A braided tensor 2-category from link homology

W. Liu, Mazel-Gee, Stroppel, Wedrich [LMSW]

1. Introduction

1.1 Link invariants from braided categories!

Given: (1) a braided monoidal category $\mathcal{C}$

(2) a dualizable object $c \in \mathcal{C}$

(3) an (appropriately framed) link $L \subseteq \mathbb{R}^3$

Get: a scalar

$$I_{c\in\mathcal{C}}(L) := \left\langle \begin{matrix}\text{evaluate "string diagram } L\text{"}\\ \text{labelled by } c\in\mathcal{C}\end{matrix} \right\rangle \in \mathrm{End}_{\mathcal{C}}(I)$$

Example: [90'] Reshetikhin-Turaev:

For $\mathcal{C} = \mathrm{Rep}^{f.d.}(U_q(\mathfrak{sl}_2))$, $c = V_{fund.}$, this recovers the Jones-polynomial: $J_L(q) := I_{c\in\mathcal{C}}(L) \in \mathrm{End}_{\mathcal{C}}(I) \cong \mathbb{C}[q^{\pm 1}]$

Other invariants for other quantum groups.

Every such link invariant $I_{c\in\mathcal{C}}(L)$ factors through a braid invariant:

Alexander's theorem: Every link is a closure of a braid: β 

$$Br_n \xrightarrow[f]{\text{use braiding of } \mathcal{C}} \mathrm{End}_{\mathcal{C}}(c^{\otimes n}) \xrightarrow[\text{trace}]{\text{use dualizability of } c\in\mathcal{C}} \mathrm{End}_{\mathcal{C}}(I) \qquad I_{c\in\mathcal{C}}(\text{closure}(\beta)) = tr(f(\beta))$$

(In words: technically useful as it separates braiding & dualizability)

Slogan: Interesting braid and link invariants arise in this way from braided monoidal categories!

Not just a question of aesthetic!

$$\begin{tikzcd}
\text{Fact. homology/ skein theory} \arrow[r, squiggly] & \text{Braided monoidal cats} \arrow[dl, leftrightarrow] \arrow[dr, squiggly] & \\
\text{4D w. } \partial \text{ 3D TQFTs} \arrow[rr, squiggly] & & \text{link polynomials}
\end{tikzcd}$$

Focus on a specific universal instance:

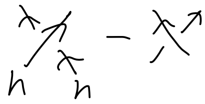
$\overset{\text{"}sl_\infty\text{"}}{\text{HOMFLYPT}}$ polynomial $\underset{\text{variables}}{\overset{\text{specialize}}{\rightsquigarrow}}$ sl_n link polynomials
in two variables

Also arise as above as $\mathrm{Inv}_H(L)$ from a braided category H with a distinguished dualizable object $h \in H$.

Let's describe (parts of) it!

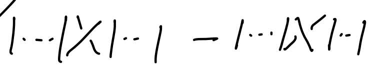
Let H_+ be the full monoidal subcategory of H generated by h.

(Note: $h^* \notin H_+$, so H_+ suffices to recover the braid group action $Br_n \longrightarrow \mathrm{End}_H(h^{\otimes n})$ but not the link invariant.)

Fact: H_+ is the free $\mathbb{Z}[q^{\pm 1}]$-linear braided monoidal cat on an object h satisfying the relation  $= (q-q^{-1}) \uparrow\uparrow$.

Unpacked: H_+ has:

objects: $h^{\otimes n}$ for $n \in \mathbb{N}_0$

$\hom(h^{\otimes n}, h^{\otimes m}) = \begin{cases} 0 & n \neq m \\ \mathbb{Z}[q^{\pm 1}][Br_n] \big/ \end{cases}$  $= (q-q^{-1})\, |\cdots|$ $=: \underset{\substack{\uparrow \\ \text{Hecke algebra}}}{H_n}$ $\quad n = m$

In these terms, the HOMFLYPT-braid group action $Br_n \longrightarrow \mathrm{End}_H(h^{\otimes n}) = H_n$ is the quotient map $Br_n \hookrightarrow \mathbb{Z}[q^{\pm 1}][Br_n] \twoheadrightarrow H_n$.

