ZMP Seminar: Knot Homology

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Part 1: Khovanov's categorification of the Jones polynomial

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Plan

1. Knots, links, and their invariants
   - The Jones polynomial
2. Khovanov's categorification of the Jones polynomial
3. Plan for the seminar

What is ... ?

Definition (Knots)

A knot is an equivalence class of smooth embeddings $K\colon S^1 \to S^3$.

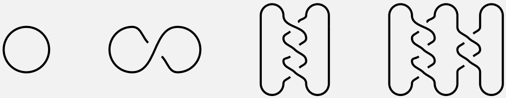
Equivalence $K_1 \sim K_2 \quad \Longleftrightarrow \quad K_2 = \varphi \circ K_1$ for some $\varphi \in \mathrm{Diff}^+(S^3)$.

Definition (Links)

A link is an equivalence class of smooth embeddings $L\colon S^1 \sqcup \cdots \sqcup S^1 \to S^3$.

Definition (Link diagrams)

A link diagram for a link L is a generic projection of $\mathrm{im}(L) \subset S^3 \to \mathbb{R}^2$.



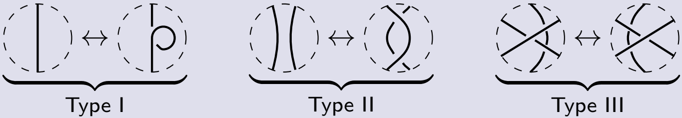
Typical questions about knots and links

- Is a given knot the unknot? Unknot detection.
- Are two given knots the same?
- What is the minimal number of crossings in a diagram of a given knot? Crossing number.
- What is the infimal length of a rope of diameter 1 into which a given knot can be tied? Rope length.
- What is the minimal possible genus of an oriented surface $S \subset S^3$ bounded by a given knot? Knot genus.
- What is the minimal possible genus of an oriented smooth surface $S \subset B^4$ bounded by a given knot in $S^3 = \partial B^4$? Smooth slice genus.
- ...

Towards computable invariants

Theorem (Reidemeister)

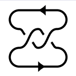
Any two diagrams of a link can be obtained from each other through planar isotopy and a finite sequence of Reidemeister moves:



A link is an equivalence class of link diagrams modulo Reidemeister moves.

The Jones polynomial

Algorithm (State sum for the Jones polynomial)

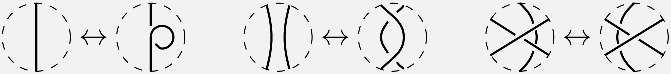
- Input: a diagram of an oriented knot or link, for example: 
- Step 1: count $n_+ = \#($$)$ and $n_- = \#($$)$
- Step 2: expand in states  $\mapsto$  $- q$ , rewrite  $\mapsto q + q^{-1}$
- Output: the coefficient of the empty diagram times $(-1)^{n_-} q^{n_+ - 2n_-}$

Example (The Jones polynomial of the Hopf link)

Theorem

The Jones polynomial is an invariant of oriented links.

Proof idea: the output of the algorithm remains constant under



Physics of the Jones polynomial

Theorem (Witten 1989)

The Jones polynomial arises as vacuum expectation value of Wilson loops in the vector representation in $\mathrm{SU}(2)$ Chern-Simons theory on S^3.

This can be significantly generalized by varying:

- the gauge group, e.g. to $\mathrm{SU}(N)$ for $N \geq 3$
- representations of the gauge group on link components $\rightsquigarrow$ colors
- the ambient 3-manifold S^3
- boundary conditions $\rightsquigarrow$ tangles (knot fragments)

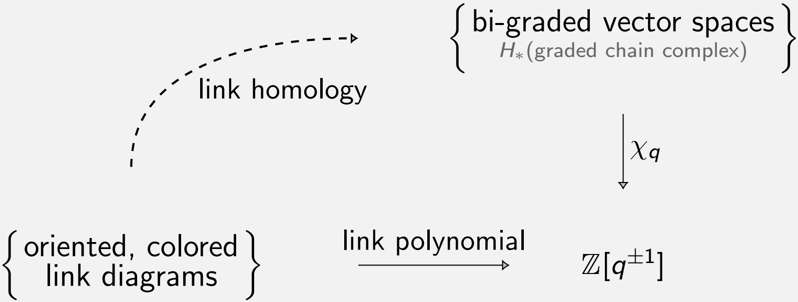
Question

Why does the Jones polynomial have integer coefficients?

