A Smooth Model for the String Group

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Outline

What is the string group?

Whitehead tower of $O(n)$:

$$\mathrm{String}(n) \to \underbrace{\mathrm{Spin}(n)}_{\pi_3 \cong \mathbb{Z}} \xrightarrow{\cong} \underbrace{\mathrm{Spin}(n)}_{\pi_2 = 0} \xrightarrow{2:1} \underbrace{\mathrm{SO}(n)}_{\pi_1 = \mathbb{Z}/2} \hookrightarrow \underbrace{O(n)}_{\pi_0 = \mathbb{Z}/2}$$

Motivation:

- Spin geometry $\rightsquigarrow$ String geometry?
- loop space geometry
- SUSY σ-modles

Observation: If $P = f^*(E\,\mathrm{Spin}(n)) \to M$ is a principal $\mathrm{Spin}(n)$ bundle, then a lift

$$\begin{array}{ccc} & & B\,\mathrm{String}(n) \\ & \overset{\widetilde{f}}{\nearrow} & \downarrow \\ M & \xrightarrow{f} & B\,\mathrm{Spin}(n) \end{array}$$

exists iff $\frac{p_1}{2}(M)$ vanishes.

String group models

May replace $\mathrm{Spin}(n)$ by an arbitrary simple 1-connected compact Lie group G.

Definition: A smooth model (for the string group) is a morphism

$$q\colon \mathrm{String}_G \to G$$

of Lie groups which is a 3-connected cover (i.e. $\pi_3(\mathrm{String}_G) = 0$ and $\pi_i(q)\colon \pi_i(\mathrm{String}_G) \xrightarrow{\cong} \pi_i(G)$ for $i \neq 3$). Analogously one defines topological models.

Lemma: $\ker(q)$ is a $K(\mathbb{Z}, 2)$ and thus String_G cannot be finite-dimensional.

$\rightsquigarrow$ consider generalisations for Lie group structures on String_G:

- topological groups
- infinite-dimensional Lie groups
- Lie 2-groups (smooth group stacks)

Towards an infinite-dimensional model

Fact: $PU := PU(\ell^2)$ is a $K(\mathbb{Z}, 2)$ and a Lie group when endowed with the norm topology.

$\Rightarrow \exists$ a *smooth* principal PU-bundle $q\colon P \to G$ representing

$$1 \in [G, BPU] \cong [G, K(3, \mathbb{Z})] \cong H^3(G, \mathbb{Z}) \cong \mathbb{Z}$$

$\Rightarrow \pi_3(P) = 0$ and $\pi_i(q)$ is an isomorphism for $i \neq 3$, so $P \to G$ could serve as a string group model.

Problems:

- No explicit construction of $P \to G$ known (only existence)!
  $\rightsquigarrow$ if anybody knows...
- No criteria for existence of Lie group structure known (compare to Spin or the abelian case)!

However, we can use $P \to G$ to construct another model.

The automorphism group of $P \to G$

Definition: $\mathrm{Aut}(P) := \{\varphi \in \mathrm{Diff}(P) : \forall g \in PU\, f(p \cdot g) = f(p) \cdot g\}$

$\rightsquigarrow Q\colon\ \mathrm{Aut}(P) \to \mathrm{Diff}(G)$ given by

$$\begin{array}{ccc} P & \xrightarrow{\varphi} & P \\ \downarrow & & \downarrow \\ G & \xrightarrow{Q(\varphi)} & G \end{array}$$

- $\mathrm{Gau}(P) := \ker(Q) \cong C^\infty(P, PU)^{PU}$ is the *gauge group* of P
- There are continuous versions $\mathrm{Aut}_c(P)$ and $\mathrm{Gau}_c(P)$ and Q extends to

$$Q_c\colon\ \mathrm{Aut}_c(P) \to \mathrm{Homeo}(G)$$

Fact: $\mathrm{Gau}(P)$, $\mathrm{Aut}(P)$ and $\mathrm{Diff}(G)$ are Lie groups and

$$\mathrm{Gau}(P) \to \mathrm{Aut}(P) \to \mathrm{Diff}(G)_{[P]}$$

is an extension of Lie groups. The corresponding Lie algebras are $\mathcal{V}_{\mathrm{vert}}(P)^{PU}$, $\mathcal{V}(P)^{PU}$ and $\mathcal{V}(G)$.

