathrm{cont}}(\mathcal{P})$. Now Corollary 3.4.9 yields

$$\mathrm{Gau}\big(\mathrm{pr}_2^*(\mathcal{P})\big) \cap \mathrm{Gau}_{\mathrm{cont}}\big(\mathrm{pr}_2^*(\mathcal{P})\big)_0 = \mathrm{Gau}\big(\mathrm{pr}_2^*(\mathcal{P})\big)_0$$

such that if $(\gamma_i)_{i\in I_n} \in \mathrm{Gau}_{\mathrm{cont}}\big(\mathrm{pr}_2^*(\mathcal{P})\big)_0$ is a continuous gauge transformation, then Proposition 3.4.7 implies that each neighbourhood of $(\gamma_i)_{i\in I_n}$ contains a smooth gauge transformation.herefore the identity component of $\mathrm{Gau}_{\mathrm{cont}}\big(\mathrm{pr}_2^*(\mathcal{P})\big) \cong C\big(\mathbb{S}^k, \mathrm{Gau}_{\mathrm{cont}}(\mathcal{P})\big)$ and hence the identity component of $C\big(\mathbb{S}^k, \mathrm{Gau}_{\mathrm{cont}}(\mathcal{P})_0\big)$ contains an element of $C\big(\mathbb{S}^k, \mathrm{Gau}(\mathcal{P})_0\big)$ (Lemma 3.4.10). Thus

$$\iota : \pi_0\Big(C\big(\mathbb{S}^k, \mathrm{Gau}(\mathcal{P})_0\big)\Big) \to \pi_0\Big(C\big(\mathbb{S}^k, \mathrm{Gau}_{\mathrm{cont}}(\mathcal{P})_0\big)\Big)$$

is surjective, which implies that

$$\pi_k(\mathrm{incl}) : \pi_k\big(\mathrm{Gau}(\mathcal{P})\big) \to \pi_k\big(\mathrm{Gau}_{\mathrm{cont}}(\mathcal{P})\big)$$

is surjective. That $\pi_k(\mathrm{incl})$ is also injective follows with Lemma 3.4.8 as in [Nee02, Theorem A.3.7].

□

Chapter 4

Central Extensions of Lie Groups

4.1 Central Extensions and Cocycles

This whole chapter is a rough summary of some results in [Nee02], although some details may differ a bit, as far as they are necessary for the understanding of the following text.

Definition. Exact Sequence, Central Extension of groups: If A, B and C are groups, then a *short exact sequence* is a sequence of homomorphisms

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C$$

such that α is injective, β is surjective and $\operatorname{im}(\alpha) = \ker(\beta)$. A *central extension* of the group G by the abelian group Z is a short exact sequence

$$Z \xrightarrow{\alpha} \hat{G} \xrightarrow{\beta} G$$

of groups, such that $\operatorname{im}(\alpha)$ is a central subgroup of $\hat{G}$.

Remark 4.1.1 One usually identifies Z with $\operatorname{im}(\alpha)$ as a central subgroup of $\hat{G}$. Then canonical factorisation yields $\hat{G}/Z \cong G$, such that one can think of $\hat{G}$ as a group built up of the two parts Z and G. That is the reason why one often writes $Z \hookrightarrow \hat{G} \overset{q}{\twoheadrightarrow} G$ to indicate that Z is a subgroup of $\hat{G}$ and that G is a quotient of $\hat{G}$.

Lemma 4.1.2 *If $Z \hookrightarrow \hat{G} \overset{q}{\twoheadrightarrow} G$ is a central extension of groups, then there exists a homomorphism $\sigma : G \to \hat{G}$ such that $q \circ \sigma = \mathrm{id}_G$ if and only if there exists an isomorphism $\alpha : Z \times G \to \hat{G}$ making the diagram*

$$\begin{array}{ccccc} Z & \longrightarrow & Z \times G & \xrightarrow{\mathrm{pr}_2} & G \\ \big\downarrow{\scriptstyle \mathrm{id}_Z} & & \big\downarrow{\scriptstyle \alpha} & & \big\downarrow{\scriptstyle \mathrm{id}_G} \\ Z & \longrightarrow & \hat{G} & \xrightarrow{q} & G \end{array} \tag{4.1}$$

commutative.

Proof: If $\alpha : Z \times G \to \hat{G}$ is such an isomorphism then

$$\sigma = \alpha\big|_{\{e\}\times G} : \{e\} \times G \to \hat{G}$$

defines a homomorphism. That $q \circ \sigma = \mathrm{id}_G$ is the same as saying that q is surjective, σ is injective and that $\sigma(G) \cap q^{-1}\big(\{g\}\big)$ contains exactly one element for all $g \in G$. This clearly is satisfied since $q^{-1}\big(\{g\}\big) = Zg'$ for an arbitrary g' such that $q(g') = g$. If conversely $\sigma : G \to \hat{G}$ is an homomorphism such that $q \circ \sigma = \mathrm{id}_G$, then

$$\alpha' : Z \times G \to \hat{G}, \quad (z, g) \mapsto z \cdot \sigma(g)$$

is a homomorphism since Z is central in $\hat{G}$. Furthermore it is injective since σ is so and $\sigma(G) \cap Z = \{e\}$ and it is surjective since $\hat{G}/Z \cong G \cong Z\sigma(G)/Z$.

□

Definition. Trivial Central Extension, Split: If $Z \hookrightarrow \hat{G} \overset{q}{\twoheadrightarrow} G$ is a central extension of G by Z, then this extension is said to be *trivial* if there exists a homomorphism $\sigma : G \to \hat{G}$ such that $q \circ \sigma = \mathrm{id}_G$. This homomorphism σ is called a *splitting* of the central extension.

The question arising is how to handle central extensions and how to characterise them. This is done by identifying them with a certain class of maps parametrising all central extensions of a given group G by an abelian group Z

Definition. Cocycle, Coboundary: Given a group G and an abelian group Z, a map $f : G \times G \to Z$ is called a *cocycle* if

$$f(x, e) = f(e, x) = e, \quad f(x, y)f(xy, z) = f(x, yz)f(y, z)$$

holds for all $x, y, z \in G$. It is called a *coboundary* if it is a cocycle and there exists a map $h : G \to Z$ satisfying $h(e) = e$ such that

$$f(x, y) = h(xy)h(x)^{-1}h(y)^{-1}$$

holds for all $x, y \in G$. Furthermore we define

$$Z^2(G, Z) := \{f : G \times G \to Z : f \text{ is a cocycle}\}$$

which is an abelian group with respect to pointwise multiplication and its subgroup

$$B^2(G, Z) := \{f : G \times G \to Z : f \text{ is a coboundary}\}.$$

Then the quotient $Z^2(G, Z)/B^2(G, Z) := H^2(G, Z)$ is denoted by $\mathrm{Ext}(G, Z)$.

Lemma 4.1.3 *If $f : G \times G \to Z$ is a cocycle, then*

$$(z, g)(z', g') \mapsto (zz'f(g, g'), gg')$$

defines a group multiplication on $Z \times G$ with identity element (e, e).

Proof: It is an easy calculation to verify that the property of f being a cocycle actually is equivalent to this map being a group multiplication.

□

Remark 4.1.4 The group $Z \times G$ with the multiplication defined above is denoted by $Z \times_f G$. Two extensions $Z \times_f G$ and $Z \times_{f'} G$ are isomorphic (i.e. there exists an isomorphism α making the diagram (4.1) commutative) if and only if the difference $(g, g') \mapsto f(g, g')f(g, g')^{-1}$ is a coboundary [Mac63, Theorem IV.4.1] and hence $\operatorname{Ext}(G, Z)$ parametrises the equivalence classes of central extensions of G by Z. The neutral element in $Z \times_f G$ is (e, e), inversion given by

$$(z, g)^{-1} = \left(z^{-1} f(g, g^{-1})^{-1}, g^{-1}\right)$$

and conjugation with (z, g) by

$$c_{(z,g)}(w, h) = \left(wf(g, g^{-1})^{-1} f(h, g) f(g^{-1}, hg), g^{-1}hg\right).$$

In the context of Lie groups we want the extensions to admit smooth local sections in the sense defined below. As we will see on later this implies the existence of continuous linear sections for the corresponding central extensions of Lie algebras. Since we will need integration methods later on we have to restrict our considerations to sequentially complete locally convex (*s.c.l.c.*) spaces and since all known abelian Lie groups Z modelled on a s.c.l.c. space $\mathfrak{z}$ are of the form $\mathfrak{z}/\Gamma$ for a discrete subgroup $\Gamma \cong \pi_1(Z)$ we will restrict in the following to this kind of abelian groups. Note that if Z is finite-dimensional, then covering theory yields that it is always of the form $\mathfrak{z}/\pi_1(Z)$.

Definition. Central Extension of Lie Groups: If G, $\hat{G}$ are Lie groups and $Z \cong \mathfrak{z}/\Gamma$ is an abelian Lie group for a s.c.l.c. space $\mathfrak{z}$ and a discrete subgroup $\Gamma \cong \pi_1(Z) \subseteq \mathfrak{z}$, then a central extension $Z \hookrightarrow \hat{G} \overset{q}{\twoheadrightarrow} G$ of groups, where the homomorphisms are assumed to be smooth, is called a *central extension of Lie groups* if the there exists an open unit neighbourhood $U \subseteq G$ and a smooth map $\sigma : U \to \hat{G}$ such that $q \circ \sigma = \mathrm{id}_U$. This map σ is called *local section.*

Remark 4.1.5 The requirement of the existence of a smooth local section implies in particular that the central extension $Z \hookrightarrow \hat{G} \overset{q}{\twoheadrightarrow} G$ has the structure of a Z-bundle with base space G.

Proposition 4.1.6 *If G is a connected Lie group, Z is an abelian Lie group and $Z \hookrightarrow \hat{G} \overset{q}{\twoheadrightarrow} G$ is a central extension of groups, then $\hat{G}$ carries the structure of a Lie group if and only if the central extension can be described by a cocycle $f : G \times G \to Z$ which is smooth on an open unit neighbourhood in $G \times G$.*

Proof: [Nee02, Proposition 4.2]

□

Since in this text we are always studying Lie algebras coming from Lie groups, we will always assume the Lie algebras to be locally convex topological Lie algebras. Then the concept of a central extension is the following.

