Seminar on Categorical Algebra: Braids, bimodules, and bicategories

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1 Introduction

The finite-dimensional representation theory of quantum groups can be organised in so-called ribbon categories, which lead to algebraic invariants of knots, links and tangles, such as the Jones polynomial, and manifold invariants with good gluing properties (TQFTs). This first part of this seminar will review the categorical algebra of monoidal categories, braidings, graphical calculi, etc. that are employed in these constructions.

The second part of this seminar concerns the research area of link homology (which got its own MSC code 57K18 recently). A link homology theory associates a chain complex of graded abelian groups to each knot or link diagram, which is invariant under Reidemeister moves up to homotopy equivalence. Its homology is a bigraded abelian group and taking a graded Euler characteristic recovers a corresponding polynomial link invariant.

The goal of the seminar is to bring the two parts together, namely to work towards a better understanding of the categorical algebra underpinning link homology theories.

2 List of talks

1. **Monoidal categories, duals and pivotal structure**. In this first talk, we introduce the notion of a monoidal category, and discuss a number of elementary examples and basic theorems. We then focus on the string diagram calculus of monoidal categories and introduce a number of diagrammatic categories. Finally, we introduce the concept of a pivotal structure leading to an oriented string diagram calculus and a notion of dimension.

2. **The Temperley-Lieb monoidal category**. We introduce the Temperley-Lieb category TL, a rich algebraic variant of the category of unoriented planar tangles. We classify pivotal structures on TL and characterize TL via a universal property. We use this universal property to relate TL to categories of representations of Lie algebras.

3. **Braided monoidal categories and the Kauffman bracket**. This talk introduces braided monoidal categories, their interaction with pivotal structures and ribbon structure, and how ribbon monoidal categories give rise to knot and link invariants. This is exemplified with the Temperley-Lieb category which leads to the Kauffman bracket and the Jones polynomial.

4. **Frobenius algebras in monoidal categories and topological quantum field theories**. This talk introduces the notion of a Frobenius algebra object in an arbitrary monoidal category, and discusses a number of concrete examples such as group algebras of finite groups, and cohomology rings $\bigoplus_k \mathrm{H}^k(M, \mathbb{C})$ of compact oriented manifolds. The second half of the talk is concerned with the definition of oriented topological quantum field theories, and their classification in one and two dimensions, coming full circle back to Frobenius algebras.

5. **Climbing the categorical ladder: Bicategories and a taste of higher categories**. This talk begins with the definition of bicategories, their relation to monoidal categories, and guiding examples such as the bicategory Cat of categories, functors and natural transformations, and Alg of algebras, bimodules and bimodule maps. Adjunctions and monads are discussed as generalizations of the duals and algebras from previous talks. Moving up in dimension, the graphical calculus of monoidal and braided monoidal bicategories in terms of surfaces in $\mathbb{R}^3$ and $\mathbb{R}^4$ (a.k.a. 'movie moves') is sketched. The talk ends with an overview of the Baez-Dolan periodic table of higher categories and their expected graphical calculi.

6. **Khovanov homology.** This talk will give an introduction to Khovanov homology as a categorification of the Jones polynomial. We will see in which sense it is invariant under Reidemeister moves and what it means for a link homology to be functorial under link cobordisms.

7. **Introduction to categorification.** In this talk we work towards understanding Khovanov homology on a more conceptual level in the framework of categorification.

8. **From Hecke to HOMFLYPT** After having met Khovanov homology, we aim towards one of the most general and mysterious link homology theories: the triply-graded Khovanov–Rozansky homology which categorifies the HOMFLYPT invariant. We prepare for this by reviewing the type A Hecke algebras and related topics.

9. **Soergel bimodules.** Soergel bimodules, introduced by Soergel form an additive monoidal category that categorifies the corresponding Hecke algebra. This talk introduces Soergel bimodules and Bott–Samelson bimodules (named after the Bott–Samelson resolution) and explains their relationships to Hecke algebra bases.

10. **Diagrammatic calculus for Soergel bimodules.** This talk introduces the diagrammatic Soergel calculus of Elias–Khovanov and Elias–Williamson.

