# Seminar on highest weight categories and tilting theory

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## 1 Introduction

Highest weight categories, cellular categories, tilting modules and related concepts play important roles in modern representation theory. In this seminar, we will study these concepts from the viewpoint of examples that are relevant for quantum topology.

**Organisational meeting: 23rd August 2022, online. paul.wedrich@uni-hamburg.de**

## 2 Participating in the seminar

Prerequisities: Participants should have completed the MSc course Advanced Algebra and should be familiar with the basics of (Lie algebra) representation theory.

To pass the seminar, participants should:

- Give a 75-minute talk on one of the topics outlined above.
- Prepare a write-up of the talk that can be shared with other participants.
- Attend other talks and contribute actively to the discussion. Please email beforehand if you cannot attend a session.

## 3 Organisation of the seminar

- Talk topics will be assigned at or after the organisational meeting.
- For every talk a short outline with literature references will be provided. In preparing the talk, participants should roughly follow the outline and consult the references provided. Some topics (esp. those marked with *) may require reading and work beyond the references provided, depending on the participant's background.
- Different sources use different notation conventions. The talk outlines contain guidance on how to stay consistent.
- The topics of the talks build on each other to some degree, so it is important to stay up-to-date with the progress of the seminar and attend ideally all talks.
- Participants are required to:

  - submit a draft write-up of their talk by email two weeks before their talk.
  - attend a preparatory meeting one week before their talk.
  - submit a corrected version of the write-up before their talk, which can be shared on Moodle.

- The language of the seminar will be English.

- The seminar is planned to take place in room 241 on Tuesdays between 12:15 and 13:45.

# 4 Schedule of talks

## 4.1 Cellular algebras

Define cellular algebras and their cell modules, following [GL96] and then introduce two classes of examples: Temperley–Lieb algebras and Hecke algebras. Explain why these are cellular. For the Hecke algebras, this requires introducing the Kazhdan–Lusztig basis and the Robinson–Schensted correspondence.

## 4.2 Representation theory of cellular algebras

Define cell modules (a.k.a. cell representations) of cellular algebras and present the most important results from Sections 2 and 3 of [GL96]. Illustrate all results using the two examples developed in the previous talk.

## 4.3 SOACCs

Present the definition of cellular categories from [Wes09] and strictly object-adapted cellular categories from [EL16], see also [EMTW20, Section 11]. Explain in which sense these concepts generalize cellular algebras. Present examples and explain how these structures aid in the computation of trace decategorifications ([EL16]).

## 4.4 Highest weight categories following Cline–Parshall–Scott

The main reference for this talk is the classical paper of Cline–Parshall–Scott [CPS88], especially §3. The goal is to define highest weight categories and then work towards and present the statement and proof of Theorem 3.6 as well as the statement (but not the proof) of Theorem 3.11.

Theorem 3.6 establishes a connection between highest weight categories and so-called quasi-hereditary algebras. Explain how the definition given in [CPS88] relates to the one presented in [Koe02, Definition 2.1]

Notation: the $A(\lambda)$ in [CPS88] should be denoted $\nabla(\lambda)$, $V(\lambda)$ should be denoted $\Delta(\lambda)$, and $S(\lambda)$ should be denoted $L(\lambda)$.

## 4.5 Ringel duality

Ringel duality is a duality on quasi-hereditary algebras. The goal of the talk is to given an expanded version of the brief summary in Section 2 of [Koe02], starting from Theorem 2. The primary reference is Ringel's paper [Rin91].

Notation: the simple modules $E(-)$ in [Rin91] should be denoted $L(-)$.

### 4.6 Standard categories following Bellamy–Thiel

Introduce the notion of standard categories following [BT22, Section 2]. This should serve as a warm-up for the following three talks. Compare this notion to semi-infinite highest weight categories from [BS18].

### 4.7 The BBG category $\mathcal{O}$ *

The goal of this talk is to give an introduction to the Bernstein–Gelfand–Gelfand category $\mathcal{O}$ and give a detailed explanation why this is an example of a standard category, as claimed in [BT22, Example 2.20]. The main reference is [Hum08].

