ZMP Seminar: *Kh* for Tangles

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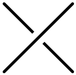
Outline

1. Review of Khovanov homology
2. Arc Algebras
3. Extending *Kh* to Tangles
4. Bar-Natan's Picture World
5. Interpreting Khovanov's Bimodules
6. Thanks!

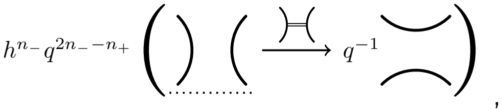
Review of *Kh*

To an oriented link diagram D, Khovanov (Kho00) associates a chain complex $\mathcal{C}_{Kh}(D)$ as follows:

1. Label all of the crossings of D as $1, \ldots, c$.
2. Form the cube of complete resolutions of D by replacing each crossing



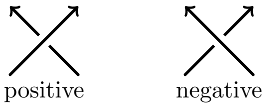
with the formal bigraded "complex"



where powers of h and q indicate shifts in the *homological* and *quantum* gradings, the underlined term is in homological grading $h = 0$, $n_\pm$ are 1 or 0 depending on whether the crossing is *positive* or *negative*, and the "differential" is a *saddle cobordism*.

Review of *Kh*

3. Here, a crossing is said to be positive or negative according to the following convention.

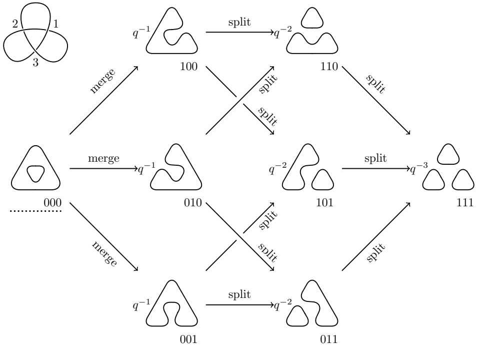


4. In both cases, the first term in our formal complex is called the 0-*resolution* of the crossing and the second is called the 1-*resolution*.
5. This numbering allows us to identify the complete resolutions $D_{\boldsymbol{v}}$ of the diagram D with the vertices $\boldsymbol{v}$ of the c-dimensional cube $\mathbf{2}^c$, where $\mathbf{2} = \{0, 1\}$.
6. Each edge of the cube of resolutions is of the form $D_{\boldsymbol{v}} \to D_{\boldsymbol{w}}$, where $\boldsymbol{v}$ and $\boldsymbol{w}$ differ in a single entry and $|\boldsymbol{v}| = |\boldsymbol{w}| - 1$ and is given by either a *merge* of two circles into one or a *split* of one circle into two.

Review of *Kh*

Example

Up to an overall quantum grading shift of q^{-3}, the cube of resolutions for the right-handed trefoil is given by



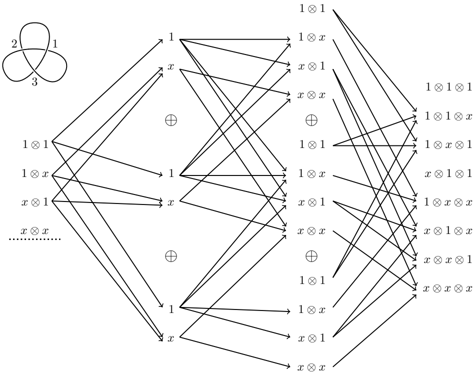
Review of Kh

7. For R a commutative, unital, ring (e.g. $\mathbb{C}$), apply the TQFT associated to the rank 2 commutative Frobenius algebra $A := R[x]/(x^2)$, with comultiplication given by $\Delta(1) = 1 \otimes x + x \otimes 1$ and $\Delta(x) = x \otimes x$, to obtain a commuting cube of maps of R-modules.
8. Caution: here, the quantum gradings of 1 and x are $\mathrm{gr}_q(1) = -1$ and $\mathrm{gr}_q(x) = 1$.
9. Insert signs so that each face *anticommutes*.
10. Sum along homological gradings (given up to global homological grading shifts by the quantities $|\boldsymbol{v}| = v_1 + \cdots + v_c$ for $\boldsymbol{v} \in \mathbf{2}^c$) to obtain the Khovanov (co)chain complex $\mathcal{C}_{Kh}(D; R)$.
11. Note: after this slide, we will assume that R is a field $R = \mathbb{F}$, for simplicity, and drop it from our notation, but the resulting invariant does depend nontrivially on the choice of R.