11. **Rouquier complexes.** We meet Rouquier complexes of Soergel bimodules and check that their composites satisfy braid relations up to canonical homotopy equivalence. This talk should follow the article of Elias–Krasner.

12. **Triply-graded link homology.** This talk follows a paper of Khovanov and takes us from the categorical braid invariant given by Rouquier complexes of Soergel bimodules to a categorification of the HOMFLYPT invariant.

13. **Bonus talk 1.** Higher categories, dualizability, and the cobordism hypothesis.

14. **Bonus talk 2.** Higher categories from Soergel bimodules.

Organisational meeting: 15th March 2022, online. paul.wedrich@uni-hamburg.de

3 Participating in the seminar

To pass the seminar, participants should:

- Give a 75-minute talk on one of the topics outlined above.
- Prepare a write-up of the talk that can be shared with other participants.
- Attend other talks and contribute actively to the discussion. Please email beforehand if you cannot attend a session.

4 Organisation of the seminar

- Talk topics will be assigned at or after the organisational meeting on 15th March 2022.
- For every talk a detailed outline with literature references will be provided. In preparing the talk, participants should roughly follow the outline and consult the references provided. Some topics may require reading and work beyond the references provided, depending on the participant's background.
- The topics of the talks build on each other to some degree, so it is important to stay up-to-date with the progress of the seminar and attend ideally all talks.
- Participants are invited to schedule a meeting with one of the seminar organizers two to three weeks before their talk. At this meeting, participants should already be familiar with the topic and have a plan for their talk.
- Participants are invited to submit a write-up of their talk one to two weeks before their talk to the organizers to receive feedback. Write-ups are due two weeks after the relevant talk and should incorporate feedback received during and after the talk.
- The language of the seminar will be English.
- The seminar is planned to take place in room 435, Geomatikum.

5 Some hints for preparing talks

- Try to single out one theorem which you might want to call the main theorem of the talk and make sure that everybody understands the statement.
- Is there a definition you might call the main definition of the talk?
- We will meet many concepts during this seminar, which might be known to some of the participants, but will be new to others. When preparing your talk, make sure to include explicit examples to illustrate new concepts.
- In some cases the talk outlines contain more than what fits into a single talk. If you are unsure about what to include, don't hesitate to ask.

6 Schedule of talks

6.1 Monoidal categories, duals and pivotal structures

In this first talk, we introduce the notion of a monoidal category, and discuss a number of elementary examples and basic theorems. We then focus on the string diagram calculus of monoidal categories and introduce a number of diagrammatic categories. Finally, we also introduce the concept of a pivotal structure leading to an oriented string diagram calculus.

The main references for this talk are Section 2 of [EGNO15] and Sections 3.1, 4.1 and 4.2 of [Sel11].

- Following [EGNO15, Section 2], define monoidal categories (using the traditional Definition 2.2.8 of [EGNO15] instead of the alternative Definition 2.2.1) and monoidal functors, and discuss a number of basic examples. These example should include, but are not limited to, the category of sets and functions, the category of vector spaces over a field, and the categories of oriented and unoriented planar tangles (which can only be sketched and does not need to be introduced rigorously).

- Recall MacLane's strictification theorem [EGNO15, Theorem 2.8.5] and summarize the main ideas of the proof appearing in [EGNO15]. Introduce the monoidal category Vec_G of vector spaces graded by a group G and grading preserving morphisms, and discuss its variant Vec_G^ω for a 3-cocycle $\omega : G^3 \to k$ where the associator $\alpha : (U \otimes V) \otimes W \to U \otimes (V \otimes W)$ is given on homogenous vectors u, v, w with degree $g, h, k \in G$, respectively, by the 'boring' associator multiplied with $\omega(g, h, k)$. Briefly explain why Vec_G^ω and $\mathrm{Vec}_G^{\omega'}$ are equivalent if ω and ω' are cohomologous. Discuss how this does not contradict MacLane's strictification theorem.

  Also state MacLane's coherence theorem [EGNO15, Theorem 2.9.2] and explain how it follows from the strictification theorem.