Definition. Central Extension of Topological Lie Algebras: An exact sequence of topological Lie algebras $\mathfrak{z} \hookrightarrow \hat{\mathfrak{g}} \overset{q}{\twoheadrightarrow} \mathfrak{g}$, where q is assumed to be a continuous Lie algebra homomorphism, is called a *central extension of Lie algebras* if $\mathfrak{z}$ is a central subalgebra of $\hat{\mathfrak{g}}$ and there exists a continuous linear map $\sigma : \mathfrak{g} \to \hat{\mathfrak{g}}$ such that $q \circ \sigma = \mathrm{id}_{\mathfrak{g}}$. Then σ is called a *continuous linear section.*

Remark 4.1.7 Since we want the topological vector space, underlying $\hat{\mathfrak{g}}$ to be isomorphic to the product $\mathfrak{z} \oplus |\mathfrak{g}|$, where $|\mathfrak{g}|$ denotes the vector space underlying $\mathfrak{g}$, we need that $\mathfrak{z}$ is complemented in $|\hat{\mathfrak{g}}|$. This means that there has to be a continuous projection onto $|\mathfrak{g}|$ and in our context this projection is given by $\sigma \circ q : \hat{\mathfrak{g}} \to \hat{\mathfrak{g}}$.

Central extensions of the topological Lie algebra $\mathfrak{g}$ by the vector space $\mathfrak{z}$ are parametrised as follows.

Definition. Lie Algebra Cocycle, Lie Algebra Coboundary: Given a topological Lie algebra $\mathfrak{g}$ and a locally convex space $\mathfrak{z}$ a map $\omega : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{z}$ is called a *cocycle* if

$$\omega\big(x, [y, z]\big) + \omega\big(y, [z, x]\big) + \omega\big(z, [x, y]\big) = 0$$

holds for all $x, y, z \in \mathfrak{g}$. It is called a *coboundary* if it is a cocycle and there exists a continuous linear map $\alpha : \mathfrak{g} \to \mathfrak{z}$ such that

$$\omega(x, y) = \alpha\big([x, y]\big)$$

holds for all $x, y \in \mathfrak{g}$. Furthermore we define

$$Z^2(\mathfrak{g}, \mathfrak{z}) := \{f : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{z} : f \text{ is a cocycle}\},$$

which is a vector space with respect to pointwise operations and its subspace

$$B^2(\mathfrak{g}, \mathfrak{z}) := \{f : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{z} : f \text{ is a coboundary}\}.$$

Then the quotient $Z^2(\mathfrak{g}, \mathfrak{z})/B^2(\mathfrak{g}, \mathfrak{z}) := H^2(\mathfrak{g}, \mathfrak{z})$ is called the *second Lie algebra cohomology* space.

Lemma 4.1.8 *If* $\omega : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{z}$ *is a continuous cocycle then*

$$\big[(z,x),(z',x')\big] := \big(\omega(x,x'),[x,x']\big)$$

defines a continuous Lie bracket on $\hat{\mathfrak{g}} := \mathfrak{z} \oplus \mathfrak{g}$, *turning it into a topological Lie algebra.*

Proof: It is clear that $[\cdot,\cdot] : \hat{\mathfrak{g}} \times \hat{\mathfrak{g}} \to \hat{\mathfrak{g}}$ is continuous, since ω and the bracket on $\mathfrak{g}$ are continuous. It is an easy calculation that this map defines a Lie bracket if and only if ω is a continuous cocycle.

□

Remark 4.1.9 The Lie algebra $\hat{\mathfrak{g}}$ with the bracket as above is denoted by $\mathfrak{z} \oplus_\omega \mathfrak{g}$. Two Lie algebra extensions $\mathfrak{z} \oplus_\omega \mathfrak{g}$ and $\mathfrak{z} \oplus_{\omega'} \mathfrak{g}$ are equivalent if and only if the difference $(x,x') \mapsto \omega(x,x') - \omega'(x,x')$ is a continuous coboundary and hence $H^2(\mathfrak{g},\mathfrak{z})$ parametrises the equivalence classes of central extensions of $\mathfrak{z}$ by $\mathfrak{g}$.
If the extension $\mathfrak{z} \hookrightarrow \hat{\mathfrak{g}} \overset{q}{\twoheadrightarrow} \mathfrak{g}$ with the linear section $\sigma : \mathfrak{g} \to \hat{\mathfrak{g}}$ is given, then $\omega_\sigma(x,y) = \big[\sigma(x),\sigma(y)\big] - \sigma\big([x,y]\big)$ is a $\mathfrak{z}$-valued cocycle and $\mathfrak{z} \oplus_{\omega_\sigma} \mathfrak{g} \to \hat{\mathfrak{g}}$, $(z,x) \mapsto z + \sigma(x)$ is an isomorphism of topological Lie algebras.

Proposition 4.1.10 *Assume that* $\mathfrak{z}$ *is a locally convex space,* $\Gamma \subseteq \mathfrak{z}$ *is a discrete subgroup and* $\mathfrak{z}/\Gamma \cong Z \hookrightarrow \hat{G} \overset{q}{\twoheadrightarrow} G$ *a central extension of Lie groups ,where* G *is connected. If* $f : G \times G \to Z$ *is the cocycle describing the central extension of groups and* $f = q_Z \circ f_\mathfrak{z}$ *for the quotient map* $q_Z : \mathfrak{z} \to \mathfrak{z}/\Gamma$ *where* $f_\mathfrak{z}$ *is smooth in a unit neighbourhood, then*

$$Df(x,y) := d^2 f_\mathfrak{z}(e,e)(x,y) - d^2 f_\mathfrak{z}(e,e)(y,x)$$

defines a continuous Lie algebra cocycle such that $\hat{\mathfrak{g}} := L(\hat{G})$ *is isomorphic to* $\mathfrak{z} \oplus_{Df} \mathfrak{g}$.

Proof: [Nee02, Lemma 4.6]

□

4.2 Integrability Criteria

The question arising now is if for a given central extension of Lie algebras $\mathfrak{z} \hookrightarrow \hat{\mathfrak{g}} \overset{q}{\twoheadrightarrow} \mathfrak{g}$ there exists a central extension $Z \hookrightarrow \hat{G} \overset{q}{\twoheadrightarrow} G$ of Lie groups such that the the cocycle $f : G \times G \to Z$, describing the central group extension, describes the Lie algebra extension by means of $Df = \omega$ for a continuous cocycle ω with $\hat{\mathfrak{g}} \cong \mathfrak{z} \oplus_\omega \mathfrak{g}$. This question will be answered in this section.

Definition. Period Map, Period Group: If G is a connected Lie group and $\Omega \in \Omega^2(G, \mathfrak{z})$ is a closed $\mathfrak{z}$-valued 2-form for the s.c.l.c. space $\mathfrak{z}$, then the *period map* $\mathrm{per}_\Omega : \pi_2(G) \to \mathfrak{z}$ is given by

$$\mathrm{per}_\Omega([\sigma]) = \int_\sigma \Omega$$

for a smooth representative σ in $[\sigma]$. The image $\mathrm{im}(\mathrm{per}_\Omega)$ is called the *period group* . If Ω is left invariant with $\Omega(e) = \omega$, then we set $\mathrm{per}_\omega := \mathrm{per}_\Omega$ and $\Pi_\omega := \mathrm{im}(\mathrm{per}_\Omega)$.

Remark 4.2.1 See [Nee02, Section A.3] for the existence of smooth representatives in each class $[\sigma] \in \pi_2(G)$. Actually for the integral to be defined one requires the map $\mathbb{S}^2 \to G$ only to be piecewise smooth for a triangulation of the compact manifold $\mathbb{S}^2$. This is important for the calculations in the proofs of the following statements, but we will not go into the details here.

Lemma 4.2.2 *The period map is well-defined, i.e.* $\int_\sigma \Omega$ *does only depend on the homotopy class of* σ*. Furthermore* per_Ω *is a group homomorphism and* Π_Ω *is a subgroup of* $\mathfrak{z}$*.*

Proof: This statement is contained in [Nee02, Lemma 5.7].

□

Theorem 4.2.3 *If G is a connected and simply connected Lie group with Lie algebra $\mathfrak{g}$, $\mathfrak{z}$ a s.c.l.c. space and $\omega \in Z^2(\mathfrak{g}, \mathfrak{z})$ a 2-cocycle, then there exists a central extension of Lie groups $Z \hookrightarrow \hat{G} \overset{q}{\twoheadrightarrow} G$ such that $Z \cong \mathfrak{z}/\Gamma$ for a discrete subgroup $\Gamma \subseteq \mathfrak{z}$ and that $\hat{\mathfrak{g}} \cong \mathfrak{z} \oplus_\omega \mathfrak{g}$ if and only if the period group Π_ω is a discrete subgroup of $\mathfrak{z}$.*

Proof: [Nee02, Theorem 7.9]

□

If G is not simply connected, then there is an additional obstruction on $\omega \in Z^2(\mathfrak{g}, \mathfrak{z})$. In the construction procedure one wants to apply the preceding theorem to the universal covering group $\tilde{G}$ of G and then 'factor out $\pi_1(G)$' as a subgroup of $\hat{G}$. That this works properly depends on the so called integration map.

Definition. Integration Map: Consider $\omega \in Z^2(\mathfrak{g}, \mathfrak{z})$ and let $\Omega \in \Omega^2(G, \mathfrak{z})$ be the corresponding left invariant 2-form. For $X \in \mathfrak{g}$ and the corresponding *right* invariant vector field X_r the form $i(X_r).\Omega$ is a closed $\mathfrak{z}$-valued 1-form, which can be integrated over $\gamma \in C^\infty_*(\mathbb{S}^1, G)$, i.e. $\int_\gamma i(X_r).\Omega$. The map

$$(\gamma, X) \mapsto \int_\gamma i(X_r).\Omega$$

can be considered for fixed $\gamma \in C^\infty_*(\mathbb{S}^1, G)$ as a linear map from $\mathfrak{g}$ to $\mathfrak{z}$. Since this map does only depend on the homotopy class of γ, it can be considered as a map $P_2(\omega) : \pi_1(G) \to \mathrm{Lin}(\mathfrak{g}, \mathfrak{z})$. This map is called *integration map* .

Lemma 4.2.4 *The cohomology class of $i(X_r).\Omega$ does only depend on the cohomology class of ω in $H^2(\mathfrak{g}, \mathfrak{z})$, i.e. if $\Omega = d\alpha$ for a left invariant 1-form α, then $i(X_r).d\alpha$ is exact. For $\gamma \in C_*(\mathbb{S}^1, G)$ the linear map*

$$\mathfrak{g} \to \mathfrak{z}, \quad x \mapsto \int_\gamma i(X_r).\Omega$$

is continuous.

Proof: The first assertion is [Nee02, Lemma 3.11]. The continuity follows from

$$\int_\gamma i(X_r).\Omega = \int_0^1 \omega\big(\mathrm{Ad}(\gamma(t)^{-1}).x, \gamma'(t)\big) dt$$

since the integrand is a continuous map $[0, 1] \times \mathfrak{g} \to \mathfrak{z}$ (cf. [Nee02, Definition 7.1]).
□

Remark 4.2.5 Since the integration map P_2 only depends on the class of ω in $H^2(\mathfrak{g}, \mathfrak{z})$ we will denote by P_2 also the factorisation through $H^2(\mathfrak{g}, \mathfrak{z})$.