Review of *Kh*

Example

Up to signs and quantum grading shifts, the complex $\mathcal{C}_{Kh}(\mathrm{RHT})$, with bases chosen s.t. right-most tensorands split under Δ, is given by



Review of *Kh*

12 Up to isomorphism, the homology $Kh(L) := H_*(\mathcal{C}_{Kh}(D))$ of this chain complex, as a bigraded R-module

$$Kh(L) = \bigoplus_{i,j\in\mathbb{Z}} Kh^{i,j}(L), \tag{1}$$

with i and j the homological and quantum gradings of the summand $Kh^{i,j}(L)$, respectively, is an isotopy invariant of the link L represented by the diagram D

13 which recovers the (unnormalized) Jones polynomial as its *graded Euler characteristic*:

$$\widehat{J}(L) = \chi_q(Kh(L)) = \sum_{i,j\in\mathbb{Z}} (-1)^i q^{-j} \dim Kh^{i,j}(L). \tag{2}$$

Review of *Kh*

Exercise

Check that the Khovanov homology of the right-handed trefoil with $\mathbb{Z}$-coefficients is given as a bigraded abelian group as follows.

$$
Kh^{i,j}(\mathrm{RHT}) = \begin{array}{c|cccc}
-j & & & & \\
9 & & & & \mathbb{Z} \\
7 & & & & \mathbb{Z}/2 \\
5 & & & \mathbb{Z} & \\
3 & \mathbb{Z} & & & \\
1 & \mathbb{Z} & & & \\
\hline
 & 0 & 1 & 2 & 3 \quad i
\end{array} \tag{3}
$$

Review of *Kh*

14. $Kh(-)$ is *functorial* with respect to link cobordisms in $S^3 \times I$, where I is a closed interval, and detects several families of knots, as well as a wealth of information about so-called "exotic" phenomena in dimension 4 (see (HS22) for a recent example)
15. but, as the previous example illustrates, computations of Khovanov homology grow very quickly (exponentially) in the number of crossings in the diagram D.

Extending *Kh*

When one has a homological invariant of links (or 3-manifolds) which is difficult to compute when the input data D is large, one might look for the following:

- A decomposition $D = D_1 \cup_\partial D_2 \cup_\partial \cdots \cup_\partial D_k$ into smaller data, where $D_i \cup_\partial D_{i+1}$ means we are "gluing" the two data D_i and D_{i+1} along some common "boundary data".
- Algebras $\mathcal{A}_1, \ldots, \mathcal{A}_{k-1}$ associated to the boundary data of the D_i
- and complexes of (bi)modules ${}_{\mathcal{A}_{i-1}}\mathcal{C}(D_i)_{\mathcal{A}_i}$
- such that

$$\mathcal{C}(D) \simeq \mathcal{C}(D_1) \otimes_{\mathcal{A}_1} \mathcal{C}(D_2) \otimes_{\mathcal{A}_2} \cdots \otimes_{\mathcal{A}_{k-1}} \mathcal{C}(D_k). \tag{4}$$

Extending *Kh*

For an arbitrary invariant, finding such a decomposition might not be possible or, if it is, may require more complicated algebraic objects — such as A_∞-algebras and (bi)modules, as is the case for Heegaard Floer invariants of 3-manifolds*

*Come to my next talk in the QTCat Seminar if you want to know more!

— but this recipe works right out of the box in the case of Khovanov homology. In this case, the input data for Khovanov homology was a link diagram D so the natural candidates for the D_i are *tangle diagrams*, in this case obtained by cutting D into slices along parallel straight lines** which avoid the crossings, so the boundary data are sets of even numbers of points.

