From link homology to TQFTs

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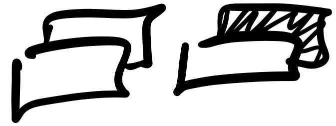
parts are joint work with S. Morrison, K. Walker,
Y. Liu, A. Mazel-Gee, D. Reutter, C. Stroppel,
M. Hogancamp, D. Rose

1. Strategies



$$\text{Link polynomial} \rightsquigarrow \text{braided monoidal cat} \xrightarrow{\text{skein theory}} \text{4D TQFT} \xrightarrow{\partial} \text{1-2-3-TQFT}$$

Jones p. $\quad \mathrm{Rep}^{f.d.} U_q(\mathfrak{sl}_2) \quad$ CYK $\quad$ WRT

$\Big\downarrow$ forget

$$\text{monoidal cat} \rightsquigarrow \text{3D TQFT}$$

TV

$$\text{Link homology} \rightsquigarrow \overbrace{\text{braided monoidal}}^{E_2}\ (\infty,2)\text{-cat} \xrightarrow{\text{skein theory}} \text{5D TQFT} \xrightarrow{\partial} \text{?-TQFT}$$

or chain complexes $\quad$ LMRSW 24 $\quad$ MWW Lasagna $\quad$ X

$\Big\downarrow$ forget

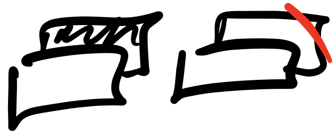
$$\overbrace{\text{monoidal}}^{E_1}\ (\infty,2)\text{-cat} \rightsquigarrow \boxed{\text{4D TQFT}}$$

today: educated guess
for 2D layer here

roughly: surface $\mapsto$ dg category

- TFT as organizational principle
- no Kh braiding required
  ↳ very general, robust construction

2. Algebraic input



„Link homology for unlinks"

$\Bbbk$ commutative ring

A commutative (extended) Frobenius algebra $/\Bbbk$ $\quad\longleftrightarrow\quad$ (unoriented) 1-2 TQFT in $\Bbbk$-mod

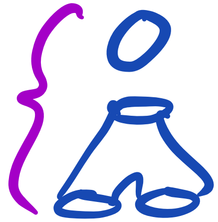
$\mathcal{F}$

$\Bbbk = \mathbb{Z}$

$A = H^*(\mathbb{CP}^1) = \mathbb{Z}[X]/(X^2)$

$\varepsilon:\ X \mapsto 1,\ 1 \mapsto 0$

abstract 1- & 2-mfds.



$\mapsto A$

$\mapsto (A^{\otimes 2} \xrightarrow{m} A)$

etc

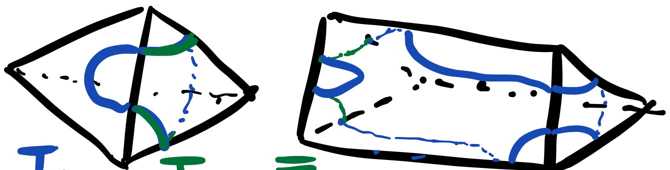
Idea 1: Holography

(B^3, L) $\xmapsto{\text{forget embedding, apply TQFT}}$ $\mathcal{F}(L) \in \Bbbk\text{-mod}$ $\quad \mathbb{Z}$

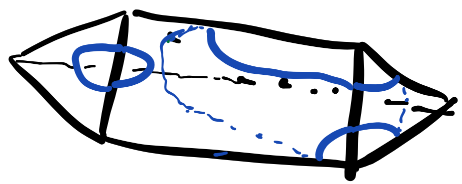
cpt. 1-submfd of $\partial B^3 = S^2$

Mental model: BN dotted cobs in B^3

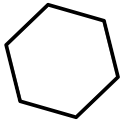
Idea 2: Stratify S^2, $B^3 \rightsquigarrow$ polyhedron, study gluing along faces



glue $\longrightarrow$



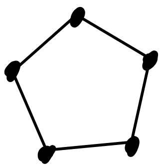
$$F(T_1 \cup T_2) \otimes F(\overline{T}_2 \cup T_3) \longrightarrow F(T_1 \cup T_3)$$

$P =$  $\rightsquigarrow$ $P \times [0,1]$ has natural stratification

„polygon" „prism"