Khovanov: the coefficients are signed sums of dimensions of vector spaces!

Plan

Link homology as categorification



A categorification against taking the graded Euler characteristic:

$$\chi_q\left(\bigoplus_{i,j\in\mathbb{Z}} H^{i,j}\right) = \sum_{i,j\in\mathbb{Z}} (-1)^i q^{-j} \dim(H^{i,j})$$

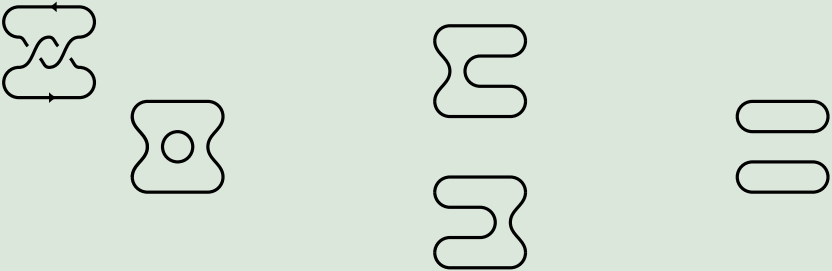
Khovanov's categorification of the Jones polynomial

Algorithm (Construction of Khovanov homology)

- Input: a diagram of an oriented knot or link.
- Step 1: count $n_+ = \#($$)$ and $n_- = \#($$)$
- Step 2: build cube of resolutions
- Step 3: apply graded 2D TFT
- Step 4: totalize to a graded chain complex
- Output: bigraded homology (with global shift)

Khovanov's categorification of the Jones polynomial

Example (The Khovanov homology of the Hopf link)

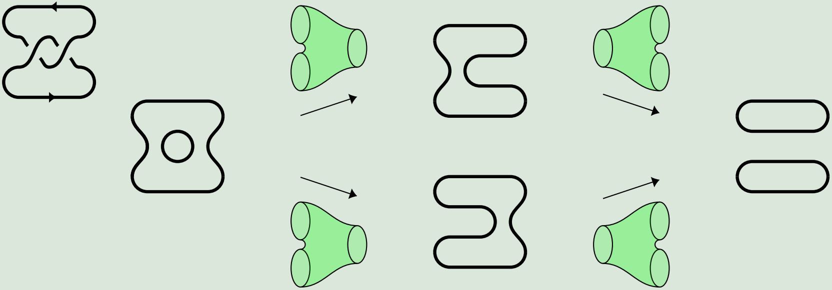


The cube of resolutions

Consider the edges in the cube of resolutions.

Khovanov's categorification of the Jones polynomial

Example (The Khovanov homology of the Hopf link)

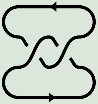


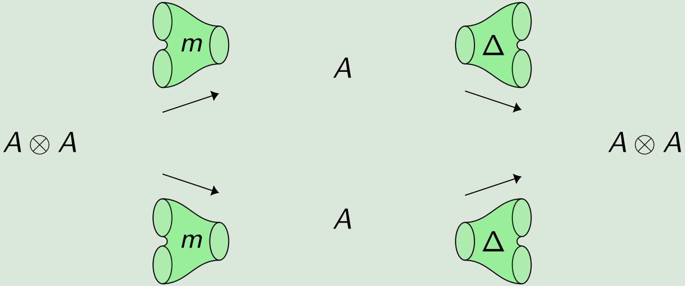
A commutative diagram of bordisms

Apply a 2-dimensional graded TFT.