**This is slightly unnatural; we'll come back to it later.

Crossingless matchings

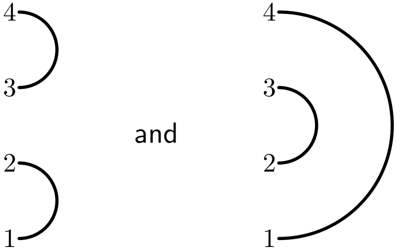
For $k \geq 1$, let $[k] = \{1, 2, \ldots, k\}$. A *matching* on $2n$ points is a 2-to-1 function $\mu : [2n] \to [n]$. We say μ is *crossingless* if $i < j < k < \ell$ and $\mu(i) = \mu(k)$ implies that $\mu(j) \neq \mu(\ell)$. The crossingless condition is easily understood graphically: we can represent a matching μ by a planar diagram consisting of $2n$ points on a fixed line and smooth embeddings of n intervals joining them*. The matching μ is then crossingless if and only if the arcs of the corresponding diagrams can be arranged to have no points of intersection.

*Up to endpoint-fixing planar isotopy of the individual arcs.

Crossingless matchings

Example

The crossingless matchings on 4 points are completely determined by the diagrams



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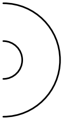
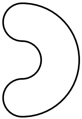
We can omit the numbering of the endpoints by declaring that they are numbered from bottom to top.

Khovanov's arc algebras

Let $\mathfrak{C}_n$ be the set of crossingless matchings on $2n$ points. Given $a, b \in \mathfrak{C}_n$, let $a^!b$ be the configuration of circles in the plane obtained by flipping the diagram for a along the vertical axis and gluing it to the diagram for b along their common boundary.

Example

If $a =$  and $b =$  , then $a^!b =$  (6)

Khovanov's arc algebras

The *arc algebra on* $2n$ *points* is the algebra H_n given as a vector space by $H_0 = \mathbb{F}$ and by

$$H_n = q^n \bigoplus_{a,b \in \mathfrak{C}_n} \mathcal{C}_{Kh}(a^! b), \tag{7}$$

if $n > 0$, or more concretely by

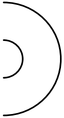
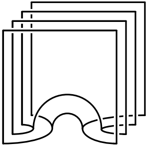
$$H_n = q^n \bigoplus_{a,b \in \mathfrak{C}_n} A^{\otimes \# \pi_0(a^! b)}, \tag{8}$$

i.e. generators of H_n are configurations of planar circles obtained by pairing diagrams for crossingless matchings, together with a labeling of each circle by either 1 or x.

Khovanov's arc algebras

Multiplication on H_n is given by the maps $\mathcal{C}_{Kh}(a^! b) \otimes_{\mathbb{F}} \mathcal{C}_{Kh}(b^! c) \to \mathcal{C}_{Kh}(a^! c)$ induced by the *minimal saddle cobordisms* $\Sigma_{a,b,c} : a^! b \sqcup b^! c \to a^! c$ which is given by the identity cobordisms $a^! \times I$ and $c \times I$ on a and c and, on $b \sqcup b^!$, is given by the cobordism $\Sigma_b : b \sqcup b^! \to \mathrm{id}_{2n}$, where id_{2n} consists of $2n$ stacked horizontal lines, given by surgery along n 0-spheres with one point on each arc in b and one on the corresponding arc in $b^!$ exactly once. All other products are zero.

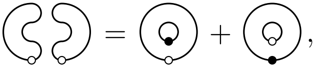
Example

If $b =$  , then $\Sigma_b =$  (9)

Khovanov's arc algebras

Example

If we declare that an empty dot on a circle represents the label 1 and a filled dot represents the label x, then, in the arc algebra H_2, we have



(10)

since the minimal saddle cobordism first merges the two circles and then splits the single merged circle into two, so the corresponding sequence of maps on elements of tensor powers of A is

$$1 \otimes 1 \overset{\text{merge}}{\longmapsto} 1 \overset{\textit{split}}{\longmapsto} 1 \otimes x + x \otimes 1. \tag{11}$$

