Lie algebras, winter semester 21-22

Paul Wedrich
paul.wedrich@uni-hamburg.de

July 19, 2022

Contents

Literature

While preparing these lecture notes, I found the following resources very useful:

- Wolfgang Soergel, lecture notes for the courses *Halbeinfache Lie-Algebren*, *Spiegelungsgruppen und Wurzelsysteme Lie-Algebren*, *Mannigfaltigkeiten und Liegruppen* (in German), available at https://home.mathematik.uni-freiburg.de//soergel/.

- Christoph Schweigert, lecture notes for the course *Einführung in die Theorie der Lieschen Algebren* (in German), available at https://www.math.uni-hamburg.de/home/schweigert/skripten.html.

- Pavel Etingof, Oleg Golberg, Sebastian Hensel, Tiankai Liu, Alex Schwendner, Dmitry Vaintrob, Elena Yudovina, *Introduction to representation theory. With historical interludes by Slava Gerovitch*, Student Mathematical Library, 59. American Mathematical Society, Providence, RI, 2011. Available via the website http://www-math.mit.edu/ etingof/.
- James E. Humphreys, *Introduction to Lie algebras and representation theory.* Second printing, revised. Graduate Texts in Mathematics, 9. Springer-Verlag, New York-Berlin, 1978.

Conventions

- $\mathbb{F}$, a field
- $\mathbb{N} = \mathbb{Z}_{>0} = \{1, 2, 3, \dots\}$ and $\mathbb{N}_0 = \mathbb{Z}_{\geq 0} = \{0, 1, 2, \dots\}$

Exercises

At the end of most subsection are exercises that will be discussed in the exercise classes. These are separated by horizontal lines. The exercises to be discussed in the exercise classes each week are indicated by margin notes. Starred exercises (*) are optional additional exercises, some of which are a bit open-ended. They will be discussed in the exercise classes, but do not count to the total number of exercises of which 40% should be solved.

Comments, corrections?

These lecture notes will continue to be updated. If you find any mistakes or have any comments, please let me know.

Acknowledgements

I would like to thank the students of my 21/22 course and especially Max Demirdilek, Sarah Hasselberg, Laura Kähler, Kirsten Lehr, Nadine Peter, Karim Ritter von Merkl, Tim Seifert, Jacob Stegemann, andDennis Totzke for valuable comments and corrections.

1 Introduction

1.1 Definition of a Lie algebra

Definition 1.1.1 A **Lie algebra** over $\mathbb{F}$ is an vector space $\mathfrak{g}$ over $\mathbb{F}$, equipped with a bilinear map

$$\begin{aligned} \mathfrak{g} \times \mathfrak{g} &\to \mathfrak{g}, \\ (x, y) &\mapsto [x, y], \end{aligned}$$

called the **Lie bracket**, such that the following identities hold for all $x, y, z \in \mathfrak{g}$:

- $[x, x] = 0$, i.e. the Lie bracket is antisymmetric, and
- $[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0$, i.e. the Lie bracket satisfies the **Jacobi identity**.

Lemma 1.1.2 For any Lie bracket on $\mathfrak{g}$ we have $[v, w] = -[w, v]$ for $v, w \in \mathfrak{g}$.

Proof. By bilinearity, we have $0 = [v + w, v + w] = [v, v] + [v, w] + [w, v] + [w, w] = [v, w] + [w, v]$. □

Remark 1.1.3 We have defined a Lie algebra to be a pair $(\mathfrak{g}, [-, -])$ of a vector space and a Lie bracket on it. However, it is common to refer to $\mathfrak{g}$ itself as a Lie algebra if the choice of Lie bracket is clear from the context.

We have the following, more general notion of an algebra:

Definition 1.1.4 An **algebra** over $\mathbb{F}$ is an vector space A over $\mathbb{F}$, equipped with a bilinear map

$$\begin{aligned} A \times A &\to A, \\ (x, y) &\mapsto m(x, y), \end{aligned}$$

called the **multiplication**.

- If m satisfies the properties of a Lie bracket ($[x, y] := m(x, y)$) from Definition 1.1.1, then (A, m) is a Lie algebra. It is said to be **abelian** if $[x, y] = 0$ for all $x, y \in A$.
- If m is associative, i.e. $m(x, m(y, z)) = m(m(x, y), z)$ for all $x, y, z \in A$, then (A, m) is called an **associative algebra**. It is said to be **abelian** if $m(x, y) = m(y, x)$ for all $x, y \in A$.
- If there is an element $1_A \in A$, such that $m(1_A, x) = m(x, 1_A) = x$ for all $x \in A$, then (A, m) is said to be **unital** and 1_A is called the **unit element** or **identity element**.

As before, we will sometimes refer to A as an algebra, even though it would be more accurate to ascribe this name to the pair (A, m).

Remark 1.1.5 It is very important to distinguish the concepts of Lie algebras and associative algebras. A Lie algebra is, in general, neither associative, nor unital, nor abelian. Conversely, the multiplication in an associative algebra typically does not satisfy the Jacobi identity. Because of this, we will use special notation for the multiplication in Lie algebras and associative algebras:

$$\begin{aligned} [x, y] &:= m(x, y) \text{ in the case of Lie algebras} \\ x \cdot y &:= m(x, y) \text{ in the case of associative algebras} \end{aligned}$$

For example, the associativity law then takes the familiar form $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ for all $x, y, z \in A$. Sometimes we will even drop the dot from the notation of an associative multiplication and simply write $xy := x \cdot y$.

Later we will see many connections between the theories of Lie algebras and associative algebras.

Definition 1.1.6 Let (A_1, m_1) and (A_2, m_2) be algebras over $\mathbb{F}$. A **morphism** (or **algebra homomorphism**) from (A_1, m_1) to (A_2, m_2) is a linear map $f : A_1 \to A_2$ that satisfies $f(m_1(x, y)) = m_2(f(x), f(y))$ for all $x, y \in A_1$. In other words, the following diagram commutes:

$$\begin{tikzcd} A_1 \times A_1 \arrow[r, "f \times f"] \arrow[d, "m_1"'] & A_2 \times A_2 \arrow[d, "m_2"] \\ A_1 \arrow[r, "f"] & A_2 \end{tikzcd}$$

Such a morphism is called an **mono-/epi-/isomorphism** if $f\colon A_1 \to A_2$ is an mono-/epi-/isomorphism of vector spaces over $\mathbb{F}$. We also say that f is injective/surjective/bijective, respectively. In the latter case, (A_1, m_1) and (A_2, m_2) are said to be **isomorphic**, in symbols $(A_1, m_1) \cong (A_2, m_2)$.