The interesting property arising from the integration map is that it permits us to integrate the adjoint action $\mathrm{ad}_{\hat{\mathfrak{g}}} : \mathfrak{g} \times \hat{\mathfrak{g}} \to \hat{\mathfrak{g}}$, $\big(x', (z, x)\big) \mapsto \big(z, [x', x]\big)$ to an action of G on $\hat{\mathfrak{g}}$, i.e. there exist a map $\mathrm{Ad}_{\hat{\mathfrak{g}}}$ making the diagram

$$\begin{array}{ccc} G & \xrightarrow{\mathrm{Ad}_{\hat{\mathfrak{g}}}} & \mathrm{Aut}(\hat{\mathfrak{g}}) \\ \downarrow & & \downarrow \\ \mathfrak{g} & \xrightarrow{\mathrm{ad}_{\hat{\mathfrak{g}}}} & \mathrm{der}(\hat{\mathfrak{g}}) \end{array}$$

commutative. That is why this map is called integration map.

Proposition 4.2.6 *If G is a connected Lie group, $\mathfrak{z}$ a s.c.l.c. space and $\omega \in Z^2(\mathfrak{g}, \mathfrak{z})$, then the adjoint action $\mathrm{ad}_{\hat{\mathfrak{g}}}$ of $\mathfrak{g}$ on $\hat{\mathfrak{g}} \cong \mathfrak{z} \oplus_\omega \mathfrak{g}$ integrates to a smooth action $\mathrm{Ad}_{\hat{\mathfrak{g}}}$ of G on $\hat{\mathfrak{g}}$ if and only if $P_2\big([\omega]\big) \in \mathrm{Hom}\big(\pi_1(G), \mathrm{Lin}(\mathfrak{g}, \mathfrak{z})\big)$ vanishes.*

Proof: [Nee02, Proposition 7.6]

□

Now we have the full information concerning the integrability of the central extension.

Theorem 4.2.7 *If G is a connected Lie group with Lie algebra $\mathfrak{g}$, $\mathfrak{z}$ a s.c.l.c. space and $\omega \in Z^2(\mathfrak{g}, \mathfrak{z})$, then there exists a central extension of Lie groups $Z \hookrightarrow \hat{G} \stackrel{q}{\twoheadrightarrow} G$ such that $Z \cong \mathfrak{z}/\Gamma$ for a discrete subgroup and that $\hat{\mathfrak{g}} \cong \mathfrak{z} \oplus_\omega \mathfrak{g}$ if and only if $\Pi_\omega \subseteq \mathfrak{z}$ is discrete and $P_2\big([\omega]\big) \in \operatorname{Hom}\big(\pi_1(G), \operatorname{Lin}(\mathfrak{g}, \mathfrak{z})\big)$ vanishes.*

Proof: [Nee02, Lemma 7.11]

□

Chapter 5

Central Extensions of Gau($\mathcal{P}$)

5.1 Results for the Current Group $C^\infty(M,K)$

In this section we present some results of [NM03] for current groups $C^\infty(M,K)$ for compact smooth manifolds M and Lie groups K. These are the results we want to generalise to gauge groups Gau($\mathcal{P}$) for K-bundles $\mathcal{P}=(K,M,P,\pi)$ with connection, compact base space M and locally exponential structure groups.
Assume that Y is a s.c.l.c. space and that M is a finite-dimensional manifold. For an open subset $U\subseteq\mathbb{R}^n$ we can identify a Y-valued 1-forms with an n-tuple $(f_1,\ldots,f_n)\in C^\infty(U,Y)^n$, representing the 1-form $\sum_{i=1}^n f_i dx^i$ on U. Hence we have $\Omega^1(U,Y)\cong C^\infty(U,Y)^n$ as vector spaces and we endow $\Omega^1(U,Y)$ with the topology making this isomorphism an isomorphism of topological vector spaces. For each chart $\varphi: U\subseteq M\to\varphi(U)\subseteq\mathbb{R}^n$ we consider the pull back

$$(\varphi^{-1})^*:\Omega^1(M,Y)\to\Omega^1\big(\varphi(U),Y\big)$$

and we endow $\Omega^1(M,Y)$ with the initial topology with respect to the pull backs coming from all charts of M.
Since Y is locally convex all $\Omega^1(U,Y)$ are so, such that $\Omega^1(M,Y)$ is locally convex, and since Y is sequentially complete all $\Omega^1(U,Y)$ are so, such that $\Omega^1(M,Y)$ is a s.c.l.c. space. The subspace $dC^\infty(M,Y)$ is closed because it coincides with the annihilator of the continuous linear maps

$$\lambda_\alpha:\Omega^1(M,Y)\to Y,\ \ \omega\mapsto\int_\alpha\omega$$

for $\alpha\in C^\infty(\mathbb{S}^1,M)$ (note that the sequentially completeness is essential for the existence of the integral). Hence

$$\mathfrak{z}_M(Y):=\Omega^1(M,Y)/dC^\infty(M,Y)$$

is also a s.c.l.c. space.

Lemma 5.1.1 *If $\mathfrak{z}$ and λ_α are chosen as above, then the maps*

$$\lambda_\alpha : \mathfrak{z} \to Y, \ \ [\omega] \mapsto \int_\alpha \omega,$$

where α runs through $C^\infty(\mathbb{S}^1, M)$, separate the points of $\mathfrak{z}$. If $M = \mathbb{S}^1$ and $\alpha = \mathrm{id}_{\mathbb{S}^1}$, then $\lambda_{\mathbb{S}^1} := \lambda_\alpha : \mathfrak{z} \to Y$ is an isomorphism.

Proof: Assume that $\lambda_\alpha([\omega]) = \lambda_\alpha([\omega'])$ for $\omega, \omega' \in \Omega^1(M,Y)$ and all $\alpha \in C^\infty(\mathbb{S}^1, M)$. Then $\lambda_\alpha([\omega]) = \lambda_\alpha([\omega'])$ also holds for all $\alpha \in C(\mathbb{S}^1, M)$, since $C^\infty(\mathbb{S}^1, M)$ is dense in $C(\mathbb{S}^1, M)$. Since $\lambda_\alpha([\omega]) = \lambda_\alpha([\omega']) \Leftrightarrow \int_\alpha \omega - \omega' = 0$, the function

$$f : M \to Y \ \ x \mapsto \int_\gamma \omega - \omega',$$

where $\gamma \in C^\infty([0,1], M)$ is a path from x_0 to x, is well-defined. The Fundamental Theorem of Calculus now yields $df = \omega - \omega'$ hence $\omega - \omega = 0 \bmod dC^\infty(M,Y)$. Since the kernel of $\lambda_{\mathbb{S}^1}$ is exactly $dC^\infty(\mathbb{S}^1)$, the map is injective. For $y \in Y$ consider the 1-form $\omega(\partial_t) = y$ with $\lambda_{\mathbb{S}^1}(\omega) = \int_{\mathbb{S}^1} \omega = y$. Hence $\lambda_{\mathbb{S}^1}$ is surjective.

□

Remark 5.1.2 A 1-form $\beta \in \Omega^1(M,Y)$ is closed if and only if for all pairs of homotopic paths α_1, α_2 the integrals of β over α_1 and α_2 coincide. Therefore the subspace $\mathrm{H}^1_{\mathrm{dR}}(M,Y) \subseteq \mathfrak{z}_M(Y)$ is the annihilator of the functionals $\lambda_{\alpha_1} - \lambda_{\alpha_2}$ for $[\alpha_1] = [\alpha_2] \in \pi_1(M)$ such that it is in particular closed. Hence $[\beta] \in \mathrm{H}^1_{\mathrm{dR}}(M,Y)$ is equivalent to the independence of $\lambda_\alpha\big([\beta]\big)$ from the homotopy class of α. Denote by k the rank of the finitely generated free abelian group

$$H_1(M)/\mathrm{tor}\big(H_1(M)\big)$$

and consider a basis given by the smooth representatives $[\alpha_1], \dots [\alpha_k]$. Then there exist smooth maps $f_i : M \to \mathbb{S}^1$ such that $\deg\big([f_i \circ \alpha_j]\big) = \delta_{ij}$. If $\delta^l(f_i) : M \to T_e\mathbb{S}^1 \cong \mathbb{R}$ denotes the left logarithmic derivative of f_i, then

$$\Phi : \mathrm{H}_{\mathrm{dR}}(M,Y) \to Y^k, \ \ [\beta] \mapsto \Big(\int_{\alpha_j} \beta\Big)_{j \in I_k}$$

is an isomorphism of topological vector spaces with continuous inverse given by

$$\Phi^{-1} : Y^k \to \mathrm{H}^1_{\mathrm{dR}}(M,Y), \ \ (y_1, \dots, y_k) \mapsto \big[\sum_{i=1}^k \delta^l(f_i \cdot y_i)\big].$$

The details and references are contained in [NM03, Remark I.3].

Lemma 5.1.3 *If $\kappa : \mathfrak{k}\times\mathfrak{k}\to Y$ is a symmetric invariant bilinear map, then*

$$\omega_{M,\kappa} : C^\infty(M,\mathfrak{k})\times C^\infty(M,\mathfrak{k})\to \mathfrak{z}_M(Y),\quad (\xi,\eta)\mapsto \big[X\mapsto \kappa(\xi, d\eta.X)\big],$$

where $\kappa(\xi, d\eta.X)$ denotes the smooth function $x\mapsto\kappa\big(\xi(x),(d\eta.X)(x)\big)$ defines a $\mathfrak{z}_M(Y)$-valued cocycle.

Proof: The cocycle condition is

$$\kappa\big([\eta,\nu], d\xi.X\big)+\kappa\big([\nu,\xi], d\eta.X\big)+\kappa\big([\xi,\eta], d\nu.X\big)\equiv 0 \bmod dC^\infty(M,Y)$$

for all $\xi,\eta,\nu\in C^\infty(M,\mathfrak{k})$. Since κ and $[\cdot,\cdot]$ are bilinear we have

$$d\kappa\big([\xi,\eta],\nu\big).X=\kappa\big(d[\xi,\eta].X,\nu\big)+\kappa\big([\xi,\eta],d\nu.X\big)= \\ \kappa\big([d\xi.X,\eta],\nu\big)+\kappa\big([\xi,d\eta.X],\nu\big)+\kappa\big([\xi,\eta],d\nu.X\big)$$

and due to the invariance and symmetry

$$d\kappa\big([\xi,\eta],\nu\big)=\kappa\big([\eta,\nu],d\xi.X\big)+\kappa\big([\nu,\xi],d\eta.X\big)+\kappa\big([\xi,\eta],d\nu.X\big).$$

□

Theorem 5.1.4 (**Reduction Theorem**) *The period group $\Pi_{\omega_{M,\kappa}}=\mathrm{im}(\mathrm{per}_{\omega_{M,\kappa}})$ is contained in the subspace $H^1_{\mathrm{dR}}(M,Y)\subseteq\mathfrak{z}_M(Y)$. Identifying it with Y^k as in the preceding remark, we have*

$$\Pi_{\omega_{M,\kappa}}\cong\Pi^k_{\omega_{\mathbb{S}^1,\kappa}}$$

and in particular $\Pi_{\omega_{M,\kappa}}$ is discrete if and only if this is the case for $M=\mathbb{S}^1$.