- Explain the string diagram graphical calculus for monoidal categories, e.g. following Section 3.1 of [Sel11], or any other reference.

- Introduce the notion of a dual of an object in a monoidal category and define rigid monoidal categories (see [EGNO15, Section 2.10]). Discuss duals in the category of (possibly infinite-dimensional) vector spaces and in the category of unoriented planar tangles. Explain in what sense rigidity is a property of a monoidal category rather than extra structure.

- In a rigid monoidal category $\mathcal{C}$, show that a choice of duality data for every object gives rise to a monoidal functor $(-)^* : \mathcal{C} \to \mathcal{C}^{\mathrm{op,mp}}$ where $\mathcal{C}^{\mathrm{op,mp}}$ denotes the opposite category with opposite tensor product. Explain that different choices of duality data lead to monoidally equivalent double dual functors.

- If time permits: Give an example of an object X in a monoidal category which has a dual and a double dual, but for which the double dual is not isomorphic to X itself. Hint: Consider the monotone map $\mathbb{Z} \to \mathbb{Z}, n \mapsto 2n$ seen as an object of the monoidal category of endofunctors of the poset $\mathbb{Z}$ seen as a category.

- Define a pivotal structure on a rigid monoidal category to be a monoidal natural isomorphism from the identity functor to the double dual functor $(-)^{**}$. Explain why the set of pivotal structures on a rigid monoidal category is either empty or a torsor over $\mathrm{Aut}^{\otimes}(\mathrm{id})$, the set of monoidal natural automorphisms of the identity functor. Show that the category of finite-dimensional vector spaces admits a unique pivotal structure. Show that for a group G, pivotal structures on the category Vec_G of finite-dimensional G-graded vector spaces correspond to group homomorphisms $G \to \mathbb{C}^{\times}$.

- Explain that the graphical calculus of pivotal categories is that of oriented planar tangles and oriented planar graphs (see e.g. Section 4.2 of [Sel11]).

6.2 The Temperley-Lieb monoidal category

The purpose of this talk is to introduce the Temperley-Lieb categories and their pivotal structures. These pivotal structures allow to define TL in terms of a universal property relating it to a number of representation theoretic categories.

- Recall the definitions of duals and of pivotal structures, and their graphical calculus from the first talk (see e.g. Sections 4.1 and 4.2 of [Sel11]).

- Define the *dimension* $\dim_p(X) \in \mathrm{End}_{\mathcal{C}}(I)$ of an object X in a rigid monoidal category $\mathcal{C}$ equipped with a pivotal structure $p : \mathrm{id}_{\mathcal{C}} \Rightarrow (-)^{**}$ as the evaluation of a loop labelled by X. Explain why $\dim_p(X)$ depends on the choice of pivotal structure.

  Explain that for the unique pivotal structure on the category of finite-dimensional vector spaces this agrees with the usual notion of dimension. Recall from the first talk that pivotal structures on the category of finite-dimensional G-graded vector spaces correspond to group homomorphisms $\lambda : G \to \mathbb{C}^{\times}$. For such a pivotal structure $\lambda : G \to \mathbb{C}^{\times}$ and a $g \in G$ compute the dimension $\dim_{\lambda}$ of the one-dimensional vector space $\mathbb{C}\langle g \rangle$ concentrated in degree g. (This last example shows that dim indeed depends on the choice of pivotal structure.)

- For a $\lambda \in \mathbb{C}$, define the $\mathbb{C}$-linear monoidal *Temperley-Lieb category* $\mathrm{TL}(\lambda)$ either combinatorially or as the $\mathbb{C}$-linearization of the category of unoriented planar tangles with loop value λ. More generally, define TL as a category over the polynomial ring $\mathbb{C}[d]$ in one indeterminate corresponding to the loop value. Some background on the Temperley–Lieb category appears in the lecture notes [Wed22].

- Show that TL has two pivotal structures. Hint: What is the data of a pivotal structure on a monoidal category freely generated by an object and generating morphisms subject to relations?