$\rightsquigarrow$ build higher algebraic structures

slides 1-3

kc) $P_k =$ 

2-functor

$BN^{\otimes k} \xrightarrow{BN(P_k)} Mor(Cat_{\mathbb{K}}^{\mathbb{Z}})$

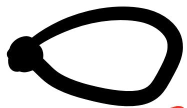
$\underline{n} \longmapsto BN(\underline{n})$

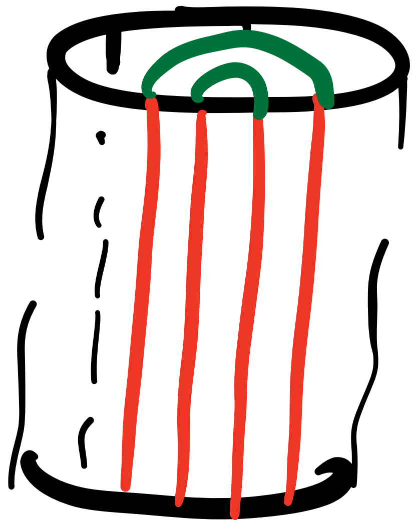
$(\square, \ldots) \longmapsto BN(\underline{n})-BN(\underline{m})$ -bimodule

$\ldots \longmapsto$ bimodule homs.