Proof: [NM03, Theorem I.6]

□

Definition. Universal Invariant Symmetric Bilinear Form: If K is a finite-dimensional Lie group with Lie algebra $\mathfrak{k}$ then we denote by $V(\mathfrak{k})$ the quotient $S(\mathfrak{k})/\mathfrak{k}.S(\mathfrak{k})$, where $S(\mathfrak{k})$ is the universal symmetric product where $\mathfrak{k}$ acts via $x.(y\vee z)\mapsto[x,y]\vee z+y\vee[x,z]$. Then $\kappa:\mathfrak{k}\times\mathfrak{k}$, $(x,y)\mapsto[x\vee y]$ is the *universal invariant symmetric bilinear form* on $\mathfrak{k}$. It has the universal property that each invariant symmetric bilinear form $f:\mathfrak{k}\times\mathfrak{k}\to Y$ factors through a unique linear map $\tilde{f}:V(\mathfrak{k})\to Y$ satisfying $f=\tilde{f}\circ\kappa$.

Theorem 5.1.5 *If K is a finite-dimensional Lie group and $\kappa:\mathfrak{k}\times\mathfrak{k}\to V(\mathfrak{k})$ is the universal invariant symmetric bilinear form on $\mathfrak{k}$, then the $\mathfrak{z}_{\mathbb{S}^1}\big(V(\mathfrak{k})\big)$-valued cocycle $\omega_{\mathbb{S}^1,\kappa}$ has discrete image.*

Proof: [NM03, Theorem II.9]

□

5.2 Twisted Loop Groups

As we have seen in the previous section, bundles over $\mathbb{S}^1$ are supposed to become important in the analysis of central extensions of $\mathrm{Gau}(\mathcal{P})$. Hence we will study their gauge groups, twisted loop groups, here and especially their homotopical properties. Since the argumentation is easier when dealing with continuous maps we will first stick to continuous twisted loop groups.

Definition. Path Group, Loop Group: If K is a topological group then the group

$$PK := \left\{\tau \in C([0,1],K) : \tau(0) = e\right\}$$

is called the (continuous) *path group* of K and the subgroup

$$\Omega K := \{\tau \in PK : \tau(0) = \tau(1)\}$$

is called the (continuous) *loop group* of K.

Remark 5.2.1 It is immediate from the definition that ΩK actually is a normal subgroup of PK. Endowed with the compact open topology $C(\mathbb{R},K)$ is a topological group, PK and ΩK are closed normal subgroups and the first homotopy group $\pi_1(K) \cong \pi_0(\Omega K)$ is isomorphic to $PK/\Omega K$.

Definition. Twisted Loop Group: If K is a topological group and $k \in K$, then

$$C_k(\mathbb{R},K) := \left\{\gamma \in C(\mathbb{R},K) : \gamma(x+n) = k^{-n}\gamma(x)k^n\right\}$$

is called (continuous) *twisted loop group* of k. If K carries in addition the structure of a Lie group, then

$$C_k^\infty(\mathbb{R},K) := \left\{\gamma \in C^\infty(\mathbb{R},K) : \gamma(x+n) = k^{-n}\gamma(x)k^n\right\}$$

is called (smooth) twisted loop group of k.

Remark 5.2.2 The continuous twisted loop group $C_k(\mathbb{R},K)$ is a topological group as a closed subgroup of $C(\mathbb{R},K)$. Note that in general functions in $C_k(\mathbb{R},K)$ do not factor through continuous functions on $\mathbb{S}^1$. If K is a finite-dimensional Lie group, $k \in K_0$ and $\mathcal{P} = (K,\mathbb{S}^1,P_k,\pi)$ is the K-bundle determined by k, then $C_k(\mathbb{R},K) \cong \mathrm{Gau}_{\mathrm{cont}}(\mathcal{P}_k)$ and $C_k(\mathbb{R},K)$ can be given the structure of a Lie group as well as $C_k^\infty(\mathbb{R},K) \cong \mathrm{Gau}(\mathcal{P}_k)$. Then we have for their Lie algebras

$$L\big(C_k(\mathbb{R},K)\big) = \{\xi \in C(\mathbb{R},\mathfrak{g}) : \xi(x+n) = \mathrm{Ad}(k^n).\xi(x)\}$$
$$L\big(C_k^\infty(\mathbb{R},K)\big) = \{\xi \in C^\infty(\mathbb{R},\mathfrak{g}) : \xi(x+n) = \mathrm{Ad}(k^n).\xi(x)\}.$$

Proposition 5.2.3 *If K is a topological group and $k \in K_0$, then the continuous twisted loop group*

$$C_k(\mathbb{R}, K) := \big\{\gamma \in C(\mathbb{R}, K) : \gamma(x+n) = k^{-n}\gamma(x)k^n \ \text{ for all } \ x \in \mathbb{R}\big\}$$

is isomorphic as a topological group to a semi direct product $\Omega K \rtimes_k K$.

Proof: Consider a curve $\tau : [0,1] \to K$ such that $\tau(0) = e$, $\tau(1) = k$ and define $\tau_k(x) := \tau(x-n)k^n$ where n is the unique integer such that $x - n \in [0,1)$. This defines a continuous curve $\tau_k : \mathbb{R} \to K$ such that $\tau_k(x+n) = \tau_k(x)k^n$ for all $x \in \mathbb{R}$. The map $\sigma_k : K \to C_k(\mathbb{R}, K)$, $\sigma_k(k')(x) = \tau_k^{-1}(x)\, k'\, \tau_k(x)$ defines a homomorphism of topological groups since it can be considered as conjugation of the constant map $k \in C(\mathbb{R}, K)$ with $\tau_k \in C(\mathbb{R}, K)$. In addition, we have $\mathrm{ev}_0 \circ \sigma = \mathrm{id}_K$ such that the exact sequence

$$\ker(\mathrm{ev}_0) \hookrightarrow C_k(\mathbb{R}, K) \overset{\mathrm{ev}_0}{\twoheadrightarrow} K$$

splits. Hence $C_k(\mathbb{R}, K)$ is isomorphic to the semi direct product $\Omega K \rtimes K$ since ev_0 is continuous and $\ker(\mathrm{ev}_0) \cong \Omega K$.

$\square$

Remark 5.2.4 The isomorphism is given by

$$\alpha : \Omega K \rtimes_k K \to C_k(\mathbb{R}, K), \quad (\gamma, k') \mapsto \gamma \cdot \sigma_k(k'),$$

where $\sigma_k : K \to C_k(\mathbb{R}, K)$ is the homomorphism from the previous proof. The splitting homomorphism $\sigma_k : K \to C_k(\mathbb{R}, K)$ provides the homomorphism

$$\delta_k : K \to \mathrm{Aut}(\Omega K), \quad \delta_k(h)(\gamma) = \sigma_k(h)^{-1}\, \gamma\, \sigma_k(h)$$

describing the semi-direct product by the multiplication on $\Omega K \times K$

$$(\gamma, h), (\gamma', h') \mapsto \big(\gamma\, \delta_k(h)(\gamma'), hh'\big).$$

Corollary 5.2.5 *If K is a topological group and $k \in K_0$, then*

$$\pi_n\big(C_k(\mathbb{R}, K)\big) \cong \pi_n(\Omega K \rtimes K) \cong \pi_n(\Omega K) \times \pi_n(K).$$

If K is a finite-dimensional Lie group and τ is smooth, the above considerations also lead to an isomorphism

$$\alpha : C^\infty_*(\mathbb{S}^1, K) \rtimes K \to C^\infty_k(\mathbb{R}, K).$$

Lemma 5.2.6 *If K is a finite-dimensional Lie group then for each $k \in K_0$ there exists a smooth curve $\tau_k : \mathbb{R} \to K$ such that $\tau_k(x+n) = \tau(x)k^n$.*

Proof: Choose a smooth curve $\tau' : [0,1] \to K$ such that $\tau'(0) = e$ and $\tau'(1) = k^{-1}$. Then consider a smooth bijective map $\lambda : [0,1] \to [0,1]$ such that $\lambda(0) = 0$, $\lambda(1) = 1$ and $\frac{d^n}{dt^n}\lambda(0) = \frac{d^n}{dt^n}\lambda(1) = 0$ for all $n \in \mathbb{N}$. Then $\tau' \circ \lambda : [0,1] \to K$ is smooth and

$$\tau : \mathbb{R} \to K, \quad x \mapsto \tau'\big(\lambda(x-n)\big)k^n,$$

where n is the unique integer such that $x - n \in [0,1)$ is a smooth curve satisfying $\tau(x+n) = \tau(x)k^n$.

□

Corollary 5.2.7 *If K is a finite-dimensional Lie group and $k \in K_0$, then the twisted loop group $C^\infty_k(\mathbb{R}, K)$ is isomorphic to the semi-direct product $C^\infty_*(\mathbb{S}, K) \rtimes_k K$. Furthermore we have*

$$\begin{aligned}\pi_n\big(C^\infty_k(\mathbb{R},K)\big) &\cong \pi_n\big(C^\infty_*(\mathbb{S},K) \rtimes_k K\big) \\ &\cong \pi_n\big(C^\infty_*(\mathbb{S}^1,K)\big) \times \pi_n(K) \cong \pi_{n+1}(K) \times \pi_n(K).\end{aligned}$$

In particular we have $\pi_1(C^\infty_k(\mathbb{R},K)) \cong \pi_1(K)$ and $\pi_2(C^\infty_k(\mathbb{R},K)) = \pi_3(K)$ since $\pi_2(K)$ is trivial for finite-dimensional Lie Groups.

