A Smooth Model for the String Group

Christoph Wockel (Hamburg)
(with Th. Nikolaus and Ch. Sachse)

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Outline

A smooth Lie group model

Promoting the model to a Lie 2-group model

Comparison with other models
What is the string group?

Whitehead tower of $O(n)$:

$$\text{String}(n) \rightarrow \text{Spin}(n) \xrightarrow{\cong} \text{Spin}(n) \xrightarrow{2:1} \text{SO}(n) \hookrightarrow O(n)$$

$\pi_3 \cong \mathbb{Z}$

$\pi_2 = 0$

$\pi_1 = \mathbb{Z}/2$

$\pi_0 = \mathbb{Z}/2$

Motivation:

- Spin geometry $\rightsquigarrow$ String geometry?
- loop space geometry
- SUSY $\sigma$-modles

Observation: If $P = f^*(E \text{Spin}(n)) \rightarrow M$ is a principal Spin$(n)$ bundle, then a lift $B \text{String}(n)$ exists iff $\frac{p_1}{2}(M)$ vanishes.
String group models

May replace $\text{Spin}(n)$ by an arbitrary simple 1-connected compact Lie group $G$.

**Definition:** A smooth model (for the string group) is a morphism

\[ q : \text{String}_G \to G \]

of Lie groups which is a 3-connected cover (i.e. $\pi_3(\text{String}_G) = 0$ and $\pi_i(q) : \pi_i(\text{String}_G) \xrightarrow{\cong} \pi_i(G)$ for $i \neq 3$). Analogously one defines topological models.

**Lemma:** $\ker(q)$ is a $K(\mathbb{Z}, 2)$ and thus $\text{String}_G$ cannot be finite-dimensional.

\[ \Rightarrow \] consider generalisations for Lie group structures on $\text{String}_G$:

- topological groups
- infinite-dimensional Lie groups
- Lie 2-groups (smooth group stacks)
Towards an infinite-dimensional model

Fact: \( PU := PU(\ell^2) \) is a \( K(\mathbb{Z}, 2) \) and a Lie group when endowed with the norm topology.

\( \Rightarrow \exists \) a smooth principal \( PU \)-bundle \( q: P \rightarrow G \) representing

\[
1 \in [G, BPU] \cong [G, K(3, \mathbb{Z})] \cong H^3(G, \mathbb{Z}) \cong \mathbb{Z}
\]

\( \Rightarrow \pi_3(P) = 0 \) and \( \pi_i(q) \) is an isomorphism for \( i \neq 3 \), so \( P \rightarrow G \) could serve as a string group model.

Problems:

- No explicit construction of \( P \rightarrow G \) known (only existence)!
  \( \leadsto \) if anybody knows...

- No criteria for existence of Lie group structure known (compare to Spin or the abelian case)!

However, we can use \( P \rightarrow G \) to construct another model.
The automorphism group of $P \rightarrow G$

**Definition:** $\text{Aut}(P) := \{ \varphi \in \text{Diff}(P) : \forall g \in PU \, f(p \cdot g) = f(p) \cdot g \}$

$\leadsto Q : \text{Aut}(P) \rightarrow \text{Diff}(G)$ given by

\[ P \xrightarrow{\varphi} P \]
\[ \Downarrow \]
\[ G \xrightarrow{Q(\varphi)} G \]

- $\text{Gau}(P) := \ker(Q) \cong C^\infty(P, PU)^{PU}$ is the gauge group of $P$
- There are continuous versions $\text{Aut}_c(P)$ and $\text{Gau}_c(P)$ and $Q$ extends to
  \[ Q_c : \text{Aut}_c(P) \rightarrow \text{Homeo}(G) \]

**Fact:** $\text{Gau}(P)$, $\text{Aut}(P)$ and $\text{Diff}(G)$ are Lie groups and

$\text{Gau}(P) \rightarrow \text{Aut}(P) \rightarrow \text{Diff}(G)[P]$

is an extension of Lie groups. The corresponding Lie algebras are $\mathcal{V}_{\text{vert}}(P)^{PU}$, $\mathcal{V}(P)^{PU}$ and $\mathcal{V}(G)$. 
The Lie group model

**Definition:** String\(_G := \text{Aut}(P)|_G\) and String\(_{G,c} := \text{Aut}_c(P)|_G\), where \(G \subset \text{Diff}(G)\) via left translation.