### 4.8 Tilting modules for quantum groups *

The goal of this talk is to give an introduction to tilting modules for quantum groups and a detailed explanation why this is an example of a standard category, following [BT22, Example 2.21]. An additional reference is [AST18].

### 4.9 Bases in standard categories

Explain why standard categories have natural bases following [BT22, Section 3].

### 4.10 Temperley–Lieb in positive characteristic

Illustrate the construction of bases in a standard category in the example of the Temperley–Lieb category in positive characteristic. The main reference is [TW21]. Along the way, define $p$-Jones–Wenzl projectors and state their main properties. Define the generating morphisms between $p$-Jones–Wenzls and outline which relations exist between their composites.

### 4.11 Additional talks: Tilting modules and the $p$-canonical basis *

Given time and interest, we will work through parts of Riche–Williamson's work [RW18].

## 5 Some hints for preparing talks

- Try to single out one theorem which you might want to call the main theorem of the talk and make sure that everybody understands the statement.
- Is there a definition you might call the main definition of the talk?
- We will meet many concepts during this seminar, which might be known to some of the participants, but will be new to others. When preparing your talk, make sure to include explicit examples to illustrate new concepts.
- In some cases, the talk outlines contain more matrial than what fits into a single talk. If you are unsure about what to include, don't hesitate to ask.

# References


[AST18] Henning Haahr Andersen, Catharina Stroppel, and Daniel Tubbenhauer. Cellular structures using $U_q$-tilting modules. *Pacific J. Math.*, 292(1):21–59, 2018. MR3708257 DOI:10.2140/pjm.2018.292.21 arXiv:1503.00224.

[BS18] Jonathan Brundan and Catharina Stroppel. Semi-infinite highest weight categories, 2018. arXiv:1808.08022.

[BT22] Gwyn Bellamy and Ulrich Thiel. Cellularity of endomorphism algebras of tilting objects. *Adv. Math.*, 404:Paper No. 108387, 2022. MR4413216 DOI:10.1016/j.aim.2022.108387 arXiv:2107.07887.

[CPS88] E. Cline, B. Parshall, and L. Scott. Finite-dimensional algebras and highest weight categories. *J. Reine Angew. Math.*, 391:85–99, 1988. MR961165.

[EL16] Ben Elias and Aaron D. Lauda. Trace decategorification of the Hecke category. *J. Algebra*, 449:615–634, 2016. MR3448186 DOI:10.1016/j.jalgebra.2015.11.028 arXiv:1504.05267.

[EMTW20] Ben Elias, Shotaro Makisumi, Ulrich Thiel, and Geordie Williamson. *Introduction to Soergel bimodules*, volume 5 of *RSME Springer Series*. Springer, Cham, [2020] ©2020. MR4220642 DOI:10.1007/978-3-030-48826-0.

[GL96] J. J. Graham and G. I. Lehrer. Cellular algebras. *Invent. Math.*, 123(1):1–34, 1996.

[Hum08] James E. Humphreys. *Representations of semisimple Lie algebras in the BGG category $\mathcal{O}$*, volume 94 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2008. MR2428237 DOI:10.1090/gsm/094.

[Koe02] Steffen Koenig. Ringel duality and Kazhdan-Lusztig theory. *Pacific J. Math.*, 203(2):415–428, 2002. MR1897907.

[Rin91] Claus Michael Ringel. The category of modules with good filtrations over a quasi-hereditary algebra has almost split sequences. *Math. Z.*, 208(2):209–223, 1991. MR1128706.

[RW18] S. Riche and G. Williamson. Tilting modules and the $p$-canonical basis. *Astérisque*, (397):ix+184, 2018. MR3805034 arXiv:1512.08296.

[TW21] Daniel Tubbenhauer and Paul Wedrich. Quivers for $\mathrm{SL}_2$ tilting modules. *Represent. Theory*, 25:440–480, 2021. MR4273168 DOI:10.1090/ert/569 arXiv:1907.11560.

[Wes09] Bruce W. Westbury. Invariant tensors and cellular categories. *J. Algebra*, 321(11):3563–3567, 2009. MR2510062 DOI:10.1016/j.jalgebra.2008.07.004 arXiv:0806.4045.