Proof: With the preceeding lemma the arguments from the continuous case can be copied to obtain a split exact sequence of topological groups

$$\ker(\mathrm{ev}_0) \hookrightarrow C^\infty_k(\mathbb{R},K) \overset{\mathrm{ev}_0}{\twoheadrightarrow} K.$$

It remains to check that $\sigma : K \to C^\infty_k(\mathbb{R},K)$, $k \mapsto \tau_k k \tau_k^{-1}$ is smooth. The topology on $C^\infty_k(\mathbb{R},K)$ is the one induced from the projections $\pi_i : C^\infty_k(\mathbb{R},K) \to C^\infty(U_i,K)$ for $U_i = [\frac{i}{2} - \varepsilon, \frac{i+1}{2} + \varepsilon]$ for $i = 0,1$ and $0 < \varepsilon < \frac{1}{2}$. Clearly σ is smooth if it is so in every component $C^\infty(U_i,K)$, where it is conjugation of constant maps with $\tau_k\big|_{U_i} \in C^\infty(U_i,K)$ and thus smooth.

□

Remark 5.2.8 As in the continuous case, we obtain the isomorphism

$$\alpha : C^\infty_*(\mathbb{S}^1,K) \rtimes_k K \to C^\infty_k(\mathbb{R},K), \quad (\gamma,k') \mapsto \gamma\sigma_k(k').$$

If $\mathcal{P}_k$ is a finite-dimensional K-bundle over $\mathbb{S}^1$, we can use this isomorphism to construct a cocycle on $\mathfrak{gau}(\mathcal{P}) \cong L\big(C^\infty_k(\mathbb{R},K)\big)$ induced by the cocycle

$$\omega_k(\xi,\eta) := \big[\kappa(\xi, d\eta)\big] \tag{5.1}$$

on $L\big(C^\infty(\mathbb{S}^1,K)\big)$.

Proposition 5.2.9 *If $\mathcal{P}_k = (K, \mathbb{S}^1, P_k, \pi)$ is a finite-dimensional K-bundle over $\mathbb{S}^1$, then the map*

$$\tilde{\omega}_\kappa : \mathfrak{gau}(\mathcal{P}_k) \times \mathfrak{gau}(\mathcal{P}_k) \to \mathfrak{z}_{\mathbb{S}^1}(Y), \quad (\xi, \eta) \mapsto \Big[\kappa\big(\xi, d\eta\big)\Big] \tag{5.2}$$

is a $\mathfrak{z}_{\mathbb{S}^1}(Y)$-valued cocycle and

$$\mathrm{per}_{\omega_\kappa} = \mathrm{per}_{\tilde{\omega}_\kappa}.$$

If $Y = V(\mathfrak{k})$ and $\kappa : \mathfrak{k} \times \mathfrak{k} \to V(\mathfrak{k})$ is universal, then and the image of the period map of $\pi_2\big(\mathrm{Gau}(\mathcal{P})_0\big)$ in $\mathfrak{z}_{\mathbb{S}^1}\big(V(\mathfrak{k})\big)$ is discrete.

Proof: Note that $k \in K_0$ (cf. Proposition 2.5.2). If $X \in \mathcal{V}(\mathbb{S}^1)$ is considered as a periodic vector field on $\mathbb{R}$ then we have for $\eta \in \mathfrak{gau}(\mathcal{P}_k) = L\big(C^\infty_k(\mathbb{R}, K)\big)$ that $d\eta(x+n).X(x+n) = \mathrm{Ad}(k^n).d\eta(x).X(x)$. Since κ is invariant $\kappa(\xi, d\eta.X)$ is periodic and thus factors through a smooth function $\kappa(\xi, d\eta.X)_M \in C^\infty(\mathbb{S}^1, K)$. The cocycle condition is calculated as in Lemma 5.1.3.
We pull back the cocycle ω on $C^\infty(\mathbb{S}, \mathfrak{k})$ with the diffeomorphism

$$C^\infty_k(\mathbb{R}, K) \xrightarrow{\alpha^{-1}} C^\infty_*(\mathbb{S}^1, K) \times K \cong C^\infty(\mathbb{S}^1, K)$$

to a cocycle on $C^\infty_k(\mathbb{R}, \mathfrak{k})$ and observe that this cocycle coincides with $\tilde{\omega}$ on $C^\infty_*(\mathbb{S}, \mathfrak{k})$. Since $\pi_2(K)$ is trivial for finite-dimensional K this yields $\mathrm{per}_{\omega_\kappa} = \mathrm{per}_{\tilde{\omega}_\kappa}$. The discreetness of $\mathrm{im}(\mathrm{per}_{\tilde{\omega}_\kappa})$ follows now from [NM03, Theorem II.8].

□

5.3 The Covariant Cocycle $\tilde{\omega}_\kappa$

We will now extend the idea from the preceding section in order to get a better understanding of the situation for arbitrary K-bundles. Hence we need an alternative formulation of the cocycle (5.2) somehow implementing the equivariance principle such that is becomes extendable to the situation of K-bundles with arbitrary compact base spaces.

Lemma 5.3.1 *If $\mathcal{P} = (K, M, P, \pi)$ is a K-bundle, $X \in \mathcal{V}(P)^K$ and $\xi \in C^\infty(P, \mathfrak{k})^K$, then $X.\xi := d\xi.X \in C^\infty(P, \mathfrak{k})^K$.*

Proof: We have

$$\begin{aligned} \xi \in C^\infty(P, \mathfrak{k})^k \quad &\Leftrightarrow \quad \mathrm{Ad}(k).\xi(p) = \xi(p \cdot k), \\ X \in \mathcal{V}(P)^k \quad &\Leftrightarrow \quad d\rho_k(p).X(p) = X(p \cdot k). \end{aligned}$$

Since $\mathrm{Ad}(k) : \mathfrak{k} \to \mathfrak{k}$ is linear this yields

$$(d\xi.X)(p \cdot k) = d\xi(p \cdot k).d\rho_k(p).X(p) = d\big(\xi \circ \rho_k\big)(p).X(p) = \\ d\big(\mathrm{Ad}(k) \circ \xi\big)(p).X(p) = \mathrm{Ad}(k).\big(d\xi(p).X(p)\big) = \mathrm{Ad}(k).(d\xi.X)(p),$$

and hence $d\xi.X \in C^\infty(P, \mathfrak{k})^K$.

□

Definition. Covariant Derivative: If $\mathcal{P} = (M, K, P, \pi)$ is a K-bundle with connection given by $\sigma : \mathcal{V}(M) \to \mathcal{V}(P)^K$, then for each $X \in \mathcal{V}(M)$ the function

$$\nabla_X^\sigma : C^\infty(P, \mathfrak{k})^K \to C^\infty(P, \mathfrak{k})^K, \;\; \xi \mapsto d\xi.\sigma(X) = \sigma(X).\xi$$

is called the *covariant derivative* of X.

Lemma 5.3.2 *If $\mathcal{P} = (K, M, P, \pi)$ is a K-bundle, $\xi, \eta \in C^\infty(P, \mathfrak{k})^K$ and $\kappa : \mathfrak{k} \times \mathfrak{k} \to Y$ is an invariant symmetric bilinear map, then the map $\kappa(\xi, \eta) : P \to Y$, $p \mapsto \kappa\big(\xi(p), \eta(p)\big)$ factors through a map $\kappa(\xi, \eta)_M : M \to Y$.*

Proof: Since $\xi, \eta \in C^\infty(P, \mathfrak{k})^K$ and κ is invariant, $\kappa\big(\xi, \eta\big)$ is constant on each fibre and hence factors through a map from M to Y.

□

Remark 5.3.3 The map $\kappa(\xi, \eta)_M$ can be obtained as

$$\kappa(\xi, \eta)_M : M \to Y, \;\; x \mapsto \kappa\Big(\xi\big(\Theta_x^{-1}(x, e)\big), \eta\big(\Theta_x^{-1}(x, e)\big)\Big)$$

for a local trivialisation $\Theta_x : \pi^{-1}(U_x) \to U_x \times K$ of a trivialising neighbourhood U_x of x.

From now on we identify the gauge algebra $\mathfrak{gau}(\mathcal{P})$ with $C^\infty(P, \mathfrak{k})^K$ (cf. Remark 3.3.14)

Lemma 5.3.4 *Let $\mathcal{P} = (K, M, P, \pi)$ be a K-bundle with connection given by $\sigma : \mathcal{V}(M) \to \mathcal{V}(P)^K$, compact base space M and locally exponential structure group K. If Y is a s.c.l.c. space and $\kappa : \mathfrak{k} \times \mathfrak{k} \to Y$ is an invariant symmetric bilinear map, then the mapping*

$$\mathfrak{gau}(\mathcal{P}) \times \mathfrak{gau}(\mathcal{P}) \to \Omega^1(M, Y), \;\; (\xi, \eta) \mapsto \Big(X \mapsto \kappa\big(\xi, \nabla_X^\sigma(\eta)\big)_M\Big),$$

represents a $\mathfrak{z}_M(Y)$-valued cocycle by composing it with the quotient map

$$q : \Omega^1(M, Y) \to \mathfrak{z}_M(Y) := \Omega^1(M, Y)/dC^\infty(M, Y).$$

Proof: First we note that $\xi, \nabla^\sigma_X(\eta) \in \mathfrak{gau}(\mathcal{P})$ implies that $\kappa\big(\xi, \nabla^\sigma_X(\eta)\big)$ factors through a smooth map $\kappa\big(\xi, \nabla^\sigma_X(\eta)\big)_M$ on M. The cocycle condition is

$$\kappa\big([\xi,\eta], \nabla^\sigma_X(\nu)\big)_M + \kappa\big([\eta,\nu], \nabla^\sigma_X(\xi)\big)_M + \kappa\big([\nu,\xi], \nabla^\sigma_X(\eta)\big)_M \equiv 0 \mod dC^\infty(M,Y)$$

for each vector field $X \in \mathcal{V}(M)$. Since $\kappa(\xi,\eta)_M$ is the factorisation of $\kappa(\xi,\eta)$ and $d\pi\big(\sigma(X)\big) = X$, we know that

$$d\kappa(\xi,\eta)_M.X = \big(d\kappa(\xi,\eta).\sigma(X)\big)_M.$$

With the product rule

$$\begin{aligned} d\kappa(\xi,\eta)_M.X = \big(d\kappa(\xi,\eta).\sigma(X)\big)_M &= \Big(\kappa\big(d\xi.\sigma(X),\eta\big) + \kappa\big(\xi, d\eta.\sigma(X)\big)\Big)_M \\ &= \kappa\big(\nabla^\sigma_X(\xi),\eta\big)_M + \kappa\big(\xi,\nabla^\sigma_X(\eta)\big)_M \end{aligned}$$

we perform the same calculation as in Lemma 5.1.3 to verify the cocycle condition.