Khovanov's categorification of the Jones polynomial

Example (The Khovanov homology of the Hopf link)



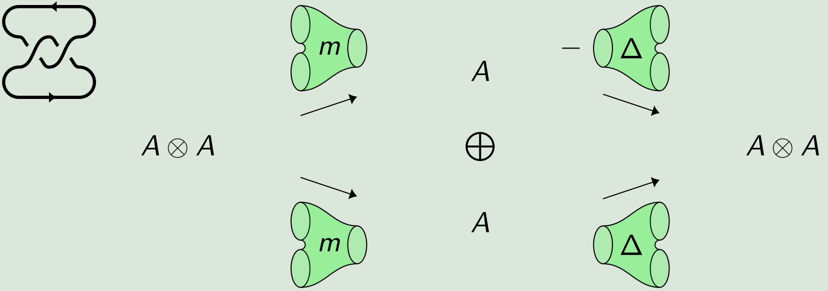


A commutative diagram of linear maps

Make anti-commutative and totalize.

Khovanov's categorification of the Jones polynomial

Example (The Khovanov homology of the Hopf link)

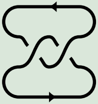


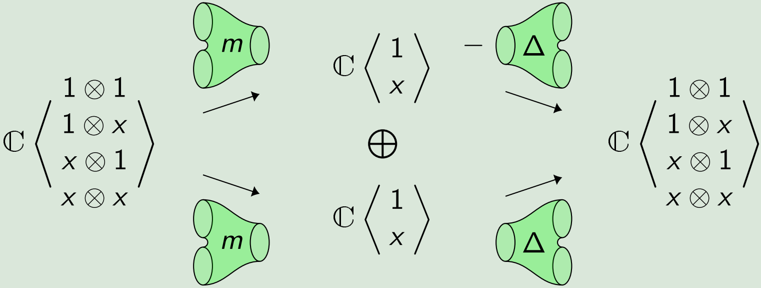
A chain complex

Take $A := H^*(\mathbb{CP}^1, \mathbb{C}) \cong \mathbb{C}[x]/\langle x^2 \rangle = \mathbb{C}\langle 1, X \rangle$, $\Delta(1) = X \otimes 1 + 1 \otimes X$.

Khovanov's categorification of the Jones polynomial

Example (The Khovanov homology of the Hopf link)



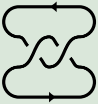


A chain complex

Take homology.

Khovanov's categorification of the Jones polynomial

Example (The Khovanov homology of the Hopf link)



$$\{0\}$$

$$\mathbb{C}\left\langle \begin{matrix} 1 \otimes x - x \otimes 1 \\ x \otimes x \end{matrix} \right\rangle \quad \bigoplus \quad \mathbb{C}\left\langle \begin{matrix} [1 \otimes 1] \\ [1 \otimes x - x \otimes 1] \end{matrix} \right\rangle$$

$$\{0\}$$

Khovanov homology $\mathsf{Kh} = \bigoplus_{i,j} \mathsf{Kh}^{i,j}$

Khovanov's categorification of the Jones polynomial

Algorithm (Construction of Khovanov homology)

- Input: a diagram of an oriented knot or link.
- Step 1: count $n_+ = \#($$)$ and $n_- = \#($$)$
- Step 2: build cube of resolutions
- Step 3: apply graded 2D TFT
- Step 4: totalize to a graded chain complex
- Output: bigraded homology (with global shift)

Theorem (Khovanov 1999)

These bigraded vector spaces computed from link diagrams

1. depend (up to isomorphism) only on the underlying link, and
2. have the Jones polynomial as graded Euler characteristic.