The Lie group model

Definition: $\mathrm{String}_G := \mathrm{Aut}(P)|_G$ and $\mathrm{String}_{G,c} := \mathrm{Aut}_c(P)|_G$, where $G \subset \mathrm{Diff}(G)$ via left translation.

Theorem [Stolz]**:** $Q_c\colon \mathrm{String}_{G,c} \to G$ is a topological model.

Theorem [NSW]**:** $Q\colon \mathrm{String}_G \to G$ is a smooth model.

Proof: Show that $\mathrm{String}_G \to \mathrm{String}_{G,c}$ is a (weak) homotopy equivalence:

$$\begin{array}{ccccccc} \cdots \longrightarrow & \pi_i(\mathrm{Gau}(P)) & \longrightarrow & \pi_i(\mathrm{String}_G) & \longrightarrow & \pi_i(G) & \rightarrow \cdots \\ & \downarrow\cong & & \downarrow & & \| & \\ \cdots \rightarrow & \pi_i(\mathrm{Gau}_c(P)) & \rightarrow & \pi_i(\mathrm{String}_{G,c}) & \rightarrow & \pi_i(G) & \rightarrow \cdots \end{array}$$

($\mathrm{Gau}_c(P)$ has the compact-open, $\mathrm{Gau}(P)$ the C^∞ topology). □

Note: $\mathrm{String}_{G,c}$ cannot be turned into a Lie group, although $\mathrm{Gau}_c(P)$ does.

Improving the model

Aim: Promote the model $\mathrm{String}_G \to G$ to a *2-group* model.

Why?

- Compare: line bundle are best studied as $U(1)$-bundles, not as maps to $|BU(1)|$ or as $\mathbb{Z}$ bundle gerbes.
  $\rightsquigarrow$ This is because $U(1)$ is the *preferred* model of $K(\mathbb{Z},1)$!
  The preferred model for $K(\mathbb{Z},2)$, the 2-group $U(1) \rightrightarrows *$.
- String theory predicts backgrounds with bundle-like structures having 3-forms as curvature.
  $\rightsquigarrow$ 2-bundles (or $U(1)$ bundle gerbes) have this structure!

Definition: A (strict) Lie 2-group $\mathcal{H}$ consists of

- a homomorphism $H \xrightarrow{\tau} K$ of Lie groups
- a smooth (right) action $K \to \mathrm{Aut}(H)$

such that

$$\tau(h.k) = k^{-1} \cdot \tau(h) \cdot k \qquad \text{(equivariance)}$$

$$h.\tau(h') = h'^{-1} \cdot h \cdot h'. \qquad \text{(Peiffer identity)}$$

Lie 2-groups

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Technical assumptions:

- always assume H and K to be metrisable!
- always assume that $\underline{\pi}_0(\mathcal{H}) := K/\tau(H)$ and $\underline{\pi}_1(\mathcal{H}) := \ker(\tau)$ have natural Lie group structures

First Examples:

- For K a Lie group $\{*\} \to K$ trivial (denoted again by K).
- For A an abelian Lie group $A \to \{*\}$ trivial (denoted BA).

Lie 2-group models

Note: There is the geometric realisation functor

$$|\cdot|\colon \textit{Lie-2-Grp} \to \textit{Top-Gp}$$

and $|BA|$ is the classifying space of A (whence the name). In particular, $|BU(1)|$ is a $K(\mathbb{Z},2)$. Moreover, $|K| = K$ (on the nose).