□

Definition. Covariant Cocycle: The cocycle constructed in the preceding lemma is called the *covariant cocycle* of the K-bundle $\mathcal{P}$ with connection σ and invariant form κ. It will be denoted by

$$\tilde{\omega}_{\kappa,\sigma} : \mathfrak{gau}(\mathcal{P}) \times \mathfrak{gau}(\mathcal{P}) \to \mathfrak{z}_M(Y), \quad (\xi,\eta) \mapsto \Big[X \mapsto \kappa\big(\xi, \nabla^\sigma_X(\eta)\big)_M\Big].$$

Remark 5.3.5 Note that in the case $M = \mathbb{S}^1$ the covariant cocycle coincides with (5.2) section since the horizontal lift of a vector field on $\mathbb{S}^1$ to a horizontal vector field on $\mathcal{P}_k$ corresponds to a periodic vector field on $\mathbb{R}$.

Lemma 5.3.6 *If $\mathcal{P} = (K, M, P, \pi)$ is a K-bundle with connection, compact base space M and locally exponential structure group K, then the covariant cocycle $\tilde{\omega}_{\kappa,\sigma}$ is continuous.*

Proof: Since $\tilde{\omega}_{k,\sigma}$ is the composition of continuous maps which can be seen in local trivialisations it is continuous.

□

Lemma 5.3.7 *If $\mathcal{P} = (K, M, P, \pi)$ is a K-bundle with connection and compact base space M, then for each pair of connections $\sigma, \sigma' : \mathcal{V}(M) \to \mathcal{V}(P)^K$ the difference $\tilde{\omega}_{\kappa,\sigma} - \tilde{\omega}_{\kappa,\sigma'}$ is a continuous coboundary, i.e. there exists a continuous linear map $\lambda : \mathfrak{gau}(\mathcal{P}) \to \mathfrak{z}$, such that $\tilde{\omega}_{\kappa,\sigma}(\xi,\eta) - \tilde{\omega}_{\kappa,\sigma'}(\xi,\eta) = \lambda\big([\eta,\xi]\big)$*

Proof: Since the horizontal component of $\sigma(X)$ coincides with the horizontal component of $\sigma'(X)$, the vector field $\sigma(X)-\sigma'(X)$ is vertical and K-invariant, hence represents an element $\nu_X \in \mathfrak{gau}(\mathcal{P}) \cong C^\infty(P,\mathfrak{k})^K$. Under this identification the action of the vector field $\sigma(X)-\sigma'(X)$ on $C^\infty(P,\mathfrak{k})^K$ changes to the pointwise adjoint action of ν_X (cf. Lemma 2.3.6). Hence we get

$$\kappa\big(\xi,\nabla^\sigma_X(\eta)\big)-\kappa\big(\xi,\nabla^{\sigma'}_X(\eta)\big)=\kappa\Big(\xi,\big(\sigma(X)-\sigma'(X)\big).\eta\Big)$$
$$=\kappa(\xi,\nu_X.\eta)=\kappa\big(\xi,[\nu_X,\eta]\big)=\kappa([\eta,\xi],\nu_X)$$

and thus $\tilde{\omega}_{\kappa,\sigma}(\xi,\eta)-\tilde{\omega}_{\kappa,\sigma'}(\xi,\eta)=\Big[X\mapsto\kappa\big([\eta,\xi],\nu_X\big)_M\Big]=\lambda\big([\eta,\xi]\big)$ for

$$\lambda:\mathfrak{gau}(\mathcal{P})\to\mathfrak{z}\;\;\xi'\mapsto\big[X\mapsto\kappa(\xi',\nu_x)_M\big].$$

□

Remark 5.3.8 Since for central extensions we are only interested in cocycles modulo coboundaries, we will suppress the dependence $[\tilde{\omega}_{\kappa,\sigma}]\in H^2(\mathfrak{gau}(\mathcal{P}),\mathfrak{z})$ on the connection σ when dealing with central extensions. Note that the behaviour of $[\tilde{\omega}_{\kappa,\sigma}]$ is not totally independent form the geometry of $\mathcal{P}$ since the previous proposition says that the sole property of the geometry on $\mathcal{P}$ that influences $[\tilde{\omega}_{\kappa,\sigma}]$ is which connections can occur.

5.4 Reduction to Bundles over $\mathbb{S}^1$

Throughout this section we will consider the period map for the covariant cocycle $\tilde{\omega}_{\kappa,\sigma}$ for a K-bundle with connection σ, compact base space and locally exponential Lie group K, defined on $\pi_2\big(\mathrm{Gau}(\mathcal{P})_0\big)$.

Lemma 5.4.1 *If $\mathcal{P}:=(K,M,P,\pi)$ is a K-bundle, $f:N\to M$ is smooth, $f^*(\mathcal{P})=(K,N,Q,\mathrm{pr}_2)$ is the pull back of $\mathcal{P}$ and $f_{\mathcal{P}}:Q\to P$ the induced homomorphism of K-bundles, then*

$$f_{\mathrm{Gau}}:\mathrm{Gau}(\mathcal{P})\to\mathrm{Gau}\big(f^*\big(\mathcal{P}\big)\big),\;\;\gamma\mapsto\gamma\circ f_{\mathcal{P}},$$

where $\mathrm{Gau}(\mathcal{P})\cong C^\infty(P,K)^K$ *and* $\mathrm{Gau}(f^*\mathcal{P})\cong C^\infty(Q,K)^K$ *is a Lie group homomorphism.*

Proof: First we observe that for $(p,n)\in Q=\{(p',n')\in P\times Q:\pi(p')=f(q')\}$ we have

$$k^{-1}\gamma\big(f_{\mathcal{P}}(p,n)\big)k=k^{-1}\gamma(p)k=\gamma(p\cdot k)=\gamma\Big(f_{\mathcal{P}}\big((p,n)\cdot k\big)\Big)$$

since $f_{\mathcal{P}}(p,n) = f(p)$ and hence $\gamma \circ f_{\mathcal{P}} \in \operatorname{Gau}\big(f^*(\mathcal{P})\big)$. Obviously f_{Gau} is a homomorphism of groups and hence it remains to check whether f_{Gau} is smooth. The local description of f_{Gau} is

$$f_{\text{Gau}}\big((\gamma_i)_{i\in I_n}\big) = (\gamma_i \circ f_i)_{i\in I_n},$$

where $f_i := f\big|_{f^{-1}(\overline{V}_i)}$ and hence componentwise f_{Gau} is the map

$$(f_i)_K : C^\infty(\overline{V}_i, K) \to C^\infty\big(f^{-1}(\overline{V}_i), K\big), \quad \gamma \mapsto \gamma \circ f_i.$$

These maps are smooth due to Corollary 3.2.9 and hence f_{Gau} is smooth.

□

Remark 5.4.2 The preceding proof shows that the Lie algebra homomorphism $df_{\text{Gau}}(e)$ induced by f_{Gau} is given by

$$df_{\text{Gau}}(e).(\xi_i)_{i=1,\ldots,n} = \big((df_i)_K(e).\xi_i\big)_{i=1,\ldots,n} = (\xi_i \circ f_i)_{i=1,\ldots,n}$$

if we consider $\mathfrak{gau}(\mathcal{P}) \cong \mathfrak{g}(\mathcal{P}) \subseteq \oplus_{i=1}^n C^\infty(\overline{V}_i, \mathfrak{k})$ in the local picture. In the global picture, where $\mathfrak{gau}(\mathcal{P}) \cong C^\infty(P, \mathfrak{k})^K$, this leads to

$$df_{\text{Gau}}(e) : \mathfrak{gau}(\mathcal{P}) \to \mathfrak{gau}\big(f^*(\mathcal{P})\big), \quad \xi \mapsto \xi \circ f_{\mathcal{P}}.$$

Lemma 5.4.3 *Let $\mathcal{P} = (K, M, P, \pi)$ be a K-bundle with connection σ, compact base space M and locally exponential structure group K, $\alpha \in C^\infty(\mathbb{S}^1, M)$ such that $\operatorname{im}(\alpha) \subseteq \operatorname{int}(M)$ and $\alpha_{\text{Gau}} : \operatorname{Gau}(\mathcal{P}) \to \operatorname{Gau}\big(\alpha^*(\mathcal{P})\big)$. If Y is a s.c.l.c. space and $\kappa : \mathfrak{k} \times \mathfrak{k} \to Y$ a symmetric invariant bilinear form, then*

$$\lambda_\alpha \circ \operatorname{per}_{\tilde{\omega}_{\sigma,M}} = \lambda_{\mathbb{S}^1} \circ \operatorname{per}_{\omega_{\mathbb{S}^1}} \circ \pi_2(\alpha_{\text{Gau}}) \tag{5.3}$$

holds, where $\pi_2(\alpha_{\text{Gau}})$ is the homomorphism on the second homotopy groups induced by α_{Gau}, $\tilde{\omega}_{\sigma,M}$ is the covariant cocycle on $\mathfrak{gau}(\mathcal{P})$ corresponding to σ and $\omega_{\mathbb{S}^1}$ is the cocycle (5.1) on $C^\infty(\mathbb{S}^1, \mathfrak{k})$.

Proof: Note that we suppressed the dependence of $\operatorname{per}_{\tilde{\omega}_{\sigma,M}}$ and $\operatorname{per}_{\omega_{\mathbb{S}^1}}$ on the bilinear form κ. We denote by Ω_M the left invariant form corresponding to $\tilde{\omega}_{\sigma,M}$ and by $\Omega_{\mathbb{S}^1}$ the left invariant form corresponding to the covariant cocycle $\tilde{\omega}_{\sigma,\mathbb{S}^1} = \alpha^*(\Omega_M)(e)$. Note that this cocycle corresponds to the pull back of the connection σ to a connection on $\alpha^*(\mathcal{P})$, e.g. obtained via pulling back the connection form from P. Since Ω_M is left invariant, so is $\lambda_\alpha \circ \Omega_M$ with $(\lambda_\alpha \circ \Omega_M)(e) = \lambda_\alpha \circ \tilde{\omega}_{\sigma,M}$. In addition, $\alpha_{\text{Gau}}^*(\Omega_{\mathbb{S}^1})$ is left invariant since

$$\begin{aligned} L_g^*\big(\alpha_{\text{Gau}}^*(\Omega_{\mathbb{S}^1})\big) &= (\alpha_{\text{Gau}} \circ L_g)^*(\Omega_{\mathbb{S}^1}) \\ &\overset{i)}{=} (L_{\alpha_{\text{Gau}}(g)} \circ \alpha_{\text{Gau}})^*(\Omega_{\mathbb{S}^1}) = \alpha_{\text{Gau}}^*\big(L_{\alpha_{\text{Gau}}(g)}^*(\Omega_{\mathbb{S}^1})\big) \overset{ii)}{=} \alpha_{\text{Gau}}^*(\Omega_{\mathbb{S}^1}), \end{aligned}$$