Khovanov's arc algebras

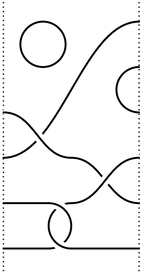
1. Since the maps $\mathcal{C}_{Kh}(a^!b) \otimes_{\mathbb{F}} \mathcal{C}_{Kh}(b^!c) \to \mathcal{C}_{Kh}(a^!c)$ defining multiplication on H_n are defined by applying a topological quantum field theory to a cobordism of configurations of circles, the multiplication on H_n is automatically associative.
   - TQFTs are insensitive to the heights of saddles (1-handles) and we can commute saddles which are far away! We'll call this *far-commutation*.
2. If $\Sigma : L_1 \to L_2$ is a link cobordism, then the induced map $Kh(\Sigma) : Kh(L_1) \to Kh(L_2)$ shifts quantum gradings by $-\chi(\Sigma)$. The quantum grading shift of q^n in the definition of H_n then ensures that multiplication is a graded map.
3. The minimal, mutually orthogonal, idempotents in H_n are the elements 1_a, for $a \in \mathfrak{C}_n$, given by the configuration $a^!a$ with each circle labeled by 1.

The tangle invariant

A $(2m, 2n)$-*tangle diagram* T is an immersion $\sqcup^k(I, \partial I) \hookrightarrow (I \times \mathbb{R}, \{0,1\} \times \mathbb{R})$ together with crossing data at each self-intersection point such that $\partial T = \{0\} \times [2m] \cup \{1\} \times [2n]$.

Example

The following is an example of a $(4, 6)$-tangle diagram.



(12)

The tangle invariant

In (Kho02), Khovanov associates to each $(2m, 2n)$-tangle diagram T a complex $\mathcal{C}_{Kh}(T)$ of *sweet* (H_m, H_n)-bimodules — here, 'sweet' means finitely generated projective as a left H_m-module and as a right H_n-module but not necessarily as an $H_m \otimes H_n^{\mathrm{op}}$-module. As a chain complex of vector spaces, this complex is defined by

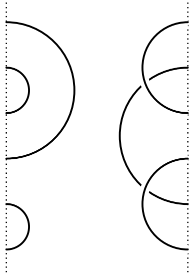
$$\mathcal{C}_{Kh}(T) = \bigoplus_{a \in \mathfrak{C}_m, b \in \mathfrak{C}_n} q^* \mathcal{C}_{Kh}(a^! T b), \tag{13}$$

where the notation q^* means that, for a vertex $\boldsymbol{v} \in \mathbf{2}^c$, we endow the summand $\mathcal{C}_{Kh}(a^! T_{\boldsymbol{v}} b)$ with an additional quantum grading shift by $n - |\boldsymbol{v}|$.

The tangle invariant

Remark

The bimodules $\mathcal{C}_{Kh}(T)$ arising in this way which *are* biprojective come from diagrams of *split* tangles, i.e. those which can be isotoped in $I \times D^2$ to be disjoint from $\{\frac{1}{2}\} \times D^2$. For example



(14)

is a diagram for a split tangle.

The tangle invariant

The actions of the algebras are given by the maps on Khovanov complexes associated to the minimal saddle cobordisms $z^! a \sqcup a^! Tb \to z^! Tb$ and $a^! Tb \sqcup b^! c \to a^! Tc$. As for associativity of multiplication in the arc algebras, by far-commutation of saddles, these actions commute with the differentials on the summands, making $\mathcal{C}_{Kh}(T)$ into a complex of bimodules.

Invariance and gluing

Theorem (Khovanov (Kho02))

Up to chain homotopy equivalence, $\mathcal{C}_{Kh}(T)$ is invariant under endpoint-fixing Reidemeister moves. If T is a $(2m, 2n)$-tangle diagram and T' is a $(2n, 2o)$-tangle diagram, then there is an isomorphism of complexes of (H_n, H_o)-bimodules

$$\mathcal{C}_{Kh}(T) \otimes_{H_n} \mathcal{C}_{Kh}(T') \cong \mathcal{C}_{Kh}(TT'). \tag{15}$$

The first part of this theorem tells us that the chain homotopy type of $\mathcal{C}_{Kh}(T)$ is an isotopy invariant of the tangle in $I \times D^2$ represented by T.