**Theorem [Stolz]:** \(Q_c : \text{String}_{G,c} \rightarrow G\) is a topological model.

**Theorem [NSW]:** \(Q : \text{String}_G \rightarrow G\) is a smooth model.

**Proof:** Show that String\(_G \rightarrow \text{String}_{G,c}\) is a (weak) homotopy equivalence:

\[
\cdots \rightarrow \pi_i(\text{Gau}(P)) \rightarrow \pi_i(\text{String}_G) \rightarrow \pi_i(G) \rightarrow \cdots \\
\downarrow \cong \quad \downarrow \quad \parallel \\
\cdots \rightarrow \pi_i(\text{Gau}_c(P)) \rightarrow \pi_i(\text{String}_{G,c}) \rightarrow \pi_i(G) \rightarrow \cdots
\]

(\(\text{Gau}_c(P)\) has the compact-open, \(\text{Gau}(P)\) the \(C^\infty\) topology).

**Note:** String\(_{G,c}\) cannot be turned into a Lie group, although \(\text{Gau}_c(P)\) does.
Improving the model

**Aim:** Promote the model $\text{String}_G \to G$ to a 2-group model.

**Why?**
- Compare: line bundles are best studied as $U(1)$-bundles, not as maps to $|BU(1)|$ or as $\mathbb{Z}$-bundle gerbes.
  - $\leadsto$ This is because $U(1)$ is the preferred model of $K(\mathbb{Z}, 1)$!
  - The preferred model for $K(\mathbb{Z}, 2)$, the 2-group $U(1) \to \ast$.
- String theory predicts backgrounds with bundle-like structures having 3-forms as curvature.
  - $\leadsto$ 2-bundles (or $U(1)$ bundle gerbes) have this structure!

**Definition:** A (strict) Lie 2-group $\mathcal{H}$ consists of
- a homomorphism $H \xrightarrow{\tau} K$ of Lie groups
- a smooth (right) action $K \to \text{Aut}(H)$

such that

\[
\tau(h.k) = k^{-1} \cdot \tau(h) \cdot k \quad \text{(equivariance)}
\]

\[
h \cdot \tau(h') = h'^{-1} \cdot h \cdot h'. \quad \text{(Peiffer identity)}
\]
Lie 2-groups

**Definition:** A (strict) Lie 2-group $\mathcal{H}$ consists of

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$$\tau(h.k) = k^{-1} \cdot \tau(h) \cdot k \quad \text{(equivariance)}$$

$$h \cdot \tau(h') = h'^{-1} \cdot h \cdot h' \quad \text{(Peiffer identity)}$$

Technical assumptions:

- always assume $H$ and $K$ to be metrisable!
- always assume that $\pi_0(\mathcal{H}) := K/\tau(H)$ and $\pi_1(\mathcal{H}) := \ker(\tau)$ have natural Lie group structures

**First Examples:**

- For $K$ a Lie group $\{\ast\} \to K$ trivial (denoted again by $K$).
- For $A$ an abelian Lie group $A \to \{\ast\}$ trivial (denoted $BA$).
Lie 2-group models

**Note:** There is the geometric realisation functor

$$| \cdot | : \text{Lie-2-Grp} \to \text{Top-Gp}$$

and $|BA|$ is the classifying space of $A$ (whence the name). In particular, $|BU(1)|$ is a $K(\mathbb{Z}, 2)$. Moreover, $|K| = K$ (on the nose).

This allows us to define 2-group models in terms of group models:

**Definition:** A Lie 2-group model (for the string group) is a Lie 2-group $\mathcal{H}$ with isomorphisms $\pi_1(\mathcal{H}) \xrightarrow{\simeq} U(1)$ and $\pi_0(\mathcal{H}) \xrightarrow{\simeq} G$ such that

$$|\mathcal{H}| \to |\pi_0(\mathcal{H})| \xrightarrow{\simeq} G$$

is a topological model.

In fact, there is a story in Lie group cohomology going on here (C. Schommer-Pries, work in progress with F. Wagemann).
Construction of the 2-group model

Recall:

- $P \to G$: principal $PU$-bundle (generator in $H^3(G, \mathbb{Z})$)
- $\text{String}_G \subseteq \text{Aut}(P)$, covering left multiplication $G \subset \text{Diff}(G)$

$\rightsquigarrow \text{Gau}(P) \cong C^\infty(P, PU)^{PU}$ has a universal central extension

$$C^\infty(G, U(1)) \to C^\infty(P, U)^{PU} \to \text{Gau}(P) \quad (\ast)$$

$\rightsquigarrow \text{String}_G \subseteq \text{Aut}(P)$ acts on $C^\infty(P, U)^{PU}$ by $f\varphi := f \circ \varphi$. This yields a Lie 2-group

\[
\begin{array}{ccc}
C^\infty(P, U)^{PU} & \xrightarrow{\tau} & \text{String}_G \\
\downarrow & & \downarrow \\
\text{Gau}(P) & & \text{Gau}(P)
\end{array}
\]

with $\pi_1(\mathcal{H}) = C^\infty(G, U(1))$.