- Compute the dimension of the generating object of TL for both these pivotal structures.

- Let $\mathcal{C}$ be a rigid monoidal category with pivotal structure $p : \mathrm{id}_{\mathcal{C}} \to (-)^{**}$. A *symmetric self-duality structure* on an object X is an isomorphism $f : X \to X^*$ so that $f = f^* \circ p_X : X \to X^{**} \to X^*$. Explain that the graphical calculus of a symmetrically self-dual object in a pivotal monoidal category is that of unoriented planar tangles.

- Show that the generating object of TL is symmetrically self-dual for precisely one of the two pivotal structures of TL. For the other pivotal structure, it is *antisymmetrically self-dual*: there exists an isomorphism $f : X \to X^{**}$ so that $f^* \circ p_X = -f$. We will therefore denote the two pivotal structures of TL by p_s and p_{as}.

- The *universal property* of TL: For $\lambda \in \mathbb{C}$, explain in what sense the pivotal monoidal category $(\mathrm{TL}(\lambda), p_s)$ is the free $\mathbb{C}$-linear pivotal monoidal category on a symmetrically self-dual object with dimension λ, and $(\mathrm{TL}(\lambda), p_{as})$ is the free $\mathbb{C}$-linear pivotal monoidal category on an anti-symmetrically self-dual object with dimension $-\lambda$. Hint: Classify $\mathbb{C}$-linear pivotal monoidal functors out of $(\mathrm{TL}(\lambda), p_s)$ and $(\mathrm{TL}(\lambda), p_{as})$.

  This universal property can be seen as a main reason for the ubiquity of TL in representation theory and quantum algebra.

- Let $\mathrm{Rep}(\mathfrak{g})$ denote the category of finite-dimensional complex representations of a Lie algebra $\mathfrak{g}$. Show that $\mathrm{Rep}(\mathfrak{g})$ has a canonical pivotal structure. Show that the data of a symmetric (resp. antisymmetric) self-duality on an object, namely a representation $\rho : \mathfrak{g} \to \mathrm{End}(V))$ of $\mathrm{Rep}(\mathfrak{g}$, is a symmetric (resp. antisymmetric) non-degenerate pairing $\langle -, - \rangle : V \otimes V \to \mathbb{C}$ so that
$$\langle \rho(x)(v), w \rangle + \langle v, \rho(x)(w) \rangle = 0. \tag{1}$$

- An *orthogonal vector space* (resp. a *symplectic vector space*) is a finite-dimensional $\mathbb{C}$-vector space V with a symmetric (resp. antisymmetric) non-degenerate pairing $\langle -, - \rangle : V \otimes V \to \mathbb{C}$. The *orthogonal Lie algebra* $\mathfrak{so}(V, \langle -, - \rangle)$ (resp. the *symplectic Lie algebra* $\mathfrak{sp}(V, \langle -, - \rangle)$) is the complex Lie algebra of endomorphisms of V fulfilling (1). V is called its vector representation.

  Since all orthogonal (resp. symplectic) vector spaces of a given dimension n are isomorphic (via an isomorphism which preserves $\langle -, - \rangle$), these complex Lie algebras are often just denoted $\mathfrak{so}_n$ and $\mathfrak{sp}_n$.

- Given an orthogonal vector space $(V, \langle -, - \rangle)$ (resp. symplectic vector space) of dimension n, use the universal property of pivotal monoidal functors out of Temperely-Lieb categories to build pivotal monoidal functors
$$(\mathrm{TL}(n), p_s) \to \mathrm{Rep}(\mathfrak{so}(V, \langle -, - \rangle))$$
$$(\mathrm{TL}(-n), p_{as}) \to \mathrm{Rep}(\mathfrak{sp}(V, \langle -, - \rangle))$$
  which send the generating object to the 'fundamental representation'.

- Show that every endomorphism of a symplectic vector space $(V, \langle -, - \rangle)$ fulfilling (1) is traceless, and hence that $\mathfrak{sp}(V, \langle -, - \rangle)$ is a Lie subalgebra of $\mathfrak{sl}(V)$, the Lie algebra of traceless endomorphisms of V.