Invariance and gluing

Sketch.

To check Reidemeister-invariance, one observes that the same subcomplexes of $\mathcal{C}_{Kh}(T)$ that arise when checking the invariance of ordinary Khovanov homology are also sub-bimodules of $\mathcal{C}_{Kh}(T)$. To see that gluing holds, note that there is a commutative diagram

$$\begin{array}{ccc} \bigoplus_{a,b,c,d} \mathcal{C}_{Kh}(a^! T b) \otimes \mathcal{C}_{Kh}(c^! T' d) & \xrightarrow{\sigma} & \mathcal{C}_{Kh}(TT') \\ \downarrow & \nearrow & \\ \mathcal{C}_{Kh}(T) \otimes_{H_n} \mathcal{C}_{Kh}(T') & & \end{array} , \tag{16}$$

where σ is the map induced by minimal saddle cobordisms in the middle. One can check that σ is H_n-middle-linear and, hence, it descends to a map $\bar{\sigma} : \mathcal{C}_{Kh}(T) \otimes_{H_n} \mathcal{C}_{Kh}(T') \to \mathcal{C}_{Kh}(TT')$ and far-commutation implies that $\bar{\sigma}$ is a chain map.

Invariance and gluing

Sketch Ctd.

Lastly, one can check that $\overline{\sigma}$ is a vector space isomorphism by first restricting to the summands

$$\bigoplus_b \mathcal{C}_{Kh}(a^! T b) \otimes_{H_n} \mathcal{C}_{Kh}(b^! T' c) \tag{17}$$

for a and b fixed, then reducing to the case that $a^! T$ has no crossings or closed components, i.e. $a^! T = e^!$ for some $e \in \mathfrak{C}_n$, in which case

$$\begin{aligned} \bigoplus_b \mathcal{C}_{Kh}(a^! T b) \otimes_{H_n} \mathcal{C}_{Kh}(b^! T' c) &= \bigoplus_b \mathcal{C}_{Kh}(e^! b) \otimes_{H_n} \mathcal{C}_{Kh}(b^! T' c) \\ &= \bigoplus_b 1_e H_n \otimes_{H_n} \mathcal{C}_{Kh}(b^! T' c) \\ &= \mathbb{F} 1_e \otimes_{\mathbb{F}} \mathcal{C}_{Kh}(e^! T' c). \end{aligned} \tag{18}$$

Invariance and gluing

Sketch Ctd.

Finally, the saddle cobordism $e \sqcup e^! \to \mathrm{id}_{2n}$ induces an isomorphism $\mathbb{F}1_e \otimes_{\mathbb{F}} \mathcal{C}_{Kh}(e^! T' c) \cong \mathcal{C}_{Kh}(e^! T' c)$. □

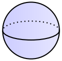
Bar-Natan's picture world

Demanding that we cut a link diagram into $(2m, 2n)$-tangle diagrams is a bit artificial. Why can't we cut a link diagram into other types of local pieces, like tangles in a disk? We can, with Bar-Natan's category of dotted cobordisms (BN05)!

1. Given a collection of evenly many (or zero) points $\boldsymbol{p}$ in ∂D^2, Bar-Natan constructs a category $\mathcal{C}ob_\bullet(\boldsymbol{p})$ whose objects are finite-direct sums of formally q-graded crossingless tangles $T \subset D^2$ with $\partial T = \boldsymbol{p}$. Note: when $\boldsymbol{p}$ is empty, we denote it by $\boldsymbol{\varnothing}$ to distinguish it from the empty manifold $\varnothing$.
2. The morphisms in $\mathcal{C}ob_\bullet(\boldsymbol{p})$ are matrices of finite sums of cobordisms (with some finite number of dots on them) in $D^2 \times I$ which are *cylindrical* near $\boldsymbol{p}$, modulo the following local relations.