If both algebras are unital with unit elements 1_{A_1} and 1_{A_2} respectively, then f is said to be **unital** if $f(1_{A_1}) = 1_{A_2}$.

Note that for a morphism $f\colon \mathfrak{g}_1 \to \mathfrak{g}_2$ between Lie algebras, the required compatibility with the Lie brackets takes the form $f([x, y]) = [f(x), f(y)]$, while for a morphism $g\colon A_1 \to A_2$ between associative algebras, the compatibility condition reads $g(x \cdot y) = g(x) \cdot g(y)$.

Definition 1.1.7 Let (A, m) be an algebra over $\mathbb{F}$. A **subalgebra** of (A, m) is a linear subspace $B \subset A$, such that $m(x, y) \in B$ for all $x, y \in B$. In this case, B is itself an algebra with multiplication given by restricting m to $B \times B \subset A \times A$. Moreover, the natural inclusion $B \hookrightarrow A$ defines a monomorphism of algebras. If A is unital with unit element 1_A, then B is called a **unital subalgebra** if $1_A \in B$. Subalgebras of Lie algebras will be called **Lie subalgebras**, or simply **subalgebras** if confusion is unlikely.

Note that a linear subspace $\mathfrak{h}$ of a Lie algebra $\mathfrak{g}$ with Lie bracket $[-,-]_{\mathfrak{g}}$ is a Lie subalgebra if and only if $[x, y]_{\mathfrak{g}} \in \mathfrak{h}$ whenever $x, y \in \mathfrak{h}$.

Exercise 1 Prove that the intersection $\bigcap_{i \in I} B_i$ of (an arbitrary number of) subalgebras B_i of a given algebra A is again a subalgebra of A and of any of the B_i. Also find an example of an algebra with two subalgebras whose union is not a subalgebra. Hint: Consider the algebra of complex $2x2$-matrices with respect to the matrix multiplication, with its subalgebra

$$U_1 = \left\{ \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix} \mid c \in \mathbb{C} \right\}$$

and another analogous subalgebra U_2.

Exercise 2 Recall or familiarize yourself with the notion of a **category** and with what it means for a category to be $\mathbb{F}$**-linear**, where $\mathbb{F}$ is a field. Verify that Lie algebras (resp. associative algebras) over $\mathbb{F}$ form a category $\mathrm{LieAlg}_{\mathbb{F}}$ (resp. $\mathrm{AssocAlg}_{\mathbb{F}}$) with morphisms as in Definition 1.1.6. Is it $\mathbb{F}$-linear?

1.2 Examples

Examples 1.2.1 The following are examples of associative unital algebras over a field $\mathbb{F}$:

(1) The polynomial ring $\mathbb{F}[x_1, \ldots, x_n]$.

(2) The vector space $\mathrm{End}_{\mathbb{F}}(V)$ of endomorphisms $f\colon V \to V$ of an $\mathbb{F}$-vector space V with multiplication given by composition $m(f, g) := f \circ g$.

(3) The vector space $\mathrm{Mat}_{n \times n}(\mathbb{F})$ of $n \times n$-matrices with entries in $\mathbb{F}$, with respect to matrix multiplication.

Example 1.2.2 Let V be any vector space over $\mathbb{F}$ and declare $[x, y] := 0$ for all $x, y \in V$, then this defines a Lie bracket on V. The Lie algebra $(V, [-,-])$ is abelian.

Remark 1.2.3 On vector spaces of dimension 0 or 1, there is a unique Lie bracket, namely the abelian one. In Exercise 3 we will see that there are exactly two Lie algebra structures up to isomorphism on any 2-dimensional vector space.

Construction 1.2.4 Let A be an associative algebra over $\mathbb{F}$. Consider the **commutator** of two elements $x, y \in A$

$$[x, y] := x \cdot y - y \cdot x.$$

This defines a bilinear map $A \times A \to A$, which is antisymmetric (easy) and satisfies the Jacobi identity (straightforward computation). Thus, $(A, [-,-])$ is a Lie algebra. L1 End

We can summarize the content of this construction in the slogan:
"Every associative algebra produces a Lie algebra, with the Lie bracket given by the commutator".

Example 1.2.5 Let V be any vector space over $\mathbb{F}$ and consider the associative algebra $\mathrm{End}_{\mathbb{F}}(V)$ of endomorphisms of V. Now apply Construction 1.2.4 to define the **general linear Lie algebra** $\mathfrak{gl}(V)$. Explicitly, the Lie bracket of two linear maps $f, g\colon V \to V$ is

$$[f, g] := f \circ g - g \circ f.$$

We will use the symbol $\mathfrak{gl}(V)$ whenever the vector space of endomorphisms of V will be considered as a Lie algebra. Conversely, we use $\mathrm{End}_{\mathbb{F}}(V)$ when we consider the endomorphisms of V as an associative algebra.

If V is finite-dimensional over $\mathbb{F}$ and we have chosen a basis $\{v_1, v_2, \ldots, v_n\}$ for $n = \dim_{\mathbb{F}}(V)$, then $\mathrm{End}_{\mathbb{F}}(V)$ is isomorphic to the vector space $\mathrm{Mat}_{n\times n}(\mathbb{F})$ of $n \times n$-matrices with entries in $\mathbb{F}$, and composition of linear maps translates into matrix multiplication. Using the matrix commutator as Lie bracket, we can also consider $\mathrm{Mat}_{n\times n}(\mathbb{F})$ as a Lie algebra, which we denote by $\mathfrak{gl}(n, \mathbb{F})$. Clearly we have $\mathfrak{gl}(n, \mathbb{F}) \cong \mathfrak{gl}(V)$.

Lie subalgebras of general linear Lie algebras $\mathfrak{gl}(V)$ or $\mathfrak{gl}(n, \mathbb{F})$ are called **linear Lie algebras**. We will see several more examples.

Example 1.2.6 (type A) For $n \in \mathbb{N}$, one checks that the linear subspace

$$\mathfrak{sl}(n, \mathbb{F}) = \{M \in \mathfrak{gl}(n, \mathbb{F}) \mid \mathrm{Tr}(M) = 0\}$$

of traceless $n \times n$-matrices is closed under the matrix commutator (Exercise 5) and thus defines a Lie subalgebra of $\mathfrak{gl}(n, \mathbb{F})$. It is called a **special linear Lie algebra** and the **classical Lie algebra of type A_{n-1}**.