Observe: $H_n \xrightarrow{q=1} \mathbb{Z}[S_n]$, can think of H_n as a deformation of $\mathbb{Z}[S_n]$.
$(H_+ \longrightarrow \mathbb{Z}[\mathrm{FinSet}^{\simeq}])$

1.2. Categorified link invariants from categorified braided cats?

K = a field of char. 0.

[99] Khovanov homology: $Kh^{i,j}(L) \in Vec_K^{f.d.}$, $i,j \in \mathbb{Z}$

(Homology of a chain complex of graded vector spaces.)

categorifies Jones polynomial:

$$J_q(L) = \chi_{graded}(Kh(L)) = \sum_{i,j} (-1)^i \dim(Kh^{i,j})\, q^j$$

There is a similar "universal" generalization:

$$\underset{\text{[Kh-Rozansky, Dunfield-Gukov-Rasmussen]}}{\overset{\text{triply graded}}{\text{HOMFLYPT homology}}} \xrightarrow[\text{[Rasmussen]}]{\text{spectral sequence}} \underset{\text{[Kh-Rozansky]}}{\overset{\text{bigraded}}{\mathfrak{sl}_n \text{ link homologies}}}$$

Can all be understood as "traces" of categorified braid invariants.

Q1: Do these categorified braid & link invariants arise from a $\underbrace{\text{"categorified" braided monoidal category}}$?

Q2: And what does mean?

vector space-valued invariants instead of numbers.

A2: Braided monoidal K-linear $(\infty,2)$-category.
$(\mathbb{E}_2\text{-})$

hom-categories are K-linear stable ∞-cats.

actually: chain cx valued invariants instead of vector spaces, invariants up to (higher) homotopies.

Again: Not just aesthetic:

$$\begin{tikzcd}
 & \mathbb{E}_2\sim(\infty,2)\text{-category} \arrow[dl, squiggly, "\text{fact. homology} \ ②"'] \arrow[dr, squiggly, "①"] & \\
\text{5D w. } \partial 4D \text{ TQFTs} \arrow[rr, squiggly, "③"] & & \text{Link homologies}
\end{tikzcd}$$

① = order of drawing.

Goal for the rest of this talk:

Describe a k-linear braided monoidal $(\infty,2)$-category $\mathcal{H}_+$ which categorifies H_+, and which gives rise to the braid invariant underlying HOMFLYPT.

(i.e.: apply Grothendieck group K_0 to its stable hom-categories recovers H_+.)

2. Towards a braided $(\infty,2)$-category $\mathcal{H}_+$

2.1. Soergel and Rouquier

Step 1: Explain braid invariant underlying HOMFLYPT homology.

In words: categorifies Braid invariant underlying HOMFLYPT-polynomial:

$$Br_n \longrightarrow H_n = \mathbb{Z}[q^{\pm 1}][Br_n]/\text{relation}$$

Def: K char. 0, $n \geq 0$.
Let $R_n := K[x_1 \ldots, x_n]$ seen as graded algebra with $|x_i| = 2$.

$(\text{grBim}_{R_n}, \otimes_{R_n})$ is an additive monoidal category.
I ignore grading from now on.

For $s_i := (i, i+1) \in S_n$, let $R_n^{s_i} :=$ s_i-invariant polynomials.

Define $B_i := R_n \otimes_{R_n^{s_i}} R_n \in \text{grBim}_{R_n}$.

Define:
(1) The category of Soergel bimodules $SBim_n \subseteq \text{grBim}_{R_n}$ as: smallest full additive, idempotent-complete, monoidal subcategory containing all B_i.