where $i)$ holds since α_{Gau} is a homomorphism and $ii)$ holds since $\Omega_{\mathbb{S}^1}$ is left invariant. Thus the $\Omega^1(\mathbb{S}^1,Y)/dC^\infty(\mathbb{S}^1,Y)$-valued 1-form $\alpha_{\mathrm{Gau}}^*(\Omega_{\mathbb{S}^1})$ is determined by its value in e which is given for $\xi,\eta\in\mathfrak{gau}(\mathcal{P})\cong C^\infty(P,\mathfrak{k})^K$ by the representative in $\Omega^1(M,Y)$

$$\Big(X\mapsto\kappa\Big(d\alpha_{\mathrm{Gau}}(e).\xi,\nabla_X\big(d\alpha_{\mathrm{Gau}}(e).\eta\big)\Big)_{\mathbb{S}^1}\Big)=\Big(X\mapsto\kappa\big(\xi\circ\alpha_{\mathcal{P}},\nabla_X(\eta\circ\alpha_{\mathcal{P}})\big)_{\mathbb{S}^1}\Big)$$

for $X\in\mathcal{V}(\mathbb{S}^1)$. Applying $\lambda_{\mathbb{S}^1}$ to the class of this 1-form yields

$$\int_{\mathbb{S}^1}\kappa\big(\xi\circ\alpha_{\mathcal{P}},\nabla_{\partial_t}(\eta\circ\alpha_{\mathcal{P}})\big)_{\mathbb{S}^1}\,dt=\int_{\alpha}\kappa\big(\xi,\nabla_{d\alpha(t).\partial_t}(\eta)\big)_M d\alpha$$

and hence $\lambda_{\mathbb{S}^1}\circ\alpha_{\mathrm{Gau}}^*(\Omega_{\mathbb{S}^1})(e)=\lambda_\alpha\circ\,\Omega_M\;(e)$. For $[\beta]\in\pi_2(\mathrm{Gau}(\mathcal{P}))$ we have thus

$$\lambda_{\mathbb{S}^1}\Big(\mathrm{per}_{\tilde{\omega}_{\mathbb{S}^1}}\Big(\pi_2\big(\alpha_{\mathrm{Gau}}([\beta])\big)\Big)\Big)=\lambda_{\mathbb{S}^1}\Big(\mathrm{per}_{\tilde{\omega}_{\mathbb{S}^1}}\big([\alpha_{\mathrm{Gau}}\circ\beta]\big)\Big)$$
$$=\lambda_{\mathbb{S}^1}\Big(\int_{\alpha_{\mathrm{Gau}}\circ\beta}\Omega_{\mathbb{S}^1}\Big)=\lambda_{\mathbb{S}^1}\Big(\int_\beta\alpha_{\mathrm{Gau}}^*\,\Omega_{\mathbb{S}^1}\Big)=\int_\beta\lambda_{\mathbb{S}^1}\circ\alpha_{\mathrm{Gau}}^*(\Omega_{\mathbb{S}^1})$$
$$=\int_\beta\lambda_\alpha\,\Omega_M=\lambda_\alpha\Big(\int_\beta\Omega_M\Big)=\lambda_\alpha\big(\mathrm{per}_{\tilde{\omega}_{\sigma,M}}([\beta])\big)$$

and since $\mathrm{per}_{\tilde{\omega}_{\sigma,\mathbb{S}^1}}=\mathrm{per}_{\omega_{\mathbb{S}^1}}$ this establishes (5.3).

□

Lemma 5.4.4 *If K is an topological group and $k:[0,1]\to K_0$ continuous, then there exist a continuous curve $S:[0,1]\to C(\mathbb{R},K)$ such that $S(t)(0)=e$*

$$S(t)(x+n)=S(t)(x)k(t^n)$$

holds for all $t\in[0,1]$ and $n\in\mathbb{N}$.

Proof: Let $\tau:[0,1]\to K$ be continuous such that $\tau(0)=e$ and $\tau(1)=k(0)$. and consider the 2-simplex $\Delta:=\big((0,0),(0,1),(1,1)\big)\subseteq\mathbb{R}^2$. Then we construct a map $\sigma:\Delta\to K$ by setting $\sigma\big((0,t)\big)=\tau(t)$, $\sigma\big((s,1)\big)=k(s)$ and requiring σ to be constant on lines perpendicular to $\big((0,0),(1,1)\big)$. Furthermore we construct $\sigma':[0,1]\times[0,1]\to K$ by setting $\sigma'(x)=\sigma(x)$ if $x\in\Delta$ and requiring σ' to be constant on lines perpendicular to $\big((1,0),(1,1)\big)$. This results in a continuous map for which $\sigma\big((s,0)\big)=e$ and $\sigma\big((s,1)\big)=k(s)$ holds. Hence

$$S':[0,1]\times\mathbb{R}\to K,\ \ (t,s)\mapsto\sigma'(t,s-n)k(s)^n,$$

where n is the unique integer such that $s-n\in[0,1)$ defines a continuous map. Since $C\big([0,1],C(\mathbb{R},K)\big)\cong C\big([0,1]\times\mathbb{R},K\big)$ [Glö03, Chapter 23] this map defines a continuous curve $S:[0,1]\to C(\mathbb{R},K)$ with the desired properties.

□

Lemma 5.4.5 *If $\mathcal{P} = (K, M, P, \pi)$ is a K-bundle with connection σ, K is finite-dimensional and $\alpha_i \in C^\infty(\mathbb{S}^1, M)$, $i = 0, 1$, are two smooth maps homotopic to each other, then the induced mappings*

$$\alpha_{i\,\mathrm{Gau}} : \mathrm{Gau}_{\mathrm{cont}}(\mathcal{P}) \to \mathrm{Gau}_{\mathrm{cont}}\big(\alpha_i^*(\mathcal{P})\big), \quad \gamma \mapsto \gamma \circ \alpha_{\mathcal{P}}$$

are homotopic, considered as maps into the topological space $\Omega K \times K$ underlying $\mathrm{Gau}_{\mathrm{cont}}\big(\alpha_i^*(\mathcal{P})\big)$.

Proof: Consider a continuous map $F : [0, 1] \to C^\infty(\mathbb{S}^1, M)$ such that $F(0) = \alpha_0$ and $F(1) = \alpha_1$. Then each K-bundle $F(t)^*(\mathcal{P})$ is described by an element $k(t) \in K_0$ determined by

$$\alpha(t)_{\mathrm{hor}}(0) \cdot k(t) = \alpha(t)_{\mathrm{hor}}(1).$$

Since α_{hor} depends continuously on t so does k such that $t \mapsto k(t)$ describes a continuous curve in K. Then the preceding lemma yields a map $S : [0, 1] \to C(\mathbb{R}, K)$ such that $S(t)(x + n) = S(t)(x)k(t)^n$ and $S(t)(0) = e$ for each $t \in [0, 1]$ and all $n \in \mathbb{N}$. Hence the homeomorphism

$$\mathrm{Gau}_{\mathrm{cont}}(F(t)^*(\mathcal{P})) \cong C_{k(t)}(\mathbb{R}, K) \to \Omega K \times K$$

is the map

$$H_t : C_{k(t)}(\mathbb{R}, K) \to \Omega K \times K, \quad \gamma \mapsto \big(S(t)\gamma S(t)^{-1}\gamma(0)^{-1}, \gamma(0)\big),$$

(cf. Proposition 5.2.3). The ΩK-component depends continuously on t since it is is a product of elements in the topological group $C(\mathbb{R}, K)$ which depend continuously on t. Hence the map

$$\tilde{F} : [0, 1] \times \mathrm{Gau}_{\mathrm{cont}}(\mathcal{P}) \to \Omega K \times K, \quad (t, \gamma) \mapsto H_t\big((\gamma \circ F(t)_{\mathrm{Gau}})\big)$$

is continuous with $\tilde{F}(0) = \alpha_{0\,\mathrm{Gau}}$ and $\tilde{F}(1) = \alpha_{1\,\mathrm{Gau}}$.

□

Corollary 5.4.6 *If $\mathcal{P} = (K, M, P, \pi)$ is a K-bundle with connection σ and compact base space M, then* $\mathrm{im}(\mathrm{per}_{\tilde{\omega}_{\kappa,\sigma}}) \subseteq H^1_{\mathrm{dR}}(M, Y)$.

Proof: Since the inclusion $\mathrm{Gau}(\mathcal{P})_0 \hookrightarrow \mathrm{Gau}_{\mathrm{cont}}(\mathcal{P})_0$ is a weak homotopy equivalence we see that $\lambda_\alpha \circ \mathrm{per}_{\tilde{\omega}_\kappa} = \lambda_{\mathbb{S}^1} \circ \mathrm{per}_{\omega_{\mathbb{S}^1}} \circ \pi_2(\alpha)$ depends only on the homotopy class of α and hence $\mathrm{per}_{\tilde{\omega}_k}\big([\beta]\big) \in H_{\mathrm{dR}^1(M,Y)}$ (cf. Remark 5.1.2).

□

Lemma 5.4.7 *If $\mathcal{P}_k = (K, \mathbb{S}^1, P_k, \pi)$ is a finite-dimensional K-bundle over $\mathbb{S}^1$ and $f \in C^\infty(\mathbb{S}^1, \mathbb{S}^1)$, then*

$$\pi_2(f_{\text{Gau}}) : \pi_2\big(\text{Gau}(\mathcal{P}_k)\big) \cong \pi_2\big(C^\infty_*(\mathbb{S}^1, K)\big) \to \pi_2\Big(\text{Gau}\big(f^*(\mathcal{P}_k)\big)\Big) \cong \pi_2\big(C^\infty_*(\mathbb{S}^1, K)\big)$$

is given by $\pi_2(f_{\text{Gau}})\big([\alpha]\big) = \deg(f) \cdot [\alpha]$.

Proof: Since $\text{incl} : \text{Gau}(\mathcal{P}_k) \to \text{Gau}_{\text{cont}}(\mathcal{P}_k)$ is a weak homotopy equivalence it suffices to consider the map

$$\pi_2(f_{\text{Gau}}) : \pi_2\big(\text{Gau}_{\text{cont}}(\mathcal{P}_k)\big) \to \pi_2\Big(\text{Gau}_{\text{cont}}\big(f^*(\mathcal{P}_k)\big)\Big), \quad \gamma \mapsto \gamma \circ f_{\text{Gau}}.$$

Due to Corollary 5.2.5 we have $\pi_2\big(\text{Gau}_{\text{cont}}(\mathcal{P}_k)\big) \cong \pi_2\big(C_*(\mathbb{S}^1, K)\big)$ such that $\pi_2(f)$ is given by $\pi_2(f)\big([\alpha]\big) = [f \circ \alpha]$ for $[\alpha] \in \pi_2\big(C_*(\mathbb{S}^1, K)\big)$ and this map equals $\deg(f)$ by [NM03, Lemma I.10].