More abstractly, whenever V is finite-dimensional over $\mathbb{F}$, one can define a Lie subalgebra $\mathfrak{sl}(V)$ of $\mathfrak{gl}(V)$ (the trace of a linear endomorphism is basis-independent) and $\mathfrak{sl}(V) \cong \mathfrak{sl}(n, \mathbb{F})$ for $n = \dim_{\mathbb{F}}(V)$.

For the following three examples, of the classical Lie algebras of types B, C, and D, let $\mathrm{char}(\mathbb{F}) \neq 2$.

Example 1.2.7 (type B) For odd $2n + 1 \in \mathbb{N}$, consider the linear subspace

$$\mathfrak{so}(2n+1, \mathbb{F}) = \{M \in \mathfrak{gl}(2n+1, \mathbb{F}) \mid SM = -M^t S\}, \quad \text{where } S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_n \\ 0 & I_n & 0 \end{pmatrix}.$$

We will see that this is a Lie subalgebra of $\mathfrak{gl}(2n + 1, \mathbb{F})$ and, in fact, of $\mathfrak{sl}(2n + 1, \mathbb{F})$. It is called an **orthogonal Lie algebra** and the **classical Lie algebra of type B_n**.

Example 1.2.8 (type C) For even $2n \in \mathbb{N}$, consider the linear subspace

$$\mathfrak{sp}(2n, \mathbb{F}) = \{M \in \mathfrak{gl}(2n, \mathbb{F}) \mid SM = -M^t S\}, \quad \text{where } S = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

We will see that this is a Lie subalgebra of $\mathfrak{gl}(2n, \mathbb{F})$ and, in fact, of $\mathfrak{sl}(2n, \mathbb{F})$. It is called a **symplectic Lie algebra** and the **classical Lie algebra of type C_n**.

Example 1.2.9 (type D) For even $2n \in \mathbb{N}$, consider the linear subspace

$$\mathfrak{so}(2n, \mathbb{F}) = \{M \in \mathfrak{gl}(2n, \mathbb{F}) \mid SM = -M^t S\}, \quad \text{where } S = \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix}.$$

We will see that this is a Lie subalgebra of $\mathfrak{gl}(2n, \mathbb{F})$ and, in fact, of $\mathfrak{sl}(2n, \mathbb{F})$. It is also called an **orthogonal Lie algebra** and the **classical Lie algebra of type D_n**.

Remarks 1.2.10 (1) The type B, C and D Lie algebras also have a basis-independent description as Lie subalgebras of $\mathfrak{gl}(V)$ for a vector space V over $\mathbb{F}$ of suitable dimension. In each case, the counterpart of the explicit matrix S is a non-degenerate bilinear form $\langle -, - \rangle\colon V \times V \to \mathbb{F}$ (symmetric for types B and D, skew-symmetric for type C) and the Lie subalgebras consist of exactly those endomorphisms $f\colon V \to V$, such that

$$\langle f(v), w\rangle = -\langle v, f(w)\rangle, \qquad \text{for all } v, w \in V. \tag{1}$$

(2) Among the classical complex Lie algebras, there are the following **exceptional isomorphisms** (that we will not all prove). The classical Lie algebras $\mathfrak{sl}(2, \mathbb{C})$, $\mathfrak{so}(3, \mathbb{C})$, $\mathfrak{sp}(2, \mathbb{C})$ are isomorphic, $\mathfrak{so}(2, \mathbb{C}) \cong \mathbb{C}$ is the 1-dimensional abelian Lie algebra, $\mathfrak{so}(5, \mathbb{C}) \cong \mathfrak{sp}(4, \mathbb{C})$, $\mathfrak{so}(6, \mathbb{C}) \cong \mathfrak{sl}(4, \mathbb{C})$, and $\mathfrak{so}(4, \mathbb{C}) \cong \mathfrak{sl}(2, \mathbb{C}) \times \mathfrak{sl}(2, \mathbb{C})$.

(3) There are also classical Lie algebras of type E_6, E_7, E_8, F_4, and G_2 that we will meet later.

Examples 1.2.11 For $n \in \mathbb{N}$, we have a (commutative) diagram of Lie subalgebras of $\mathfrak{gl}(n, \mathbb{F})$ as follows:

$$\begin{tikzcd}
 & & \mathfrak{n}(n,\mathbb{F}) \arrow[r, hook] \arrow[d, hook] & \mathfrak{sl}(n,\mathbb{F}) \arrow[d, hook] \\
\mathfrak{s}(n,\mathbb{F}) \arrow[r, hook] & \mathfrak{h}(n,\mathbb{F}) \arrow[r, hook] & \mathfrak{t}(n,\mathbb{F}) \arrow[r, hook] & \mathfrak{gl}(n,\mathbb{F})
\end{tikzcd}$$

where

- $\mathfrak{t}(n,\mathbb{F}) := \{M \in \mathfrak{gl}(n,\mathbb{F}) \mid M \text{ is a upper triangular matrix}\}$
- $\mathfrak{n}(n,\mathbb{F}) := \{M \in \mathfrak{gl}(n,\mathbb{F}) \mid M \text{ is a strictly upper triangular matrix}\}$
- $\mathfrak{h}(n,\mathbb{F}) := \{M \in \mathfrak{gl}(n,\mathbb{F}) \mid M \text{ is a diagonal matrix}\}$
- $\mathfrak{s}(n,\mathbb{F}) := \{\lambda I_n \in \mathfrak{gl}(n,\mathbb{F}) \mid \lambda \in \mathbb{F}\}$

It is straightforward to verify that these subspaces are closed under the matrix commutator and, thus, Lie subalgebras.

Exercise 3 Prove the assertion from Remark 1.2.3 that there are exactly two Lie algebra structures on any 2-dimensional vector space. Give an example of a Lie algebra structure on a 3-dimensional vector space that is neither abelian nor isomorphic to $\mathfrak{sl}(2,\mathbb{F})$.

Exercise 4 Verify that the commutator in an associative $\mathbb{F}$-algebra A defines a Lie bracket on the underlying vector space of A as claimed in Construction 1.2.4.

Exercise 5 (on type A) Verify that the commutator of two square matrices of trace zero is again trace zero, as claimed Example 1.2.6. Compute $\dim_{\mathbb{F}}(\mathfrak{sl}(n,\mathbb{F}))$, find a basis of $\mathfrak{sl}(n,\mathbb{F})$ (consisting of matrices with very few non-zero entries), and then evaluate the Lie bracket on all pairs of basis elements.