fix / choose

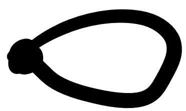
slide 1

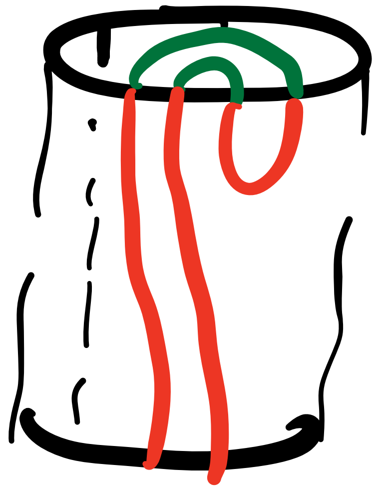
1a) $P_1 =$  $2n$



$2n$

$\rightsquigarrow$ arc ring H^n

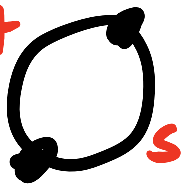
1b) $P_1 =$ 

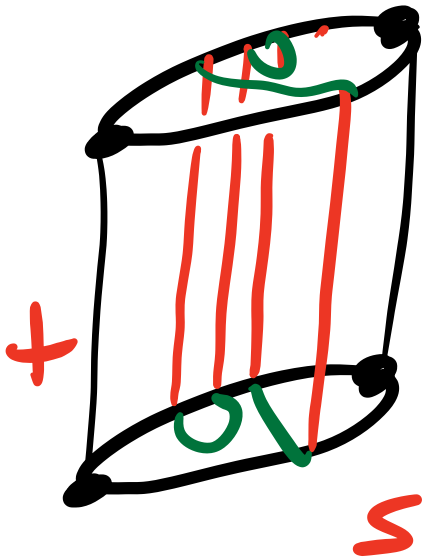


$2m$

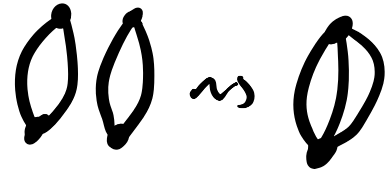
$\rightsquigarrow$ H^n-H^m-bimodule

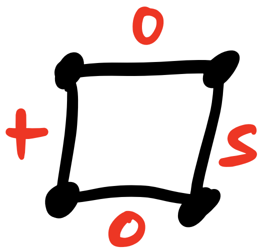
slide 2

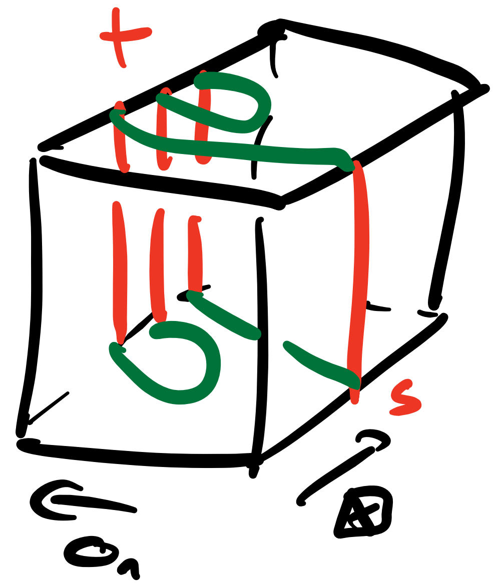
2a) P_2 = 



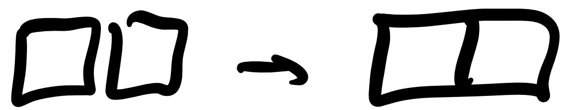
$\rightsquigarrow$ 2-category BN

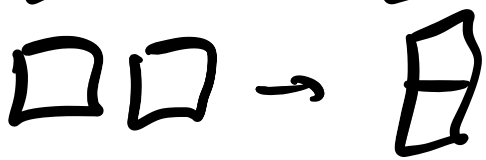
o_1: 

4a) P_2 = 



$\rightsquigarrow$ monoidal 2-category $BN_{\boxtimes}$

o_1: 

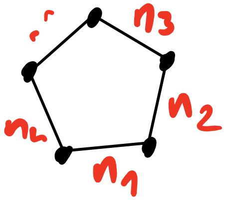
⊠ 

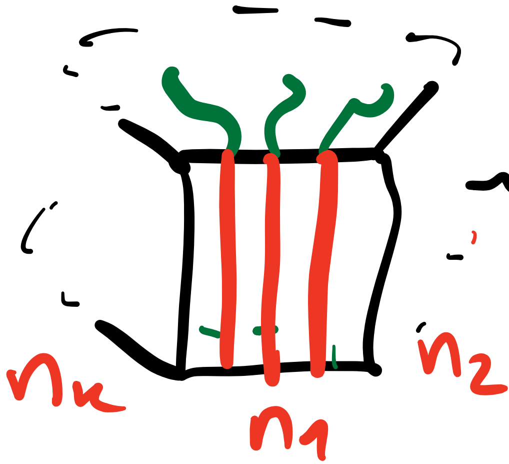
For $\mathbb{Z}[X]/(X^2)$

this categorifies TL over $\mathbb{Z}[q^{\pm 1}]$ with $O = q+q^{-1}$

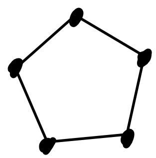
slide 3

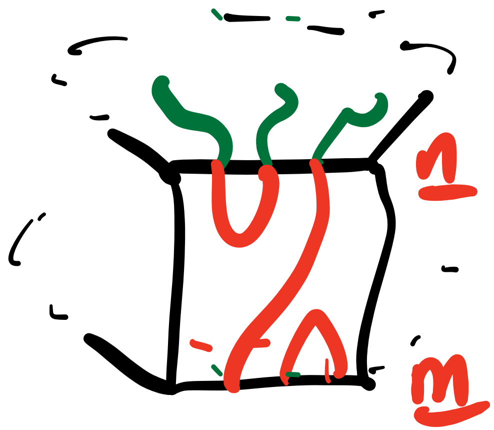
$k \in \mathbb{N}$

k a) $P_k =$ 

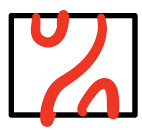


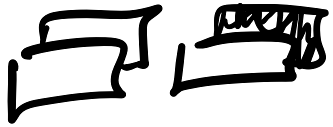
$\rightsquigarrow$ category $BN(n_1, \dots, n_k) =: BN(\underline{n})$

k b) $P_k =$ 



$\rightsquigarrow$ $BN(\underline{n})$ - $BN(\underline{m})$ -bimodule parametrized by k-tuple

( , ...)



Packaging all polygons

$\{BN(P_k)\}_{k \geq 1}$ is

- $\otimes$ ⇝ contract/subdivide edges ⇝ a simplicial object
- $\mathbb{Z}/k\mathbb{Z} \circlearrowright BN(P_k)$ rotate k-gon ⇝ a cyclic object
- gluing two polygons along an edge
  $BN(P_{k+\ell}) \simeq BN(P_{k+1}) \otimes_{BN(P_2)} P(P_{\ell+1})$ ⇝ an algebra for the cyclic A_∞-operad

cf. 2-Segal obj Dyckerhoff-Kapranov

Q ⇝

self-gluing along a pair of edges ⇝ an algebra for the modular operad

≙ a TCFT

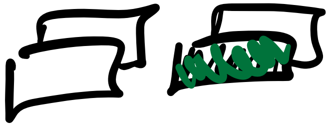
Getzler-Kapranov
Costello
Brochier-Woike