This allows us to define 2-group models in terms of group models:

Definition: A Lie 2-group model (for the string group) is a Lie 2-group $\mathcal{H}$ with isomorphisms $\underline{\pi}_1(\mathcal{H}) \xrightarrow{\cong} U(1)$ and $\underline{\pi}_0(\mathcal{H}) \xrightarrow{\cong} G$ such that

$$|\mathcal{H}| \to |\underline{\pi}_0(\mathcal{H})| \xrightarrow{\cong} G$$

is a topological model.

In fact, there is a story in Lie group cohomology going on here (C. Schommer-Pries, work in progress with F. Wagemann).

Construction of the 2-group model

Recall:

- $P \to G$: principal PU-bundle (generator in $H^3(G, \mathbb{Z})$)
- $\mathrm{String}_G \subseteq \mathrm{Aut}(P)$, covering left multiplication $G \subset \mathrm{Diff}(G)$

$\rightsquigarrow$ $\mathrm{Gau}(P) \cong C^\infty(P, PU)^{PU}$ has a universal central extension

$$C^\infty(G, U(1)) \to C^\infty(P, U)^{PU} \to \mathrm{Gau}(P) \qquad (*)$$

$\rightsquigarrow$ $\mathrm{String}_G \subseteq \mathrm{Aut}(P)$ acts on $C^\infty(P, U)^{PU}$ by $f^\varphi := f \circ \varphi$. This yields a Lie 2-group

$$\begin{array}{ccc} C^\infty(P, U)^{PU} & \xrightarrow{\quad\tau\quad} & \mathrm{String}_G \\ & \searrow \quad \nearrow & \\ & \mathrm{Gau}(P) & \end{array}$$

with $\underline{\pi}_1(\mathcal{H}) = C^\infty(G, U(1))$.

Proposition: String_G acts smoothly on the bundle

$$U(1) \to \widehat{\mathrm{Gau}(P)} \to \mathrm{Gau}(P)$$

associated to $(*)$ along the homomorphism

$$I_G \colon C^\infty(G, U(1)) \to U(1), \quad f \mapsto \int_G f d\mu.$$

Why is this a 2-group model?

Definition: The 2-group $\mathtt{STRING}_G$ is given by the homomorphism

$$\widehat{\mathrm{Gau}(P)} = C^\infty(P,U)^{PU} \times_{C^\infty(G,U(1))} U(1) \xrightarrow{\tau\circ \mathrm{pr}_1} \mathrm{String}_G$$

and the action

$$[f,\lambda]^\varphi := [f\circ\varphi,\lambda].$$

$\rightsquigarrow$ want to check that this is a Lie 2-group model for String:

- $\underline{\pi}_1(\mathtt{STRING}) = \ker\left(\widehat{\mathrm{Gau}(P)} \to \mathrm{Gau}(P)\right) = U(1)$ (by constr.)
- $\underline{\pi}_0(\mathtt{STRING}) = \mathrm{coker}\left(\mathrm{Gau}(P) \to \mathrm{String}_G\right) = G$ (by constr.)
- remains to show that $|\mathtt{STRING}| \to G$ is a topological model

Note: There exists a canonical inclusion $\mathrm{String}_G \to \mathtt{STRING}_G$, given by

$$\begin{array}{ccc} \{*\} & \xrightarrow{\simeq} & \widehat{Gau(P)} \\ \downarrow & & \downarrow \\ \mathrm{String}_G & \xrightarrow{\simeq} & \mathrm{String}_G \end{array}$$

Why is this a 2-group model?

Proposition: Both horizontal maps in

$$\begin{array}{ccc} \{*\} & \xrightarrow{\simeq} & \widehat{Gau(P)} \\ \downarrow & & \downarrow \\ \mathsf{String}_G & \xrightarrow{\simeq} & \mathsf{String}_G \end{array}$$

are in fact (weak) homotopy equivalences.

Proof: Show that $U(1) \to \widehat{\mathrm{Gau}(P)} \to \mathrm{Gau}(P)$ universal (recall $\mathrm{Gau}(P)$ is a $K(\mathbb{Z},2)$).

Theorem [NSW]**:** $|\,\mathsf{String}_G\,| \to |\,\mathtt{STRING}_G\,|$ is a (weak) homotopy equivalence and thus $\mathtt{STRING}_G$ is a Lie 2-group model.