Exercise 6 (on types, B, C, and D) First explain the precise relationship between the matrices S that appear in Example 1.2.7-1.2.9 and the non-degenerate bilinear form $\langle -, - \rangle$ in Remarks 1.2.10.(1). Then verify that the subspace of endomorphisms $f \in \mathfrak{gl}(V)$ satisfying (1) is closed under the commutator. Deduce that $\mathfrak{so}(2n+1,\mathbb{F})$, $\mathfrak{sp}(2n,\mathbb{F})$, and $\mathfrak{so}(2n,\mathbb{F})$ as described in Example 1.2.7-1.2.9 are indeed Lie algebras.

Exercise 7 (on type B) Explicitly describe the elements $M \in \mathfrak{so}(2n+1,\mathbb{F}) \subset \mathfrak{gl}(2n+1,\mathbb{F})$ appearing in Example 1.2.7 as $(1+n+n) \times (1+n+n)$ block matrices with blocks satisfying certain conditions. Use this to deduce $\mathfrak{so}(2n+1,\mathbb{F}) \subset \mathfrak{sl}(2n+1,\mathbb{F})$, compute $\dim_{\mathbb{F}}(\mathfrak{so}(2n+1,\mathbb{F}))$, and find a basis.

Exercise 8 (on type C) Explicitly describe the elements $M \in \mathfrak{sp}(2n,\mathbb{F}) \subset \mathfrak{gl}(2n,\mathbb{F})$ appearing in Example 1.2.8 as $(n+n) \times (n+n)$ block matrices with blocks satisfying certain conditions. Use this to deduce $\mathfrak{sp}(2n,\mathbb{F}) \subset \mathfrak{sl}(2n,\mathbb{F})$, compute $\dim_{\mathbb{F}}(\mathfrak{sp}(2n,\mathbb{F}))$, and find a basis. Finally, note $\mathfrak{sp}(2,\mathbb{F}) = \mathfrak{sl}(2,\mathbb{F})$.

Exercise 9 (on type D) Explicitly describe the elements $M \in \mathfrak{so}(2n,\mathbb{F}) \subset \mathfrak{gl}(2n,\mathbb{F})$ appearing in Example 1.2.9 as $(n+n) \times (n+n)$ block matrices with blocks satisfying certain conditions. Use this to deduce $\mathfrak{so}(2n,\mathbb{F}) \subset \mathfrak{sl}(2n,\mathbb{F})$, compute $\dim_{\mathbb{F}}(\mathfrak{so}(2n,\mathbb{F}))$, and find a basis. End Week 2 Exerc.

Exercise 10 Verify the assertions made in Examples 1.2.11 and compute the dimensions of all six Lie algebras as functions of $n \in \mathbb{N}$.

1.3 Toolkit

Construction 1.3.1 Let $\mathfrak{g}_1, \mathfrak{g}_2$ be Lie algebras over $\mathbb{F}$. Then one defines a Lie bracket on $\mathfrak{g}_1 \times \mathfrak{g}_2$ by declaring for $x_i, y_i \in \mathfrak{g}_i$:

$$[(x_1, x_2), (y_1, y_2)] := ([x_1, y_1], [x_2, y_2]).$$

This is called the (direct) **product** of $\mathfrak{g}_1$ and g_2. Both appear as Lie subalgebras of $\mathfrak{g}_1 \times \mathfrak{g}_2$ and, by construction $[\mathfrak{g}_1, \mathfrak{g}_2] = 0$. (Here and in the following, if A, B are subspaces of a Lie algebra, then $[A, B]$ denotes the subspace spanned by brackets $[a, b]$ where $a \in A, b \in B$.)

Definition 1.3.2 (1) A linear subspace I of a Lie algebra $\mathfrak{g}$ is called an **ideal** if $x \in \mathfrak{g}$ and $y \in I$ together imply $[x, y] \in I$. Every Lie algebra $\mathfrak{g}$ has itself and 0 (the zero vector space) as ideals.

(2) If $\mathfrak{g}$ has no other ideals besides 0 and $\mathfrak{g}$ and is not abelian, then it is called **simple**.

Note that every ideal is a Lie subalgebra, but not necessarily the other way round.

Lemma 1.3.3 Let $f : \mathfrak{g}_1 \to \mathfrak{g}_2$ be a Lie algebra morphism. Then $\ker(f)$ is an ideal in $\mathfrak{g}_1$ and $\mathrm{im}(f)$ is a Lie subalgebra of $\mathfrak{g}_2$.

Proof. If $x \in \mathfrak{g}_1$ and $y \in \ker(f)$, then $f([x, y]) = [f(x), f(y)] = [f(x), 0] = 0$ and thus $[x, y] \in \ker(f)$, i.e. $\ker(f)$ is an ideal. If $x', y' \in \mathrm{im}(f)$, then $[x', y'] = [f(x), f(y)] = f([x, y]) \in \mathrm{im}(f)$ for some $x, y \in \mathfrak{g}_1$. So $\mathrm{im}(f)$ is a Lie subalgebra of $\mathfrak{g}_2$. □

Remark 1.3.4 A consequence of the lemma is that a non-abelian Lie algebra $\mathfrak{g}$ is simple if and only if every Lie algebra morphism out of $\mathfrak{g}$ is either zero or injective. (For the "if"-direction, see Construction 1.3.9.)

Example 1.3.5 Both Lie algebras $\mathfrak{g}_1, \mathfrak{g}_2$ are ideals in $\mathfrak{g}_1 \times \mathfrak{g}_2$.

Example 1.3.6 The **center** of a Lie algebra $\mathfrak{g}$ is defined as $Z(\mathfrak{g}) = \{z \in \mathfrak{g} \mid [z, x] = 0 \text{ for all } x \in \mathfrak{g}\}$. It is an ideal in $\mathfrak{g}$. We have $\mathfrak{g} = Z(g)$ if and only if $\mathfrak{g}$ is abelian.

Example 1.3.7 If $\mathfrak{g}$ is a Lie algebra, then it follows from the Jacobi identity that the subspace $[\mathfrak{g}, \mathfrak{g}]$ spanned by brackets is an ideal in $\mathfrak{g}$. It is called the **derived Lie algebra** of $\mathfrak{g}$. We have $[\mathfrak{g}, \mathfrak{g}] = 0$ if and only if $\mathfrak{g}$ is abelian.