  If V is two-dimensional, this embedding $\mathfrak{sp}_2 \hookrightarrow \mathfrak{sl}_2$ is in fact an isomorphism and the induced equivalence $\mathrm{Rep}(\mathfrak{sp}_2) \cong \mathrm{Rep}(\mathfrak{sl}_2)$ maps the vector representation to the vector representation.

- The functor $(\mathrm{TL}(-2), p_{as}) \to \mathrm{Rep}(\mathfrak{sp}_2) \cong \mathrm{Rep}(\mathfrak{sl}_2)$ is almost an equivalence. State without proof that it is fully faithful. However, show that it is not essentially surjective. Hint: There are morphisms $p : a \to a$ in $\mathrm{TL}(\lambda)$ with $p^2 = p$ (so called idempotent morphisms) which cannot be written as a composite $f \circ g$ of morphisms $f : b \to a$ and $g : a \to b$ with $g \circ f = \mathrm{id}_b$. On the other hand, show that every idempotent morphism in $\mathrm{Rep}(\mathfrak{sl}_2)$ splits in this way.

- Define the idempotent completion $\mathrm{Kar}(\mathcal{C})$ of a category $\mathcal{C}$ and discuss its universal property. In particular, it follows that the functor $\mathrm{TL}(-2) \to \mathrm{Rep}(\mathfrak{sl}_2)$ extends to a monoidal functor $\mathrm{Kar}(\mathrm{TL}(-2) \to \mathrm{Rep}(\mathfrak{sl}_2)$. State without proof that this monoidal functor is an equivalence.

This motivates thinking of Kar(TL(λ)) for more general $\lambda \in \mathbb{C}$ as a 'quantum deformation' of Rep($\mathfrak{sl}_2$).

6.3 Braided monoidal categories and the Kauffman bracket

This talk introduces braided monoidal categories, their interaction with pivotal structures, and the way they give rise to knot and link invariants. As the main example we continue to use the Temperley-Lieb monoidal category leading to the Kauffman bracket and Jones polynomial.

References for this talk are [Sel11], [JS91].

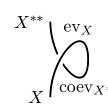
- Following Sections 3.3 and 3.5 of [Sel11], define braided and symmetric monoidal categories, and give basic examples of braided and symmetric monoidal categories, including the category of vector spaces, the category of *super vector spaces* (i.e. $\mathbb{Z}/2$-graded vector spaces with an interesting symmetric monoidal structure), the category of chain complexes and the category of oriented tangles.

- Explain the relation between braided monoidal categories and the braid groups, and discuss the graphical calculus of braid groups (see Section 3.3 of [Sel11] and [JS91]). In particular, discuss Example 3.6 of [Sel11] and show how the axioms of a braided monoidal category imply the third Reidemeister move (a.k.a. the Yang-Baxter equation). What is the free braided monoidal category on one generating object?

- Show that the set of braidings on the Temperley-Lieb category TL(λ) for $\lambda \in \mathbb{C}$ coincides with the set of solutions to the equation $A^2 + A^{-2} = -\lambda$. Given such a solution A, we will write TL(λ, A) for the Temperley-Lieb category TL(λ) equipped with the braiding determined by A.

- Let $\mathcal{C}$ be a rigid braided monoidal category with chosen duality data for every object. Define the *Drinfeld isomorphism* $u_X : X \to X^{**}$ of an object X in $\mathcal{C}$ as the isomorphism (omitting associator isomorphisms)

$$u_X := (\mathrm{id}_{X^{**}} \otimes \mathrm{ev}_X) \circ (c_{X,X^{**}} \otimes \mathrm{id}_{X^*}) \circ (\mathrm{id}_X \otimes \mathrm{coev}_{X^*}) =$$



  Show that if $\mathcal{C}$ is a symmetric monoidal category, u defines a monoidal natural isomorphism from the identity to the double dual functor, and hence a pivotal structure. If $\mathcal{C}$ is not symmetric, u_X is still natural in X but is not monoidal anymore.