Proof: Show that adding a contractible space of "morphisms" does not affect the geometric realisation. This relies heavily on the homotopy theory of topological metrisable manifolds [Palais '66].

String bundles and string connections

Aim: Do differential geometry with Lie 2-groups by using the theory of 2-bundles and connections.

Proposition: The inclusion $\mathsf{String}_G \to \mathtt{STRING}_G$ induces a functor

$$\mathsf{Bun}_{\mathsf{String}_G}(M) \to 2\text{-}\mathsf{Bun}_{\mathtt{STRING}_G}(G)$$

which induces a bijection on isomorphism classes.

Theorem [Nikolaus-Waldorf]**:** If $\mathcal{H} \to \mathcal{H}'$ is a morphism between 2-group models, then the induced functor

$$2\text{-}\mathsf{Bun}_{\mathcal{H}}(G) \to 2\text{-}\mathsf{Bun}_{\mathcal{H}'}(G)$$

is an equivalence of 2-groupoids.

Open: Corresponding statements for 2-bundles with connections.

Other existing models

- [BCSS '07] start with the contractible cover $P_eG \to G$, construct an action of P_eG on $\widehat{\Omega G}$ turning

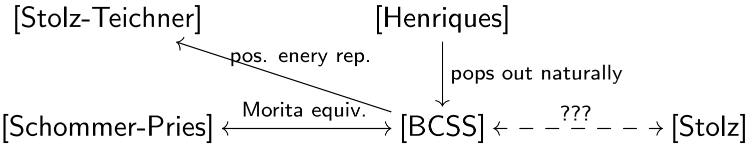
$$\begin{array}{ccc} \widehat{\Omega G} & \xrightarrow{\tau} & P_eG \\ & \searrow \quad \nearrow & \\ & \Omega G & \end{array}$$

  into a Lie 2-group and show that this is a 2-group model.
- [Stolz-Teichner '04] associate the above along a positive energy representation $\rho\colon \Omega G \to PU$.
- [Schommer-Pries '10] classifies central extensions of smooth group stacks

$$[*/U(1)] \to E \to [G]$$

  and relates this to $H^3_{\text{Lie}}(G, U(1)) \cong H^4(|BG|, \mathbb{Z}) \cong \mathbb{Z}$.
- [Henriques '08] develops integration procedure for L_∞-algebras and applies this to the string Lie 2-algebra.
- [Stolz '96]: $\text{String}_G \to G$ (topological/smooth model)

Relation between the models



Where "Morita equivalence" has to be understood as follows:

- take cover $(U_i)_{i=1,..,n}$ of G with sections $\sigma_i \colon U_i \to P_e G$
- $\gamma_{ij} := \sigma_i \cdot \sigma_j^{-1} \colon U_{ij} \to \Omega G$ is a Čech cocycle for the smooth principal bundle $P_e G \to G$

$\Rightarrow$ Get a Morita equivalence

$$\begin{array}{ccc} \bigsqcup U_{ij} & \xrightarrow{\gamma_{ij} \times \sigma_i} & \Omega G \ltimes P_e G \\ \downarrow\downarrow & & \downarrow\downarrow \\ \bigsqcup U_i & \xrightarrow{\sigma_i} & P_e G \end{array}$$

(Morita equiv. of Lie groupoids $\leftrightarrow$ diffeomorphism of manifolds)

Relation between the models

Where "Morita equivalence" has to be understood as follows:

- take good cover $(U_i)_{i=1,..,n}$ of G with sections $\sigma_i \colon U_i \to P_e G$
- $\gamma_{ij} := \sigma_i \cdot \sigma_j^{-1} \colon U_{ij} \to \Omega G$ is a Čech cocycle for the smooth principal bundle $P_e G \to G$
- assume $(U_i)_{i=1,..,n}$ to be good $\Rightarrow \gamma_{ij}$ has lifts $\widehat{\gamma}_{ij} \colon U_{ij} \to \widehat{\Omega G}$
- $\widehat{\gamma}_{ij} \cdot \widehat{\gamma}_{jk} \cdot \widehat{\gamma}_{ik}^{-1} \colon U_{ijk} \to U(1)$ is a Čech cocycle and defines a Lie groupoid $\bigsqcup U(1) \times_h U_{ij} \rightrightarrows \bigsqcup U_i$.