Remark 1.3.8 Let I, J be ideals in a Lie algebra $\mathfrak{g}$. Then $I \cap J$ and $I + J = \{x + y \mid x \in I, y \in J\}$ and $[I, J]$ are ideals in $\mathfrak{g}$.

Construction 1.3.9 If I is an ideal in $\mathfrak{g}$, then we consider the quotient vector space $\mathfrak{g}/I$ and equip it with the Lie bracket:

$$[x + I, y + I] := [x, y] + I$$

This is well-defined and independent of the representatives $x, y \in \mathfrak{g}$ since I is an ideal. The resulting Lie algebra $\mathfrak{g}/I$ is called the **quotient** of $\mathfrak{g}$ by I. The canonical quotient map $q : \mathfrak{g} \to \mathfrak{g}/I$ is a Lie algebra morphism. L2

Note that we have a short exact sequence: End

$$I \hookrightarrow \mathfrak{g} \twoheadrightarrow \mathfrak{g}/I$$

In this case $\mathfrak{g}$ is called an **extension** of $\mathfrak{g}/I$ by I.

Theorem 1.3.10 The standard isomorphism theorems hold for Lie algebras.

(1) Let $f : \mathfrak{g}_1 \to \mathfrak{g}_2$ be a morphism of Lie algebras, then

$$\mathfrak{g}_1/\ker(f) \cong \mathrm{im}(f).$$

Moreover, if I is any ideal of $\mathfrak{g}_1$ and $I \subset \ker(f)$, then there exists a unique morphism $g : \mathfrak{g}_1/I \to \mathfrak{g}_2$ such that $f = g \circ q$ where $q : \mathfrak{g}_1 \to \mathfrak{g}_1/I$ is the quotient map.

(2) If I and J are ideals in $\mathfrak{g}_1$ and $I \subset J$, then J/I is an ideal of $\mathfrak{g}_1/I$ and $(\mathfrak{g}_1/I)/(J/I) \cong \mathfrak{g}_1/J$.

(3) If I and J are ideals in $\mathfrak{g}_1$, then $(I + J)/J \cong I/(I \cap J)$.

Proof. Exercise 12. □

We list some standard notions that may be used later.

Definition 1.3.11 Let g be a Lie algebra. The **normalizer** of a Lie subalgebra $\mathfrak{h}$ in $\mathfrak{g}$ is defined as $N_{\mathfrak{g}}(\mathfrak{h}) := \{x \in \mathfrak{g} \mid [x, \mathfrak{h}] \subset \mathfrak{h}\}$. This is a subalgebra in $\mathfrak{g}$, namely the largest one containing $\mathfrak{h}$ as an ideal. If $N_{\mathfrak{g}}(\mathfrak{h}) = \mathfrak{h}$, then $\mathfrak{h}$ is said to be **self-normalizing**.

Definition 1.3.12 Let g be a Lie algebra. The **centralizer** of a subset S in $\mathfrak{g}$ is defined as $C_{\mathfrak{g}}(S) := \{x \in \mathfrak{g} \mid [x, S] = 0\}$. This is again a subalgebra in $\mathfrak{g}$. Note that $C_{\mathfrak{g}}(\mathfrak{g}) = Z(\mathfrak{g})$ and for a subalgebra $\mathfrak{h}$ we always have $C_{\mathfrak{g}}(\mathfrak{h}) \subset N_{\mathfrak{g}}(\mathfrak{h})$.

Exercise 11 Prove the assertions in Remark 1.3.8.

Exercise 12 Prove Theorem 1.3.10. You may use the standard isomorphism theorems for vector spaces.

Exercise 13 Let $\mathrm{char}(\mathbb{F}) = 0$ or $\mathrm{char}(\mathbb{F}) = p$ a prime not dividing $n \in \mathbb{N}$. Prove that $\mathfrak{gl}(n, \mathbb{F}) = \mathfrak{sl}(n, \mathbb{F}) + \mathfrak{s}(n, \mathbb{F})$ with $[\mathfrak{sl}(n, \mathbb{F}), \mathfrak{s}(n, \mathbb{F})] = 0$.

Exercise 14 If $\mathrm{char}(\mathbb{F}) \neq 2$ show that $[\mathfrak{sl}(n, \mathbb{F}), \mathfrak{sl}(n, \mathbb{F})] = \mathfrak{sl}(n, \mathbb{F})$ for all $n \geq 1$. What happens if $\mathrm{char}(\mathbb{F}) = 2$?

1.4 Representations of Lie algebras

Definition 1.4.1 Let $\mathfrak{g}$ be a Lie algebra and V a vector space, both over $\mathbb{F}$. A **representation** of $\mathfrak{g}$ on V is defined to be a Lie algebra morphism $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$. In this case, we also say $\mathfrak{g}$ **acts** on V by ρ. Specifically, if $x \in \mathfrak{g}$, then $\rho(x) \in \mathfrak{gl}(V) = \mathrm{End}_{\mathbb{F}}(V)$, and the action of x on v is defined as $x \cdot v := \rho(x)(v) \in V$. If ρ is injective, then the representation is called **faithful**.

Sometimes we abusively refer to V as a representation of $\mathfrak{g}$ if the Lie algebra morphism ρ is clear from the context.

Remark 1.4.2 A representation of $\mathfrak{g}$ on V can equivalently described via its action. This is the data of the bilinear map $\mathfrak{g} \times V \to V$, $(x, v) \mapsto x \cdot v$, which satisfies the equation $x \cdot (y \cdot v) - y \cdot (x \cdot v) = [x, y] \cdot v$ for all $x, y \in \mathfrak{g}$ and $v \in V$.

Examples 1.4.3 (1) Every vector space V admits an action of $\mathfrak{gl}(V)$ by $\rho = \mathrm{id}\colon \mathfrak{gl}(V) \to \mathfrak{gl}(V)$.

(2) Let $\mathfrak{g}$ be any of the linear Lie algebras from the previous section. These were defined as Lie subalgebras of some $\mathfrak{gl}(n, \mathbb{F}) \cong \mathfrak{gl}(\mathbb{F}^n)$. Then the inclusion $\rho\colon \mathfrak{g} \to \mathfrak{gl}(\mathbb{F}^n)$ is a Lie algebra morphism that defines the **vector representation** of $\mathfrak{g}$ on $\mathbb{F}^n$.