**Proposition:** $\text{String}_G$ acts smoothly on the bundle

$$U(1) \to \widehat{\text{Gau}(P)} \to \text{Gau}(P)$$

associated to $(\ast)$ along the homomorphism

$$I_G: C^\infty(G, U(1)) \to U(1), \quad f \mapsto \int_G f d\mu.$$
Why is this a 2-group model?

**Definition:** The 2-group $\text{STRING}_G$ is given by the homomorphism

$$
\hat{\text{Gau}(P)} = C^\infty(P, U)^{PU} \times C^\infty(G, U(1)) U(1) \xrightarrow{\tau \circ \text{pr}_1} \text{String}_G
$$

and the action

$$[f, \lambda]_\varphi := [f \circ \varphi, \lambda].$$

We want to check that this is a Lie 2-group model for $\text{String}$:

- $\pi_1(\text{STRING}) = \ker (\hat{\text{Gau}(P)} \to \text{Gau}(P)) = U(1)$ (by constr.)
- $\pi_0(\text{STRING}) = \text{coker} (\text{Gau}(P) \to \text{String}_G) = G$ (by constr.)
- remains to show that $|\text{STRING}| \to G$ is a topological model

**Note:** There exists a canonical inclusion $\text{String}_G \to \text{STRING}_G$, given by

$$\{\ast\} \xrightarrow{\simeq} \hat{\text{Gau}(P)}$$

$$\xrightarrow{\simeq}$$

$$\text{String}_G \xrightarrow{\simeq} \text{String}_G$$
Why is this a 2-group model?

**Proposition:** Both horizontal maps in
\[
\begin{array}{ccc}
\{\ast\} & \xrightarrow{\sim} & \hat{\text{Gau}}(P) \\
\downarrow & & \downarrow \\
\text{String}_G & \xrightarrow{\sim} & \text{String}_G
\end{array}
\]
are in fact (weak) homotopy equivalences.

**Proof:** Show that \( U(1) \to \hat{\text{Gau}}(P) \to \text{Gau}(P) \) universal (recall \( \text{Gau}(P) \) is a \( K(\mathbb{Z}, 2) \)).

**Theorem [NSW]:** \( |\text{String}_G| \to |\text{STRING}_G| \) is a (weak) homotopy equivalence and thus \( \text{STRING}_G \) is a Lie 2-group model.

**Proof:** Show that adding a contractible space of “morphisms” does not affect the geometric realisation. This relies heavily on the homotopy theory of topological metrisable manifolds [Palais '66].
String bundles and string connections

**Aim:** Do differential geometry with Lie 2-groups by using the theory of 2-bundles and connections.

**Proposition:** The inclusion \( \text{String}_G \rightarrow \text{STRING}_G \) induces a functor

\[
\text{Bun}_{\text{String}_G}(M) \rightarrow 2\text{-Bun}_{\text{STRING}_G}(G)
\]

which induces a bijection on isomorphism classes.

**Theorem** [Nikolaus-Waldorf]: If \( \mathcal{H} \rightarrow \mathcal{H}' \) is a morphism between 2-group models, then the induced functor

\[
2\text{-Bun}_{\mathcal{H}}(G) \rightarrow 2\text{-Bun}_{\mathcal{H}'}(G)
\]

is an equivalence of 2-groupoids.

**Open:** Corresponding statements for 2-bundles with connections.
Other existing models

- [BCSS '07] start with the contractible cover $P_e G \to G$, construct an action of $P_e G$ on $\hat{\Omega} G$ turning

  \[
  \begin{array}{ccc}
  \hat{\Omega} G & \xrightarrow{\tau} & P_e G \\
  \downarrow & & \downarrow \\
  \Omega G & \xrightarrow{} & \Omega G
  \end{array}
  \]

  into a Lie 2-group and show that this is a 2-group model.

- [Stolz-Teichner '04] associate the above along a positive energy representation $\rho : \Omega G \to PU$.

- [Schommer-Pries '10] classifies central extensions of smooth group stacks $\left[*/U(1)\right] \to E \to [G]$ and relates this to $H^3_{\text{Lie}}(G, U(1)) \cong H^4(|BG|, \mathbb{Z}) \cong \mathbb{Z}$.

- [Henriques '08] develops integration procedure for $L_\infty$-algebras and applies this to the string Lie 2-algebra.