$\Rightarrow$ Get a Morita equivalence

$$\begin{array}{ccc} \bigsqcup U(1) \times_h U_{ij} & \xrightarrow{\iota \cdot \widehat{\gamma}_{ij} \times \sigma_i} & \widehat{\Omega G} \rtimes P_e G \\ \downarrow\downarrow & & \downarrow\downarrow \\ \bigsqcup U_i & \xrightarrow{\sigma_i} & P_e G \end{array}$$

$\Rightarrow$ induces smooth group structure on the associated smooth stack

$$\left[\bigsqcup U(1) \times_h U_{ij} \rightrightarrows \bigsqcup U_i \right] .$$

$\rightsquigarrow$ Can do the same with the model $\widehat{\mathrm{Gau}(P)} \to \mathrm{String}\, G$.

Comparison to the BCSS model:

Pass to the associated stacks to apply Schommer-Pries' result:

$$\left[\begin{array}{c}\widehat{\Omega G} \rtimes P_e G \\ \downarrow \quad \downarrow \\ P_e G\end{array}\right] \xleftarrow{\cong} \left[\begin{array}{c}\bigsqcup U_{ij} \times_h U(1) \\ \downarrow \quad \downarrow \\ \bigsqcup U_i\end{array}\right] \cong \left[\begin{array}{c}\bigsqcup U'_{ij} \times_{h'} U(1) \\ \downarrow \quad \downarrow \\ \bigsqcup V_i\end{array}\right] \xrightarrow{\cong} \left[\begin{array}{c}\widehat{\mathrm{Gau}(P)} \rtimes \mathrm{String}_G \\ \downarrow \quad \downarrow \\ \mathrm{String}_G\end{array}\right]$$

$\Rightarrow$ The BCSS model and the NSW model are equivalent as (infinite-dimensional) smooth stacks.

$\Rightarrow$ The BCSS model and the NSW are equivalent as Lie 2-groups [Noohi].

$\Rightarrow$ There exists a Lie 2-group $H \xrightarrow{\tau} K$ and smooth morphisms

$$\begin{array}{ccccc}\widehat{\Omega G} & \longleftarrow & H & \longrightarrow & \widehat{\mathrm{Gau}(P)} \\ \downarrow \tau_{BCSS} & & \downarrow \tau & & \downarrow \tau_{NSW} \\ P_e G & \longleftarrow & K & \longrightarrow & \mathrm{String}_G\end{array}$$

$\rightsquigarrow$ Explicit construction? Any ideas?

C^*-algebras vs. von Neumann algebras

Since the Stolz-Teichner construction ('04) von Neumann algebras are considered to yield meaningful representations of String.

The present model seems to be closer to C^* algebras:

$PU \curvearrowright K$ (for K=compact operators of ℓ^2), so we get a C^*-algebra bundle

$$\mathcal{K} := P \times_{PU} K$$

and an action $\mathrm{Gau}(P) \subseteq \mathrm{Aut}(P) \curvearrowright \Gamma(\mathcal{K})$.

$\Rightarrow$ For each string manifold M (i.e. $\frac{p_1}{2}(M) = 0$) and each string lift $\tilde{P} \to M$ of a spin bundle we get a bundle $\tilde{P} \times_{\mathrm{String}_G} \Gamma(\mathcal{K})$ of C^*-algebras over M.

Problem: This does not seem to be meaningful, since the action of String_G is linear.

Note: For a 2-group model $\mathcal{H} = (H \xrightarrow{\tau} K)$, interesting representations come from the outer action of K on $\mathrm{Rep}_\lambda(H)$, where λ is a fixed character for the $U(1)$-action.

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