(3) Any Lie algebra $\mathfrak{g}$ acts on any vector space V over the same $\mathbb{F}$ by the **trivial action** $x \cdot v = 0$ for all $x \in \mathfrak{g}$ and $v \in V$. The 1-dimensional vector space $\mathbb{F}$ with this trivial action is called **the trivial representation** (in the physics literature sometimes **singlet**) of $\mathfrak{g}$.

Definition 1.4.4 (1) Let $\rho_1\colon \mathfrak{g} \to \mathfrak{gl}(V_1)$ and $\rho_2\colon \mathfrak{g} \to \mathfrak{gl}(V_2)$ be two representations of $\mathfrak{g}$. A **morphism** of $\mathfrak{g}$-representations (also called a **$\mathfrak{g}$-intertwiner**) from V_1 to V_2 is a linear map $f\colon V_1 \to V_2$ such that $\rho_2(x) \circ f = f \circ \rho_1(x)$ as maps $V_1 \to V_2$ for all $x \in \mathfrak{g}$. In terms of actions, this means $x \cdot f(v) = f(x \cdot v)$ for all $x \in \mathfrak{g}$ and $v \in V$. If f is furthermore an isomorphism of vector spaces, then we say that V_1 and V_2 are isomorphic $\mathfrak{g}$-representations, in symbols $V_1 \cong V_2$.

(2) A vector subspace U of a representation V of $\mathfrak{g}$ is called a **subrepresentation** of V if $x \cdot u \in U$ for every $x \in \mathfrak{g}$ and $u \in U$. In this case, the natural inclusion $U \hookrightarrow V$ is an injective morphism of $\mathfrak{g}$-representations. Moreover, there exists a unique structure of a $\mathfrak{g}$-representation on the quotient vector space V/U such that the quotient map $q\colon V \to V/U$ is an intertwiner. It is called the **quotient representation**.

(3) A representation V of $\mathfrak{g}$ is called **irreducible** or **simple** if $V \neq 0$ (it is not the zero vector space) and its only subrepresentations are 0 and V itself.

(4) If V_1 and V_2 are $\mathfrak{g}$-representation, then the **direct sum** $V_1 \oplus V_2$ naturally carries the structure of a $\mathfrak{g}$-representation by declaring $x \cdot (v_1 + v_2) = (x \cdot v_1) + (x \cdot v_2)$ for $x \in \mathfrak{g}$ and $v_1 \in V_1, v_2 \in V_2$.

(5) A representation V of $\mathfrak{g}$ is called **indecomposable** if $V \cong V_1 \oplus V_2$ implies that either $V_1 \cong 0$ or $V_1 \cong V$.

Any direct summand of a representation is also a subrepresentation, but not necessarily the other way round. Conversely, any irreducible representation is also indecomposable, but not necessarily the other way round.

Remarks 1.4.5 (1) For $\mathfrak{g}$-representations V, W we write $\mathrm{Hom}_{\mathfrak{g}}(V, W) := \mathrm{Hom}_{\mathrm{Rep}(\mathfrak{g})}(V, W)$ for the vector spaces of $\mathfrak{g}$-intertwiners from V to W. In the case $V = W$ we abbreviate $\mathrm{End}_{\mathfrak{g}}(V) := \mathrm{End}_{\mathrm{Rep}(\mathfrak{g})}(V) := \mathrm{Hom}_{\mathfrak{g}}(V, V)$.

(2) Given $f \in \mathrm{Hom}_{\mathfrak{g}}(V_1, V_2)$ and $g \in \mathrm{Hom}_{\mathfrak{g}}(V_2, V_3)$, then $g \circ f \in \mathrm{Hom}_{\mathfrak{g}}(V_1, V_3)$. To verify this, let the representations be given by $\rho_i\colon \mathfrak{g} \to \mathfrak{gl}(V_i)$ for $i = 1, 2, 3$ and for $x \in \mathfrak{g}$ we thus check:

$$\rho_3(x) \circ (g \circ f) = g \circ \rho_2(x) \circ f = (g \circ f) \circ \rho_1(x)$$

(3) For any $\mathfrak{g}$-representation V there are two naturally associated vector spaces with trivial $\mathfrak{g}$-action. First the subrepresentation of **$\mathfrak{g}$-invariants**

$$V^{\mathfrak{g}} := \{v \in V \mid x \cdot v = 0 \text{ for all } x \in \mathfrak{g}\}$$

and second the quotient representation of **$\mathfrak{g}$-coinvariants**

$$V_{\mathfrak{g}} := V/\mathfrak{g} \cdot V.$$

(4) Let V, W be representations of a Lie algebra $\mathfrak{g}$ over $\mathbb{F}$. Then the space of $\mathbb{F}$-linear maps $\mathrm{Hom}_{\mathbb{F}}(V, W)$ carries a $\mathfrak{g}$-action defined by:

$$(x \cdot f)(v) := x \cdot f(v) - f(x \cdot v) \quad \text{for all } x \in \mathfrak{g}, v \in V, f \in \mathrm{Hom}_{\mathbb{F}}(V, W)$$

The $\mathfrak{g}$-intertwiners between $\mathfrak{g}$-representations V, W are exactly the $\mathfrak{g}$-invariants among the linear maps from V to W, in formulas

$$\mathrm{Hom}_{\mathfrak{g}}(V, W) = \mathrm{Hom}_{\mathbb{F}}(V, W)^{\mathfrak{g}}$$

see Exercise 15.

(5) Choosing for W the trivial representation $W = \mathbb{F}$ in (4), one obtains a $\mathfrak{g}$-representation on the dual space $V^* = \mathrm{Hom}_{\mathbb{F}}(V, \mathbb{F})$ called the **contragradient representation**. For $x \in \mathfrak{g}$ and $f \in V^*$, one has

$$(x \cdot f)(v) = (-f)(x \cdot v) \quad \text{for } v \in V.$$

(6) Choosing for V the trivial representation $V = \mathbb{F}$ in (4), one obtains an isomorphism of $\mathfrak{g}$-representations:

$$\begin{aligned} W &\longrightarrow \mathrm{Hom}_{\mathbb{F}}(\mathbb{F}, W), \\ w &\longmapsto (\lambda \longmapsto \lambda w). \end{aligned}$$

(7) Let V, W be representations of a Lie algebra $\mathfrak{g}$ over $\mathbb{F}$. Then the tensor product vector space $V \otimes W$ carries an action of $\mathfrak{g}$, defined on elementary tensors by $x \cdot (v \otimes w) = (x \cdot v) \otimes w + v \otimes (x \cdot w)$ for $x \in \mathfrak{g}, v \in V, w \in W$. For finite-dimensional V, W, one then has $\mathrm{Hom}_{\mathbb{F}}(V, W) \cong W \otimes V^*$, see Exercise 16. L3 End

Lemma 1.4.6 Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{F}$ and $f \in \mathrm{End}_{\mathfrak{g}}(V)$ an endomorphism of a $\mathfrak{g}$-representation V. For every $\lambda \in \mathbb{F}$ the generalized eigenspace

$$\overline{V}_{\lambda}(f) := \bigcup_{n \geq 0} \ker(f - \lambda \mathrm{id}_V)^n$$

is a subrepresentation of V.