Khovanov homology as a functor

- link diagrams $\mapsto$ graded chain complexes
- Reidemeister moves $\mapsto$ homotopy equivalences
- movies of link diagrams $\mapsto$ chain maps
- $+$ existence of certain compatibility homotopies

Theorem (Khovanov 2002, ..., Morrison–Walker–W. 2019)

Khovanov homology extends to a functor:

$$\begin{array}{ccc} \left\{\begin{array}{c}\text{links embedded in } S^3 \\ \text{link cob. in } S^3 \times I/\text{isotopy}\end{array}\right\} & \xrightarrow{\quad \mathrm{Kh} \quad} & K^b(\mathrm{gr}^{\mathbb{Z}}\mathrm{Vect}) \\ \downarrow & & \downarrow \chi_q \\ \{\text{links embedded in } S^3/\text{isotopy}\} & \xrightarrow{\text{Jones polynomial}} & \mathbb{Z}[q^{\pm 1}] \end{array}$$

The Jones polynomial is the Euler characteristic of a link homology theory

Plan

Plan for the seminar

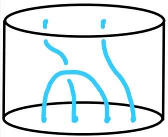
We will have six talks:

1. Khovanov's categorification of the Jones polynomial (PW)
2. Khovanov homology for tangles (Jesse Cohen)
3. Physics of knot homology (Joerg Teschner & Co.)
4. Braid group actions in geometry and physics (Tobias Dyckerhoff)
5. Soergel bimodules and Rouquier complexes (David Reutter)
6. From knot homology to TFT (PW)

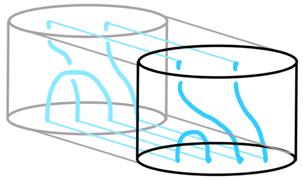
Part 2: Khovanov homology for tangles

Tangles form a 2-category:

- Objects: finite sets of points in $\mathbb{D}^2$
- 1-morphisms: tangles in $\mathbb{D}^2 \times I$



- 2-morphisms: tangle cobordism in $\mathbb{D}^2 \times I \times I/\text{isotopy}$

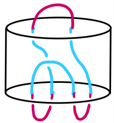


Does Khovanov homology extend to a functor between 2-categories ?

Part 2: Khovanov homology for tangles

Khovanov's 2001 functor-valued invariant of tangles:

- Cube of resolutions for tangle diagrams ✓
- 2D TFT does not extend to non-closed 1-manifolds.
  Workaround: consider all possible planar closures: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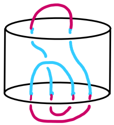


- Change-of-closure controlled by Khovanov's arc rings H^n.
- Closure chain complexes for tangle $\rightsquigarrow$ sweet dg bimodule for arc rings $\rightsquigarrow$ exact functor between triangulated categories associated to disk with even numbers of points. (Homotopy cats of gr.proj./derived cats/stable module cats for H^n).

Construction has several variations and geometric incarnations!

Part 3: Physics of knot homology

Fact

The Jones polynomial is uniquely determined by its value on the unknot and the oriented skein relation

$$q^2 V(\nwarrow\!\!\!\!\nearrow) - q^{-2} V(\nearrow\!\!\!\!\nwarrow) = (q - q^{-1}) V(\uparrow\uparrow)$$

Varying the skein relation recovers other link polynomials:

- The Alexander-Conway polynomial:
$$\Delta(\nwarrow\!\!\!\!\nearrow) - \Delta(\nearrow\!\!\!\!\nwarrow) = (q - q^{-1}) \Delta(\uparrow\uparrow)$$
- The HOMFLY-PT polynomial:
$$a P_\infty(\nwarrow\!\!\!\!\nearrow) - a^{-1} P_\infty(\nearrow\!\!\!\!\nwarrow) = (q - q^{-1}) P_\infty(\uparrow\uparrow)$$
- The $\mathfrak{gl}_N$ polynomial:
$$q^N P_N(\nwarrow\!\!\!\!\nearrow) - q^{-N} P_N(\nearrow\!\!\!\!\nwarrow) = (q - q^{-1}) P_N(\uparrow\uparrow)$$

Question

Are there corresponding link homology theories?