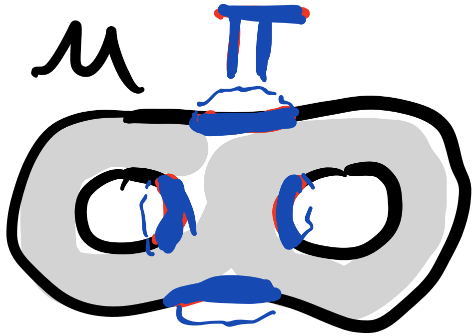
Q: valued where exactly?

Linear OK ✓

or

dg ⇝ $\hat{\otimes}$ for self-gluing

3. The invariant 



Def: A *marked surface* (Σ, M) consists of:

- Σ compact, oriented surface, no closed cpts
- $M \subseteq \partial\Sigma$ open submfd hitting every cpt of $\partial\Sigma$

s.t. $\Pi := \partial\Sigma \setminus M$ is a finite union of closed intervals

Set $BN(\Pi) := BN^{\otimes |\pi_0(\Pi)|}$

↑ $\otimes$ of BN 2-categories, one for each unmarked $\partial\Sigma$-interval

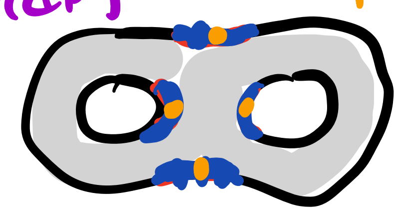
Thm (HRW24)

For every marked surface (Σ, M)

$\exists$ 2-functor $BN(\Pi) \to Mor(dgCat_k^{\mathbb{Z}})$

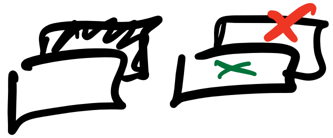
- compatible with gluing & MCG-action
- extending polygon 2-functor
- "bonus statement" for $H^*(\mathbb{CP}^1)$

focus today

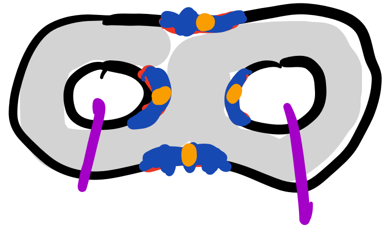
On objects:  $P \subseteq \Pi$ $\mapsto \mathcal{C}(\Sigma, P)$ dg cat

H^0: $BN(\Sigma, P)$ K_0: $TL(\Sigma, P)$

4. Use guide for $\mathcal{C}(\Sigma, P)$

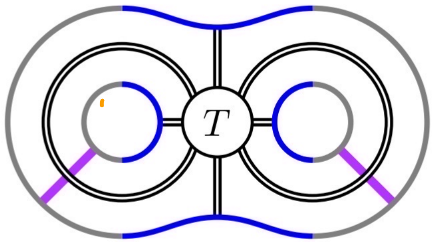


Auxiliary choice: Γ "tesselation"
set of arcs cutting Σ into disks $\Gamma \cap \Pi = \emptyset$

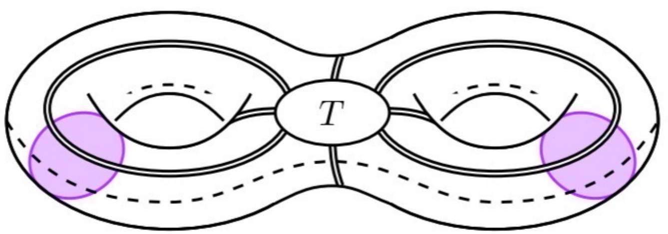


$\mathcal{C}(\Sigma, P, \Gamma)$:

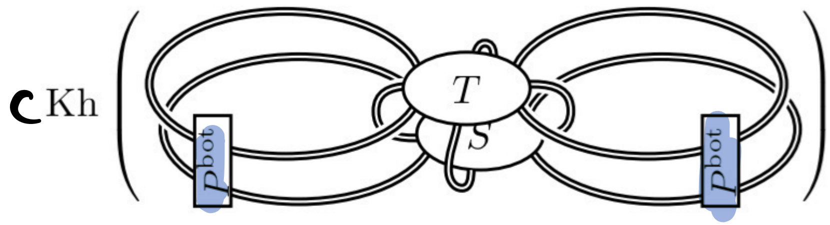
- objects: tangles T in Σ with $\partial T = P$, $T \pitchfork \Gamma$



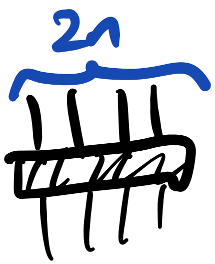
- hom complex $S \to T$: $\mathrm{CRW}(T \cup_{\partial T} \bar{S})$



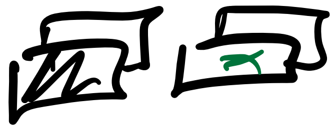
$\rightsquigarrow$



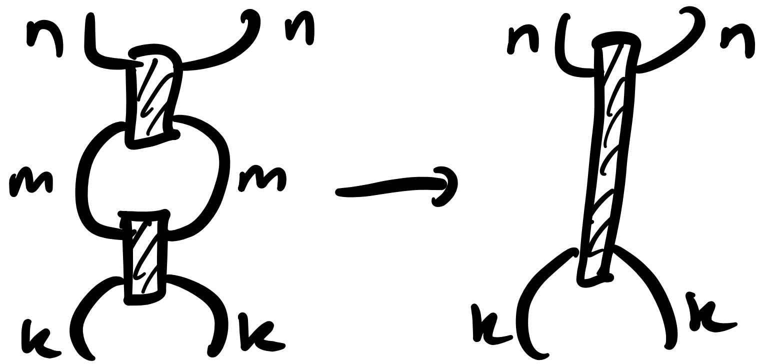
$\rightsquigarrow$ locally fin-dim cohomology!



Rozansky's bottom projector $\sim$ projective resolution of H_n H_n H_n



• composition uses new „sideways" multiplication

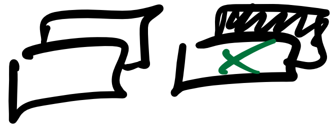


Eilenberg-Zilber shuffle product on Bar-cx

associativity requires care

Idea: Γ cut Σ into polygons

 implements $\hat{\otimes}$ to reassemble



if tesselation ↓

5 Independence of Γ:

- • $e(\Sigma, P, \Gamma) \xrightarrow[\text{coarsening}]{\text{quasi-equiv} \; \simeq} e(\Sigma, P, \Gamma' \{\gamma\})$

Proof idea: cocutting  is htpe when $\otimes$ with projectives $\simeq$ through dg sem

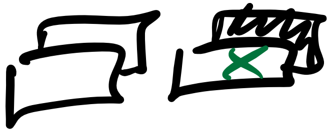
• functors for simultaneous coarsening "commute":

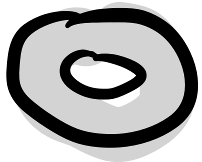
functor $T(\Sigma, \mu) \to$ dg categories
tesselation poset

• take (homotopy) colimit $e(\Sigma, P)$

• $e(\Sigma, P) \simeq e(\Sigma, P, \Gamma)$ since $|N(T(\Sigma, \mu))|$ is contractible

(Hanen 86)

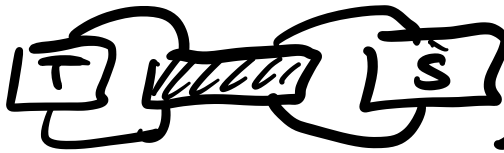
 ← square

Example: $e($  $, \phi) = \mathrm{Tr}(BN)$

dg horizontal trace

Gorsky–Hogancamp–W '20

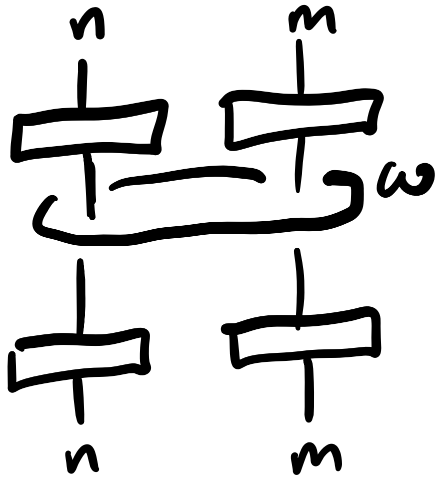
(any number of seams)