Proof. If $x \in \mathfrak{g}$ and $v \in \ker(f - \lambda \mathrm{id}_V)^n$ for some $n \geq 0$, then we have

$$(f - \mathrm{id}_V)^n (x \cdot v) = x \cdot ((f - \mathrm{id}_V)^n (v)) = x \cdot 0 = 0$$

so $x \cdot v \in \ker(f - \lambda \mathrm{id}_V)^n \subset \overline{V}_{\lambda}(f)$. □

Example 1.4.7 Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{F}$. A 1-dimensional representation of $\mathfrak{g}$ is a Lie algebra morphism $\rho\colon \mathfrak{g} \longrightarrow \mathfrak{gl}(\mathbb{F})$. If we identify $\mathfrak{gl}(\mathbb{F}) \cong \mathbb{F}$, this is equivalent to the data of a linear form $\lambda \in \mathfrak{g}^*$, such that $\lambda|_{[\mathfrak{g},\mathfrak{g}]} = 0$. Thus we have a bijection:

$$(\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])^* \leftrightarrow \begin{cases} \text{1-dimensional representations} \\ \text{up to isomorphism} \end{cases}$$

The elements on the left are called **characters** of $\mathfrak{g}$. If a 1-dimensional representation corresponds to $\lambda \in (\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])^*$, then its contragradient representation corresponds to $-\lambda$.

Exercise 15 Verify both assertions made in Remarks 1.4.5.(4).

Exercise 16 Verify both assertions made in Remarks 1.4.5.(7).

Exercise 17 (may split) Let U, V, W be $\mathfrak{g}$-representations, all over $\mathbb{F}$. Show that the swap of tensor factors $U \otimes V \cong V \otimes U$ as well as the two canonical isomorphisms

$$U \otimes (V \otimes W) \xrightarrow{\cong} (U \otimes V) \otimes W$$
$$\mathrm{Hom}_{\mathbb{F}}(U, \mathrm{Hom}_{\mathbb{F}}(V, W)) \xrightarrow{\cong} \mathrm{Hom}_{\mathbb{F}}(U \otimes V, W)$$

of vector spaces are $\mathfrak{g}$-intertwiners. Now take $\mathfrak{g}$-invariants to deduce $\mathrm{Hom}_{\mathfrak{g}}(U, \mathrm{Hom}_{\mathbb{F}}(V, W)) \cong \mathrm{Hom}_{\mathfrak{g}}(U \otimes V, W)$.

Exercise 18 (may split) Let U, V, W, X be $\mathfrak{g}$-representations, all over $\mathbb{F}$. Show that the following are $\mathfrak{g}$-intertwiners:

$$\text{“composing linear maps”} \quad \mathrm{Hom}_{\mathbb{F}}(U, V) \otimes \mathrm{Hom}_{\mathbb{F}}(V, W) \to \mathrm{Hom}_{\mathbb{F}}(U, W)$$
$$\text{“tensoring linear maps”} \quad \mathrm{Hom}_{\mathbb{F}}(U, V) \otimes \mathrm{Hom}_{\mathbb{F}}(W, X) \to \mathrm{Hom}_{\mathbb{F}}(U \otimes W, V \otimes X)$$

Exercise 19 Let V be a $\mathfrak{g}$-representation and $r \geq 0$. Show that there exists a unique action of $\mathfrak{g}$ on the exterior power $\bigwedge^r V$, such that the canonical projection $V^{\otimes r} \twoheadrightarrow \bigwedge^r V$ is a $\mathfrak{g}$-intertwiner. Similarly, show that there exists a unique action of $\mathfrak{g}$ on the symmetric power $S^r V$, such that $V^{\otimes r} \twoheadrightarrow S^r V$ is a $\mathfrak{g}$-intertwiner.

Exercise 20 (*) Recall or familiarize yourself with the notion of a **symmetric monoidal closed category**. Do representations of a Lie algebra $\mathfrak{g}$ form a symmetric monoidal closed $\mathbb{F}$-linear category? End Week 3 Exerc.

1.5 The adjoint representation and derivations

Every Lie algebra has distinguished representation, which is very important for the further theoretical development: the adjoint representation.

Lemma 1.5.1 Let $\mathfrak{g}$ be any Lie algebra and $\mathfrak{gl}(\mathfrak{g})$ the general linear Lie algebra of endomorphism of the underlying vector space of $\mathfrak{g}$. Then $\mathfrak{g}$ acts on the vector space $\mathfrak{g}$ by the **adjoint representation**, which is given by:

$$\begin{aligned} \mathrm{ad}\colon \mathfrak{g} &\to \mathfrak{gl}(\mathfrak{g}) \\ x &\mapsto \mathrm{ad}_x, \end{aligned}$$

where $\mathrm{ad}_x(y) := [x, y]$ for $y \in \mathfrak{g}$.

Proof. We check that ad is indeed a morphism of Lie algebras. Let $x, y, z \in \mathfrak{g}$ and compute:

$$\begin{aligned} \mathrm{ad}_{[x,y]}(z) &= [[x, y], z] \\ &= -[z, [x, y]] \\ &= [x, [y, z]] + [y, [z, x]] \\ &= [x, [y, z]] - [y, [x, z]] \\ &= \mathrm{ad}_x([y, z]) - \mathrm{ad}_y([x, z]) \\ &= \mathrm{ad}_x(\mathrm{ad}_y(z)) - \mathrm{ad}_y(\mathrm{ad}_x(z)) \\ &= [\mathrm{ad}_x, \mathrm{ad}_y](z) \end{aligned}$$

Here we have used antisymmetry twice and the Jacobi identity once. □

Note the two different uses of the brackets in the proof, first as a (formal) bracket in $\mathfrak{g}$ and finally as commutator of linear maps in $\mathfrak{gl}(\mathfrak{g})$ in the last line.