Part 3: Physics of knot homology

Initial proposal Gukov–Schwarz–Vafa, Dunfield–Gukov–Rasmussen 2005:

- Poincaré polynomials of knot homology $\equiv$ refined count of open BPS states $\equiv$ cohomology of all-genus open Gromov–Witten moduli.
- Geometric setup depends on type of knot homology. (Ambient Calabi–Yau 3-fold, Lagrangian boundary condition)
- Geometric transitions $\implies$ relations between knot homologies
- Closed BPS acts on open $\implies$ additional structure on knot homology

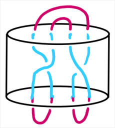
Many predictions have been verified mathematically.

Newer physics perspectives on knot homology:

- Witten 2011: Fivebranes and knots. (Via gauge theory in 5D.)
- Aganagic 2020-23: Knot Categorification from Mirror Symmetry.

Part 4: Braid group actions in geometry and physics

An example: Seidel–Smith symplectic Khovanov homology



- $2n$-punctured disc $\mapsto$ symplectic fibre bundle $\mathcal{Y}$ over $\mathrm{Conf}_{2n}(\mathbb{C})$.

  Adjoint quotient map on transverse slice in $\mathfrak{sl}_{2n}$ to the adjoint action at a nilpotent with equal Jordan blocks.
- closures $a, b \mapsto$ Lagrangians $L_a, L_b \subset \mathcal{Y}_{t_0}$, each $\underset{\text{diff}}{\cong} (S^2)^n$.
- parallel transport along braid $\beta \mapsto$ symplectic automorphism of fibre.
- symplectic Khovanov homology $\mathrm{Kh}_{\mathrm{symp}}(\widehat{\beta}) := HF^*(\beta(L_a), L_b)$

Agreement with ordinary Khovanov homology Abouzaid–Smith 2013-15.

Part 5: Soergel bimodules and Rouquier complexes

$$\mathrm{Br}_n \xrightarrow{\text{linearization}} \text{Hecke algebra} \xrightarrow{\text{Jones-Ocneanu trace}} \text{HOMFLYPT polynomial}$$

categorifies to:

$$\mathrm{Br}_n \xrightarrow{\text{Rouquier cx.}} \text{Soergel bimodules} \xrightarrow{HH_\bullet} \text{HOMFLYPT homology}$$

- This controls all known “algebraic” knot homology theories.
- Laboratory for homotopy coherent knot homology & related TFT.

Part 6: From knot homology to TFT

A local TFT is an assignment:

$$\text{spacetimes} \to \text{algebraic structures}$$

compatible with gluing, determined by value on Euclidean space.

Algebraic structures capturing local TFTs in $k + n + \varepsilon$ dimensions:

$E_k \backslash n$	0	1	2
E_0	vector spaces	categories	2-categories
E_1	algebras	monoidal cats	monoidal 2-cats
E_2	comm. algebras	braided cats	braided 2-cats
E_3	—	sym. mon. cats	sylleptic 2-cats
E_4	—	—	sym. mon. 2-cats
$\vdots$	—	—	—

Part 6: From knot homology to TFT

A local TFT is an assignment:

$$\text{spacetimes} \to \text{algebraic structures}$$

compatible with gluing, determined by value on Euclidean space.

Examples of observables:

$E_k \backslash n$	0	1	2
E_0			
E_1			
E_2		Jones polynomial	Khovanov homology
E_3	—		
E_4	—	—	
$\vdots$	—	—	—

Part 6: From knot homology to TFT

A local TFT is an assignment:

$$\text{spacetimes} \to \text{algebraic structures}$$

compatible with gluing, determined by value on Euclidean space.

Examples of TFTs:

$E_k \backslash n$	0	1	2
E_0			
E_1		Turaev–Viro	new
E_2		Crane–Yetter	new
E_3	—		
E_4	—	—	
$\vdots$	—	—	—