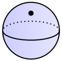
Bar-Natan's picture world

3. Sphere:



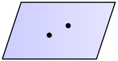
$= 0$

4. Dotted Sphere:



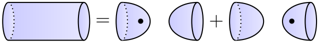
$= 1$

5. Two Dots:



$= 0$

6. Neck Cutting:



Bar-Natan's picture world

More explicitly, these relations tell us that

1. If a cobordism in this category has a connected component consisting of a sphere, then we may replace it with the cobordism obtained by deleting the sphere, at the expense of multiplying the cobordism by 0 or 1, depending on whether the sphere is undotted or dotted, respectively.
2. If a cobordism has two dots on a single component, then it is equivalent to the zero morphism.
3. We may reduce the genus of a cobordism in this category by cutting an annulus $I \times S^1$ along $\{\frac{1}{2}\} \times S^1$ and gluing a disk on either side, at the expense of summing over the two ways of inserting a dot on either side of the circle on which this surgery is performed.

Bar-Natan's picture world

Bar-Natan then shows that the homotopy type of the flattened cubical complex $[\![T]\!]$ in $\mathcal{C}ob_{\bullet}(\boldsymbol{p})$ obtained by applying the same cube-of-resolutions construction used to define $\mathcal{C}_{Kh}$ to a tangle diagram $T \subset D^2$ with $\partial T = \boldsymbol{p}$ is an isotopy invariant of the tangle represented by T and compose in a natural way under a "planar algebraic" notion of tensor product. The later statement requires tools from the theory of colored operads to make precise, however the following theorems make this approach especially useful for computations and tell us that these tangle invariants recover Khovanov homology in a natural way.

Bar-Natan's picture world

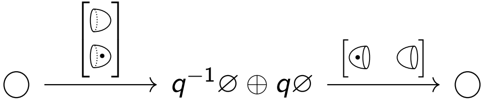
Theorem (Delooping, Bar-Natan (BN07))

The object in $\mathcal{C}ob_{\bullet}(\varnothing)$ given by a single circle is isomorphic to the object $q^{-1}\varnothing \oplus q\varnothing$.

Bar-Natan's picture world

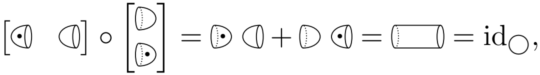
Proof.

The two maps



(19)

determine mutually inverse isomorphisms $\bigcirc \leftrightarrow q^{-1}\varnothing \oplus q\varnothing$ in $\mathcal{C}ob_{\bullet}(\varnothing)$. To see this, we compute



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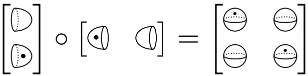
by the neck cutting relation. □

Bar-Natan's picture world

Proof.

Next, we compute

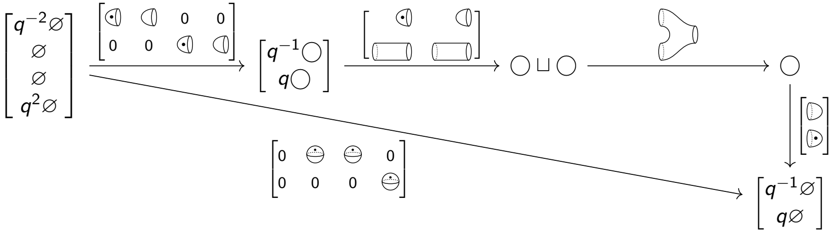


$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \mathrm{id}_{q^{-1}\varnothing \oplus q\varnothing} \qquad (21)$$

by the dotted sphere and relations. □

Bar-Natan's picture world

As an application, note that there is a commutative diagram (using column vector notation for direct sums in the interest of saving space)



(22)

in $\mathcal{C}ob_{\bullet}(\varnothing)$ which makes manifest the multiplication structure on $A(\bigcirc) \cong q^{-1}\mathbb{F} \oplus q\mathbb{F}$. Note that applying the TQFT associated to A to the empty 1-manifold $\varnothing$ in the plane yields $\mathbb{F}$. Caution: Bar-Natan uses the opposite quantum grading convention to the one used in (Kho02).