- [Stolz '96]: String$_G \to G$ (topological/smooth model)
Relation between the models

Where “Morita equivalence” has to be understood as follows:

- take cover \((U_i)_{i=1,\ldots,n}\) of \(G\) with sections \(\sigma_i: U_i \to P_e G\)
- \(\gamma_{ij} := \sigma_i \cdot \sigma_j^{-1}: U_{ij} \to \Omega G\) is a Čech cocycle for the smooth principal bundle \(P_e G \to G\)

⇒ Get a Morita equivalence

\[
\bigsqcup U_{ij} \xrightarrow{\gamma_{ij} \times \sigma_i} \Omega G \times P_e G \\
\bigsqcup U_i \xrightarrow{\sigma_i} P_e G
\]

(Morita equiv. of Lie groupoids \(\leftrightarrow\) diffeomorphism of manifolds)
Relation between the models

Where “Morita equivalence” has to be understood as follows:

- take good cover \((U_i)_{i=1,...,n}\) of \(G\) with sections \(\sigma_i: U_i \to P_e G\)

- \(\gamma_{ij} := \sigma_i \cdot \sigma_j^{-1}: U_{ij} \to \Omega G\) is a Čech cocycle for the smooth principal bundle \(P_e G \to G\)

- assume \((U_i)_{i=1,...,n}\) to be good \(\Rightarrow \gamma_{ij}\) has lifts \(\hat{\gamma}_{ij}: U_{ij} \to \hat{\Omega} G\)

- \(\hat{\gamma}_{ij} \cdot \hat{\gamma}_{jk} \cdot \hat{\gamma}_{ik}^{-1}: U_{ijk} \to U(1)\) is a Čech cocycle and defines a Lie groupoid \(\bigsqcup U(1) \times_h U_{ij} \to \bigsqcup U_i\).

\[\bigsqcup U(1) \times_h U_{ij} \xrightarrow{\nu \cdot \hat{\gamma}_{ij} \times \sigma_i} \hat{\Omega} G \times P_e G\]

\[\bigsqcup U_i \xrightarrow{\sigma_i} P_e G\]

\[\Rightarrow \text{Get a Morita equivalence}\]

\[\Rightarrow \text{induces smooth group structure on the associated smooth stack}\]

\[\left[\bigsqcup U(1) \times_h U_{ij} \xrightarrow{\nu \cdot \hat{\gamma}_{ij} \times \sigma_i} \hat{\Omega} G \times P_e G\right]\]

\[\leadsto \text{Can do the same with the model } \hat{\text{Gau}}(P) \to \text{String } G.\]
Comparison to the BCSS model:

Pass to the associated stacks to apply Schommer-Pries' result:

\[
\begin{align*}
\left[ \widehat{\Omega G} \times P^e G \right] & \xrightarrow{\cong} \left[ \bigsqcup U_{ij} \times_h U(1) \right] \xrightarrow{\cong} \left[ \bigsqcup U'_{ij} \times_{h'} U(1) \right] \\
& \mapsto \left[ \text{Gau}(P) \right] \\
\end{align*}
\]

\Rightarrow \text{The BCSS model and the NSW model are equivalent as (infinite-dimensional) smooth stacks.}

\Rightarrow \text{The BCSS model and the NSW are equivalent as Lie 2-groups [Noohi].}

\Rightarrow \text{There exists a Lie 2-group } H \xrightarrow{\tau} K \text{ and smooth morphisms}

\[
\begin{align*}
\widehat{\Omega G} & \xleftarrow{\tau_{\text{BCSS}}} H \xrightarrow{\tau} \text{Gau}(P) \\
& \xrightarrow{\tau_{\text{NSW}}} String_G \\
\end{align*}
\]

\rightleftharpoons \text{Explicit construction? Any ideas?}
\textbf{C*-algebras vs. von Neumann algebras}

Since the Stolz-Teichner construction (’04) von Neumann algebras are considered to yield meaningful representations of String.

The present model seems to be closer to \( C^* \) algebras:

\[ PU \rtimes K \] (for \( K=\text{compact operators of } \ell^2 \)), so we get a \( C^* \)-algebra bundle

\[ \mathcal{K} := P \times_{PU} K \]

and an action \( \Gamma(\mathcal{K}) \).\( \rtimes \Gamma(\mathcal{K}) \).

\[ \Gamma(\mathcal{K}) \]

⇒ For each string manifold \( M \) (i.e. \( p_{1/2}(M) = 0 \)) and each string lift \( \tilde{P} \rightarrow M \) of a spin bundle we get a bundle \( \tilde{P} \times_{\text{String}_G} \Gamma(\mathcal{K}) \) of \( C^* \)-algebras over \( M \).

\textbf{Problem:} This does not seem to be meaningful, since the action of \( \text{String}_G \) is linear.

\textbf{Note:} For a 2-group model \( \mathcal{H} = (H \xrightarrow{\tau} K) \), interesting representations come from the outer action of \( K \) on \( \text{Rep}_\lambda(H) \), where \( \lambda \) is a fixed character for the \( U(1) \)-action.