- A *twist* (or *balancing*) on a braided monoidal category $\mathcal{C}$ with braiding $c_{X,Y} : X \otimes Y \to Y \otimes X$ is a (non-monoidal) natural isomorphism $\theta : \mathrm{id}_{\mathcal{C}} \to \mathrm{id}_{\mathcal{C}}$ so that

$$\theta_{X \otimes Y} = c_{Y,X} \circ c_{X,Y} \circ (\theta_X \otimes \theta_Y).$$

- Show that for any rigid braided monoidal category $\mathcal{C}$, the assignment $\{p_X : X \to X^{**}\}_X \mapsto \{\theta_X := p_X^{-1} \circ u_X : X \to X\}_X$ defines a bijection between pivotal structures p on $\mathcal{C}$ and twists θ on $\mathcal{C}$. (See e.g. Lemma 4.20 of [Sel11].)

- A braided category is symmetric if and only if the identity natural isomorphism $\{\mathrm{id}_X\}_X$ defines a twist. In particular, any symmetric monoidal category has a canonical pivotal structure.
- For $\lambda \in \mathbb{C}$ and $A \in \mathbb{C}^\times$ a solution to $A^2 + A^{-2} = -\lambda$, show that the twists of $\mathrm{TL}(\lambda, A)$ corresponding to the two pivotal structures p_s and p_{as} of $\mathrm{TL}(\lambda)$ are given on the generating object X by $\theta_X = \pm A^{-3}$.
- A twist on a braided monoidal category (or equivalently, a pivotal structure on a braided monoidal category) is called *ribbon* if $(\theta_X)^* = \theta_{X^*}$. Explain why both twists on $\mathrm{TL}(\lambda, A)$ are ribbon.
- Follow Section 4.7 of [Sel11] (and the references therein) and explain the graphical calculus of ribbon braided monoidal categories (there called 'tortile categories'). Explain how this graphical calculus allows to express morphisms in a ribbon braided monoidal category in terms of oriented and normally framed tangles (also known as 'ribbons'). In particular, what is the free ribbon category on one generating object?
- Explain how it follows from the graphical calculus that any object X in a ribbon braided category $\mathcal{C}$ gives rise to an invariant of normally framed oriented (a.k.a. ribbon) knots K valued in $\mathrm{End}_{\mathcal{C}}(I)$.
- For the generating object of $\mathrm{TL}(\lambda, A)$ equipped with its pivotal structure p_{as}, this framed knot invariant is known as the *Kauffman bracket*. Compute the Kauffman bracket for a number of example knots.

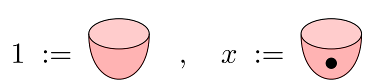
Further references for this talk are Chapter 8 of [EGNO15] and [JS93] and [Tur94].

6.4 Frobenius algebras in monoidal categories and topological quantum field theories

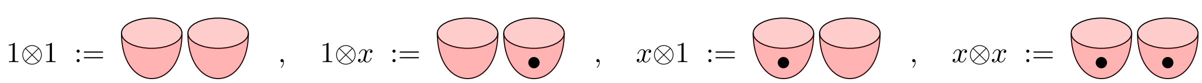
The purposes of this talk is to introduce the notion of algebra object and Frobenius algebra object in a monoidal category, and introduce some basic concepts behind topological quantum field theories. A good reference for this talk is [Koc04].

- Define algebra objects $(A, m : A \otimes A \to A, u : I \to A)$ in monoidal categories. Discuss examples in Vect and in the category of oriented planar tangles.
- A *pairing* between objects A and B in a monoidal category is a morphism $A \otimes B \to I$. A pairing is *non-degenerate* if $A \otimes B \to I$ is the counit of a duality between A and B. A *Frobenius algebra* in a monoidal category $\mathcal{C}$ is an algebra object (A, m, u) equipped with a morphism $\mathrm{tr} : A \to I$ such that the pairing $A \otimes A \xrightarrow{m} A \xrightarrow{\mathrm{tr}} I$ is non-degenerate.
- Give examples of Frobenius algebras in Vec, these should include the algebra $\mathrm{Mat}_n(\mathbb{C})$ with $\mathrm{tr} : \mathrm{Mat}_n(\mathbb{C}) \to \mathbb{C}$ given by the trace, and the group algebra $\mathbb{C}[G]$ of a finite group G.
- For M a closed oriented manifold, show that the cohomology ring $\bigoplus_n \mathrm{H}^n(M; \mathbb{C})$ has the structure of a Frobenius algebra in the monoidal category of $\mathbb{C}$-vector spaces. Hint: Poincare duality.