Bar-Natan's picture world

Exercise

Find a commutative diagram in $\mathcal{C}ob_{\bullet}(\varnothing)$, using delooping to remove all circle components as in the diagram above, encoding the comultiplication structure on A.

Bar-Natan's picture world

Theorem (Gaussian elimination, Bar-Natan (BN07))

Let $\varphi : b_1 \to b_2$ be an isomorphism in an additive category $\mathcal{C}$, then any chain complex in the matrix category $Mat(\mathcal{C})$ containing a four-term segment of the form

$$\cdots\ A \xrightarrow{\begin{bmatrix}\alpha \\ \beta\end{bmatrix}} \begin{bmatrix} b_1 \\ B \end{bmatrix} \xrightarrow{\begin{bmatrix}\varphi & \delta \\ \gamma & \varepsilon\end{bmatrix}} \begin{bmatrix} b_2 \\ C \end{bmatrix} \xrightarrow{\begin{bmatrix}\zeta & \eta\end{bmatrix}} D\ \cdots \tag{23}$$

is homotopy equivalent to the one in which this segment has been replaced by the simplified complex

$$\cdots\ A \xrightarrow{\beta} B \xrightarrow{\varepsilon - \gamma\varphi^{-1}\delta} C \xrightarrow{\eta} D\ \cdots . \tag{24}$$

Bar-Natan's picture world

Combining these, Bar-Natan gives an algorithm for computing Khovanov homology with sub-exponential, rather than exponential, complexity.

So why should we still care about Khovanov's bimodule invariants?

If Bar-Natan's approach is so useful, why should we still want to consider bimodules?

1. It turns out that the homotopy category $\mathcal{K}_n$ of complexes of graded H_n-modules categorifies the space of $U_q(\mathfrak{sl}_2)$-invariants of $V^{\otimes n}$, where V is the fundamental representation.
2. For $(2m, 2n)$-tangle diagrams, we get interesting functors $\mathcal{C}_{Kh}(T) \otimes_{H_n} -$ from $\mathcal{K}_n \to \mathcal{K}_m$.
3. If $L = T_1 \cup_\partial \cdots \cup_\partial T_k$, then the decomposition $\mathcal{C}_{Kh}(L) \cong \mathcal{C}_{Kh}(T_1) \otimes_{H_{n_1}} \otimes \cdots \otimes_{H_{n_k}} \mathcal{C}_{Kh}(T_k)$ categorifies the decomposition of the Jones polynomial into $\mathfrak{sl}_2$ Reshetikhin-Turaev invariants of tangles.

So why should we still care about Khovanov's bimodule invariants?

4. $\mathcal{C}_{Kh}(T)$ extends to a 2-functor from the 2-category of tangles and tangle cobordisms to the 2-category of complexes of bimodules over the algebras H_n (see (Kho02) and (Cap08)).
5. For $n > 0$, $HH(H_n)$ is infinite-dimensional (but finite-dimensional in each homological grading), so H_n has lots of nontrivial deformations.
6. There is an analogue of H_n constructed as the endomorphism A_∞-algebra of a collection of Lagrangians in the Fukaya category of the (n, n)-nilpotent slice. This A_∞-algebra is formal (AS16), as are certain key A_∞-bimodules over it (AS18) and, consequently, a symplectic analogue of Khovanov homology coincides with ordinary Khovanov homology.

So why should we still care about Khovanov's bimodule invariants?

7. For any $(2n, 2n)$-tangle diagram T, the Hochschild homology $HH(\mathcal{C}_{Kh}(T))$ of H_n with coefficients in the bimodule $\mathcal{C}_{Kh}(T)$ is an invariant of the closure $\overline{T}$ of T in $S^1 \times S^2$.
8. The invariants in the previous bullet point are expected to be related to a new $(3 + \varepsilon)$-TFT (see Paul's talk on 25.01.2024).

Thank you!

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