Fact: [Soergel '90s?] $SBim_n$ is additive monoidal K-linear 1-category with $\underset{\uparrow}{Gr}^{split}(SBim_n) = H_n$.
split Grothendieck ring

[The $[B_i] \in Gr^{split}(SBim_n) \simeq H_n$ give the el's $\left(|\cdot\cdot| \times |\cdot\cdot| + q^{-1} |\cdots| \right)$.]
$\rightsquigarrow |\cdot\cdot| \times || = [B_i] - q^{-1}[id]$ not represented by an object of $SBim_n$

(2) $\mathbb{H}_n := K^b(SBim_n) := \left\{ \begin{matrix} \text{bounded} \\ \text{chain cx in } SBim_n \\ \text{chain maps} \\ \text{chain homotopies} \\ \vdots \end{matrix} \right\}$ a stable k-linear monoidal $(\infty,1)$-cat. with $\underset{\uparrow}{Gr}(\mathbb{H}_n) = H_n$
Groth.-ring

$$F(\sigma) = \left(\underset{\underset{R \otimes_{R^i} R}{\sim}}{B_i} \longrightarrow \mathrm{id}[-1] \right) \qquad R \otimes_{R^i} R \xrightarrow{m} R$$

Theorem [Rouquier '04]

There is a monoidal functor $Br_n \xrightarrow{F} \mathbb{H}_n$ which categorifies $Br_n \to H_n$.

This is the braid group action underlying HOMFLYPT homology.

[Main Thm $\Rightarrow$ $\mathbb{H}_n = \mathrm{End}_{\mathbb{H}_+}(h^{\otimes n})$ and F is braid action induced from braiding on $\mathbb{H}_+$.]

Remark: Recall how H_n was a "deformation" of $\mathbb{Z}[S_n]$. Analogous:

A deformation of boring action:

$$\begin{tikzcd} Br_n \arrow[r, "F"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & \mathbb{H}_n \subseteq K^b(grBim_{K[x_1 \cdots x_n]}) \arrow[d, "\Pi_\infty"] \\ S_n \arrow[r] & D^b(Bim_{K[x_1 .. x_n]}) \end{tikzcd}$$

Π_∞ = consider bimodule up to quasi-isomorphism

$\sigma \longmapsto$ bimodule induced by algebra iso $K[x_1 .. x_n] \circlearrowright \sigma$ = swap variables

2.2. The main theorem

Define
$$\mathbb{H}_+ := \begin{cases} \text{obj} = \mathbb{N}_0 \\ \hom(m,n) = \begin{cases} \mathbb{H}_n & n = m \\ 0 & \text{else} \end{cases} \end{cases}$$
K-linear $(\infty,2)$-cat.

Thm A [w. LMSW] The functors $\mathbb{H}_n \times \mathbb{H}_m \xrightarrow{\otimes_K} \mathbb{H}_{n+m}$ induce a monoidal structure on $\mathbb{H}$.

The functor
$$\mathbb{H} \longrightarrow D(\text{Morita}) := \begin{cases} \text{obj: graded (flat) } K\text{-algebras} \\ \text{mor: } D({}_A grbim_B)^{B\text{-perf}} \end{cases}$$

$\otimes = \otimes_K$

(perfect as right B-module)

obj: $n \longmapsto K[x_1 .. x_n]$

hom-cats: $\mathbb{H}_n \subset K^b(grbim_{K[x_1 .. x_n]}) \xrightarrow{\Pi_\infty} D(grbim_{K[x_1 .. x_n]})$

is monoidal.

Think: $\mathbb{H} \to D(\text{Morita}) \subseteq \left\{ \begin{matrix} \text{stable } K\text{-linear } \infty\text{-cats} \\ \text{with } R\text{-action} \end{matrix} \right\}$ is a fiber functor.

Main Theorem: (w LMRSW):

contractible ∞-groupoid

$\exists!$ braided structure on $\mathbb{H}$ s.t.

(1) braiding $(1\otimes 1 \to 1\otimes 1) \in \hom_{\mathbb{H}}(2,2) = \mathbb{H}_2$ is the Rouquier complex $F(\times)$.

(2) $\mathbb{H} \xrightarrow{\pi_\infty} D(\text{Morita})$ is braided. (ie $\pi_\infty(\text{braiding}) =$ "boring" symmetric braiding on $D(\text{Morita})$)

Note: cf. with Reps of quantum groups:

$\mathrm{Rep}(U_q\mathfrak{g}) \to \mathrm{Vec}$ is monoidal & faithful but not braided

$\mathbb{H} \longrightarrow 2\mathrm{Vec}$ braided but not faithful but

$\mathrm{SBim} \hookrightarrow \mathbb{H} \longrightarrow 2\mathrm{Vec}$ is faithful but not braided