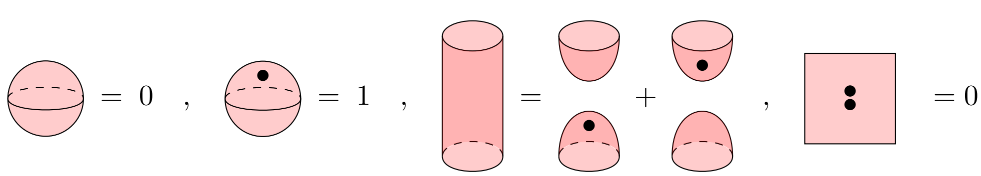
- Show that a Frobenius algebra in a monoidal category may equivalently be defined as an object A equipped with an algebra structure $(m : A \otimes A \to A, u : I \to A)$ and a coalgebra structure $(\Delta : A \to A \otimes A, \mathrm{tr} : A \to I)$ fulfilling a certain compatibility condition, known as the *Frobenius equation*.
- Define commutative algebra objects in braided monoidal categories, and discuss some examples.
- Let M be a closed oriented manifold with cohomology in even degrees. Show that $\bigoplus_n \mathrm{H}^n(M, \mathbb{C})$ defines a commutative Frobenius algebra object in $\mathrm{Vec}_{\mathbb{C}}$. Discuss what goes wrong if M has cohomology in odd degrees and how this could be fixed for even-dimensional M by passing to the category of *super vector spaces*, $\mathbb{Z}/2$-graded vector spaces equipped with an interesting symmetric braiding.
- As an example, discuss the commutative Frobenius algebra structure on $\mathbb{C}[x]/x^{n+1} \cong \bigoplus_k \mathrm{H}^k(\mathbb{C}P^n, \mathbb{C})$.
- Define oriented topological quantum field theories (of arbitrary dimension) valued in a symmetric monoidal category $\mathcal{C}$. Explain the classification of oriented one-dimensional TQFTs.
- Explain the classification of oriented two-dimensional TQFTs in terms of commutative Frobenius algebras.
- Define the *window element* $w : I \to A$ of a Frobenius algebra $(A, m, u, \Delta, \mathrm{tr})$ to be the composite $m \circ \Delta \circ u : I \to A$. Compute the window elements of the Frobenius algebra $\mathbb{C}[G]$ for G a finite group, for $\mathbb{C}[x]/x^{n+1}$ and for $\bigoplus_k \mathrm{H}^k(M, \mathbb{C})$ where M is a closed oriented manifold with cohomology in even degrees.
- Express the value of the TQFT associated to a commutative Frobenius algebra on a genus g surface in terms of the window element and the trace. Using the computation of the window elements from above, compute this value in all discussed examples.
- Explain the following fun way of computing with the two-dimensional TQFT corresponding to $\mathbb{C}[x]/(x^2)$. First we associate with 1 and x the following pictures:

  Then for $\mathcal{F}(\bigcirc\bigcirc) \cong \mathcal{F}(\bigcirc) \otimes \mathcal{F}(\bigcirc)$ we have a basis given by pictures:

  Now verify that the maps that $\mathcal{F}$ associates to cap, cup, and (upside-down) pairs of pants cobordisms can be computed on pictures as follows. Pick a basis element, stack the desired cobordism on top of it, and simplify the picture using the following relations

until you get a linear combination of basis elements. Use these relations to compute the value on a genus g surface and compare it with the computation from the previous item.

Do you see how to generalize this graphical notation to the TQFT associated to $\mathbb{C}[x]/x^{n+1}$?

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