$\mathrm{Hom}_{\mathrm{Tr}(BN)}(\hat{S}, \hat{T}) = Kh($  $)$

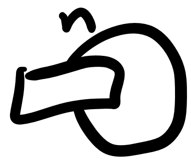
6. Bonus statement for $\mathbb{Z}[x]_{(x^2)}$

Lemma:

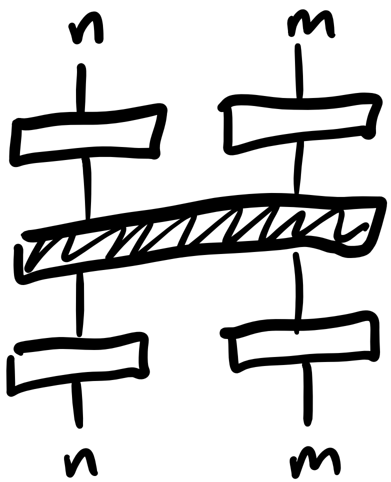
decat



$=$ $\dfrac{\delta_{n,m}}{}$



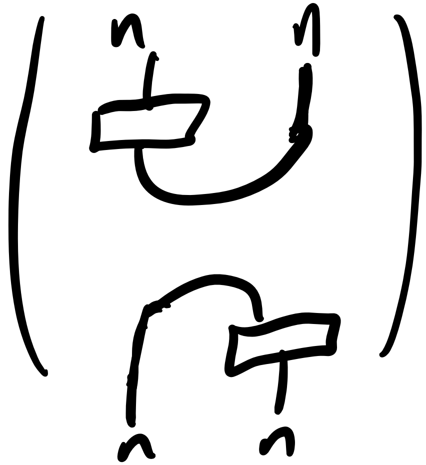
cat:



$\simeq$

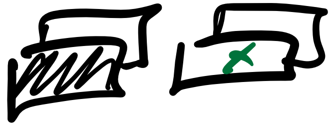
$0 \qquad n \neq m$

$CH_{\mathrm{End}(\,\cdot\,)}$

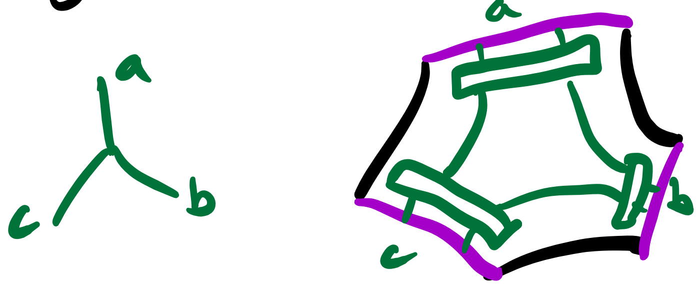


Cooper–Krushkal
categorified Jones-Wenzl
projector

Thm (HRW 24)



1) $\mathcal{C}(\Sigma, P)$ can be completed to $\mathcal{C}^-(\Sigma, P)$ generated by Cooper–Krushkal spin networks



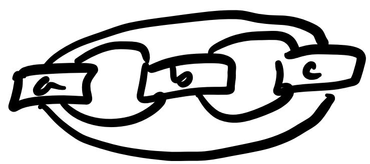
(after choosing triangulation Γ)

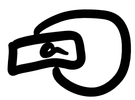
2) $\mathcal{C}(\Sigma, P) \cong \mathcal{C}(\Sigma, P)^{op}$ extends to a symmetrized hom pairing

$$\mathcal{C}^-(\Sigma, P) \times \mathcal{C}^-(\Sigma, P) \xrightarrow{(-,-)} \mathrm{Ch}^-(\Bbbk\text{-mod}^{\mathbb{Z}})$$

for which the spin networks γ are <u>orthogonal</u>

with $\chi((\gamma,\gamma)) := \dfrac{\prod_{\text{vertices in } \gamma} \; {}^{a}\!\!\perp_{b}^{\;c}}{\prod_{\text{edges } a \text{ in } \gamma}}$ $\in \mathbb{Q}[q, q^{-1}]$





continuation of WRT inv.