□

We now have collected the material to come to the main result of this section.

Theorem 5.4.8 **(Reduction Theorem)** *If $\mathcal{P} = (K, M, P, \pi)$ is a finite-dimensional K-bundle, compact base space M, Y a s.c.l.c. space and $\kappa : \mathfrak{k} \times \mathfrak{k} \to Y$ a symmetric invariant bilinear map, then $\Pi_{\tilde{\omega}_{M,\kappa,\sigma}} = \text{im}(\text{per}_{\tilde{\omega}_{M,\kappa,\sigma}}) \subseteq Y^k$, where k denotes the rank of $H_1(M)/\text{tor}\big(H_1(M)\big)$, is isomorphic to $\Pi^k_{\omega_{\mathbb{S}^1},\kappa} \subseteq Y^k$.*

Proof: As before we suppress the dependence of the cocycles on κ. The isomorphism between Y^k and $H^1_{\text{dR}}(M, Y)$ is given by

$$H^1_{\text{dR}}(M, Y) \to Y^k, \quad [\beta] \mapsto \Big(\int_{\alpha_i} \beta\Big)_{i \in I_n}$$

for $\alpha_i \in C^\infty_*(\mathbb{S}^1, M)$, where $\big([\alpha_i]\big)_{i \in I_n}$ is a basis of $H_1(M)/\text{tor}\big(H_1(M)\big)$. The inverse isomorphism $Y^k \to H^1_{\text{dR}}(M, Y)$ is then given by $(y_i)_{i \in I_n} \mapsto \sum_{i=1}^n \delta(f_i) \cdot y_i$ where $f_i \in C^\infty(M, \mathbb{S}^1)$ is such that $\deg(\alpha_i \circ f_j) = \delta_{ij}$ (c.f. Remark 5.1.2). Note that we do not have to make additional assumptions about the mapping properties w.r.t. boundaries since $\partial\, \mathbb{S}^1 = \varnothing$. Hence we get with the preceding and Lemma 5.4.3

$$\begin{aligned} \lambda_{\alpha_i} \circ \text{per}_{\tilde{\omega}_{M,\sigma}} \circ \pi_2(f_{j\,\text{Gau}}) &= \lambda_{\mathbb{S}^1} \circ \text{per}_{\omega_{\mathbb{S}^1}} \circ \pi_2(\alpha_{i\,\text{Gau}}) \circ \pi_2(f_{j\,\text{Gau}}) \\ &= \lambda_{\mathbb{S}^1} \circ \text{per}_{\omega_{\mathbb{S}^1}} \circ \pi_2\big((f_j \circ \alpha_i)_{\text{Gau}}\big) = \deg(f_j \circ \alpha_i) \cdot \lambda_{\mathbb{S}^1} \circ \text{per}_{\omega_{\mathbb{S}^1}} \\ &= \delta_{ij} \cdot \lambda_{\mathbb{S}^1} \circ \text{per}_{\omega_{\mathbb{S}^1}} \end{aligned}$$

since $\mathrm{per}_{\tilde{\omega}_{\mathbb{S}^1}} = \mathrm{per}_{\omega_{\mathbb{S}^1}}$ is a homomorphism and $\lambda_{\mathbb{S}^1}$ is linear. Applying the inverse isomorphism $Y^k \to H^1_{\mathrm{dR}}(M,Y)$, this leads to

$$\mathrm{per}_{\tilde{\omega}_{M,\sigma}}\Big(\mathrm{im}\big(\pi_2(f_j)\big)\Big) = \big[\delta(f_j)\big]\mathrm{im}(\lambda_{\mathbb{S}^1} \circ \mathrm{per}_{\omega_{\mathbb{S}^1}}).$$

and hence we have

$$\mathrm{im}(\mathrm{per}_{\tilde{\omega}_{M,\sigma}}) \supseteq \bigoplus_{i=1}^{n} \big[\delta(f_i)\big] \cdot \lambda_{\mathbb{S}^1}\big(\mathrm{im}(\mathrm{per}_{\omega_{\mathbb{S}^1}})\big) \cong \mathrm{im}(\mathrm{per}_{\omega_{\mathbb{S}^1}})^n.$$

On the other hand $\lambda_{\alpha_i} \circ \mathrm{per}_{\tilde{\omega}_{M,\sigma}} = \mathrm{per}_{\omega_{\mathbb{S}^1}} \circ \pi_2(\alpha_i)$ implies directly

$$\mathrm{per}_{\tilde{\omega}_{M,\sigma}} \subseteq \big[\delta(f_i)\big] \cdot \lambda_{\mathbb{S}^1}\big(\mathrm{im}(\mathrm{per}_{\omega_{\mathbb{S}^1}})\big),$$

such that

$$\mathrm{im}(\mathrm{per}_{\tilde{\omega}_{M,\sigma}}) \subseteq \bigoplus_{i=1}^{n} \big[\delta(f_i)\big] \cdot \lambda_{\mathbb{S}^1}\big(\mathrm{im}(\mathrm{per}_{\omega_{\mathbb{S}^1}})\big) \cong \mathrm{im}(\mathrm{per}_{\omega_{\mathbb{S}^1}})^n.$$

□

5.5 Integrability of $\tilde{\omega}_\kappa$

We now turn to the question, whether for a given K-bundle the central extension of Lie algebras $\mathfrak{z}(Y) \twoheadrightarrow \hat{\mathfrak{g}} := \mathfrak{z}(Y) \oplus_{\tilde{\omega}_\kappa} \mathfrak{g} \overset{\mathrm{pr}_2}{\twoheadrightarrow} \mathfrak{g}$ integrates to a central extension of Lie groups. There may be two obstructions, one is that the image of the associated period map may not have discrete image and the other is that the adjoint action of $\mathfrak{g}$ on $\hat{\mathfrak{g}}$ may not integrate to an action of G.

Theorem 5.5.1 *If $\mathcal{P} = (K, M, P, \pi)$ is a finite-dimensional K-bundle with compact base space M and $\kappa : \mathfrak{k} \times \mathfrak{k} \to V(\mathfrak{k})$ universal, then the image of the period map associated to the covariant cocycle $(\xi, \eta) \mapsto \Big[\kappa\big(\xi, \nabla_X(\eta)\big)\Big]$ has discrete image.*

Proof: We only have to put former results together. Theorem 5.4.8 implies that $\mathrm{im}(\mathrm{per}_{\tilde{\omega}_\kappa})$ is discrete if and only if this is the case for $\mathrm{per}_{\omega_{\mathbb{S}^1}}$ and Proposition 5.2.9 implies that it is discrete if K is finite-dimensional and $\mathfrak{k} \times \mathfrak{k} \to V(\mathfrak{k})$ universal.

□

We now turn to the second question.

Lemma 5.5.2 *If $\mathcal{P} = (K, M, P, \pi)$ is a K-bundle, $f \in C^\infty(P, K)^K$ and $X \in \mathcal{V}(P)^K$, then $\delta^l(f)(X) \in C^\infty(P, \mathfrak{k})^K$.*

Proof: Unwinding the definitions we get

$$\begin{aligned}
&\big(\delta^l(f)(X)\big)(p\cdot k) = d\lambda_{f(p\cdot k)}\big(f(p\cdot k)\big).df(p\cdot k).X(p\cdot k)\\
&= d\lambda_{f(p\cdot k)}\big(f(p\cdot k)\big).df(p\cdot k).d\rho_k(p).X(p) = d\lambda_{f(p\cdot k)}\big(f(p\cdot k)\big).d(f\circ\rho_k)(p).X(p)\\
&= d\lambda_{f(p\cdot k)}\big(f(p\cdot k)\big).d(c_k\circ f).X(p) = d\lambda_{c_k(f(p))}\Big(c_k\big(f(p)\big)\Big).dc_k\big(f(p)\big).df(p).X(p)\\
&\qquad = d\big(\lambda_{c_k(f(p))}\circ c_k\big)\big(f(p)\big).df(p).X(p) = d\big(c_k\circ\lambda_{f(p)^{-1}}\big).df(p).X(p)\\
&\qquad\qquad = \big(\mathrm{Ad}(k).\delta^l(f)(X)\big)(p).
\end{aligned}$$

□

Proposition 5.5.3 *Let $\mathcal{P} = (K, M, P, \pi)$ be a K-bundle with connection and compact base space, Y be a s.c.l.c. space and $\kappa : \mathfrak{k}\times\mathfrak{k} \to Y$ be an invariant symmetric bilinear form and define*

$$\Theta : C^\infty(P,K)^K \to \mathrm{Lin}\big(C^\infty(P,\mathfrak{k})^K, \mathfrak{z}_M(Y)\big),\ \ \Theta(f)(\xi) = \Big[\kappa\big(\delta^l(f),\xi\big)_M\Big].$$

Then we obtain for the covariant cocycle $\tilde{\omega}_\kappa(\xi,\eta) = \Big[\kappa\big(\xi, \nabla_X(\eta)\big)\Big]$ an automorphic action of $C^\infty(P,K)^K$ on $\widehat{\mathfrak{gau}(\mathcal{P})} := \mathfrak{z}_M(Y) \oplus_{\tilde{\omega},\kappa} \mathfrak{gau}(\mathcal{P})$.

Proof: The calculations in [NM03, Proposition III.3] which are purely algebraic apply to the considered case as well.

□

Theorem 5.5.4 *If $\mathcal{P} = (K, M, P, \pi)$ is a finite-dimensional K-bundle with compact base space M and $\kappa : \mathfrak{k}\times\mathfrak{k} \to V(\mathfrak{k})$ universal, then there exists a central extension of Lie groups $Z \twoheadrightarrow \widehat{\mathrm{Gau}(\mathcal{P})}_0 \twoheadrightarrow \mathrm{Gau}(\mathcal{P})_0$ such that $Z \cong \mathfrak{z}_M(Y)/\Gamma$ for a discrete subgroup $\Gamma \subseteq \mathfrak{z}_M(Y)$ and the diagram*

$$\begin{array}{ccccc}
Z & \longrightarrow & \widehat{\mathrm{Gau}(\mathcal{P})}_0 & \longrightarrow & \mathrm{Gau}(\mathcal{P})_0\\
\downarrow & & \downarrow & & \downarrow\\
\mathfrak{z}_M(Y) & \longrightarrow & \widehat{\mathfrak{gau}(\mathcal{P})} & \longrightarrow & \mathfrak{gau}(\mathcal{P})
\end{array}$$

commutes.

Proof: This is the previous lemma, Theorem 5.5.1 and [Neeb 02, Lemma VII.11].

□

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