Remarks 1.5.2 Let $\mathfrak{g}$ be a Lie algebra.

(1) The ideals of $\mathfrak{g}$ are exactly the subrepresentations of the adjoint representation.

(2) The center of $\mathfrak{g}$ is the kernel of the adjoint representation, i.e. $Z(\mathfrak{g}) = \ker(\mathrm{ad})$.

Having discussed the adjoint representation, it is now worth to take a short detour and consider the concept of derivations.

Definition 1.5.3 Let (A, m) be an algebra over $\mathbb{F}$ in the sense of Definition 1.1.4. A **derivation** of A is a linear map $\delta\colon A \to A$ that satisfies $\delta(m(x, y)) = m(\delta(x), y) + m(x, \delta(y))$ for $x, y \in A$.

Lemma 1.5.4 The derivations of an algebra (A, m) form a Lie subalgebra $\mathrm{Der}(A)$ of $\mathfrak{gl}(A)$.

Proof. It is straightforward to check that linear combinations of derivations are again derivations. It remains to check that the commutator of two derivations δ_1, δ_2 is again a derivation. For this let $x, y \in A$.

$$\begin{aligned}
[\delta_1, \delta_2](m(x, y)) &= \delta_1(\delta_2(m(x, y))) - \delta_2(\delta_1(m(x, y))) \\
&= \delta_1(m(\delta_2(x), y) + m(x, \delta_2(y))) - \delta_2(m(\delta_1(x), y) + m(x, \delta_1(y))) \\
&= m(\delta_1(\delta_2(x)), y) + m(\delta_2(x), \delta_1(y)) + m(\delta_1(x), \delta_2(y)) + m(x, \delta_1(\delta_2(y))) \\
&\quad - m(\delta_2(\delta_1(x)), y) - m(\delta_1(x), \delta_2(y)) - m(\delta_2(x), \delta_1(y)) - m(x, \delta_2(\delta_1(y))) \\
&= m([\delta_1, \delta_2](x), y) + m(x, [\delta_1, \delta_2](y))
\end{aligned}$$

□

Lemma 1.5.5 Let $\mathfrak{g}$ be a Lie algebra, then for $x \in \mathfrak{g}$, we have $\mathrm{ad}_x \in \mathrm{Der}(\mathfrak{g})$. Such derivations of $\mathfrak{g}$ are called **inner derivations**.

Proof. Let's check that ad_x is indeed a derivation. For this let $y, z \in \mathfrak{g}$.

$$\begin{aligned}
\mathrm{ad}_x([y, z]) &= [x, [y, z]] \\
&= [[x, y], z] + [y, [x, z]] \\
&= [\mathrm{ad}_x(y), z] + [y, \mathrm{ad}_x(z)]
\end{aligned}$$

□

Example 1.5.6 A short exact sequence of Lie algebras $\mathfrak{g}_1 \hookrightarrow \mathfrak{g} \to \mathfrak{g}_2$ is called **split** if there exists a Lie algebra morphism $\sigma\colon \mathfrak{g}_2 \to \mathfrak{g}$ such that $q \circ \sigma = \mathrm{id}_{\mathfrak{g}_2}$. Such an σ is necessarily injective and $\mathfrak{g} \cong \mathfrak{g}_1 + \mathfrak{g}_2$. The latter is a direct sum of vector spaces but $\mathfrak{g}_2$ need not be an ideal in $\mathfrak{g}_1 + \mathfrak{g}_2$. In that case we only have a **semi-direct** product of Lie algebras. However, since $\mathfrak{g}_1$ is an ideal, we have $[x_2, x_1] \in \mathfrak{g}_1$ for $x_1 \in \mathfrak{g}_1$ and $x_2 \in \mathfrak{g}_2$, i.e. we have a representation of $\mathfrak{g}_2$ on $\mathfrak{g}_1$. More specifically $\mathfrak{g}_2$ acts by derivations on $\mathfrak{g}_1$, i.e. the corresponding Lie algebra morphism satisfies $\rho\colon \mathfrak{g}_2 \to \mathrm{Der}(\mathfrak{g}_1) \subset \mathfrak{gl}(\mathfrak{g}_1)$. (To see this, one performs a calculation very similar to the proof of Lemma 1.5.5.)

2 Crash course on $\mathfrak{sl}(2,\mathbb{F})$-representation theory

In this subsection we let $\mathbb{F}$ be a field of $\mathrm{char}(\mathbb{F}) = 0$.

2.1 Finite-dimensional simple representations

Here we study finite-dimensional representations of $\mathfrak{sl}(2,\mathbb{F})$, which will serve as a warm-up for the further development and also as technical tool in later parts of the course.

Recall that $\mathfrak{sl}(2,\mathbb{F})$ denotes the Lie algebra of trace zero 2×2 matrices with entries in $\mathbb{F}$ and with the Lie bracket given by the matrix commutator. A basis is given by the following three matrices:

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

The Lie bracket on $\mathfrak{sl}(2,\mathbb{F})$ satisfies:

$$[h,e] = 2e, \quad [h,f] = -2f, \quad [e,f] = h \tag{2}$$

and this determines the value of the Lie bracket on any pair of elements of $\mathfrak{sl}(2,\mathbb{F})$ since we automatically have $[e,e] = [f,f] = [h,h] = 0$ and $[x,y] = -[y,x]$ for any $x,y \in \mathfrak{sl}(2,\mathbb{F})$. More generally, any triple of elements e,h,f of a Lie algebra $\mathfrak{g}$ that satisfy (2) is called an $\mathfrak{sl}(2)$-**triple**.

Let $\rho\colon \mathfrak{sl}(2,\mathbb{F}) \longrightarrow \mathfrak{gl}(V)$ be a representation, then it is convenient to abbreviate

$$E := \rho(e), \quad H := \rho(h), \quad F = \rho(f)$$

We may consider E,H,F as elements of the associative algebra $\mathrm{End}_{\mathbb{F}}(V)$. In the physics literature the elements E and F are sometimes called **creation** and **annihilation operators**. In fact, three operators E,H,F on a vector space V over $\mathbb{F}$ form an $\mathfrak{sl}(2,\mathbb{F})$ representation if and only if

$$HE - EH = 2E, \quad HF - FH = -2F, \quad EF - FE = H.$$

For a fixed $\mathfrak{sl}(2,\mathbb{F})$-representation V and $\mu \in \mathbb{F}$ we write $V_\mu := \ker(H - \mu\mathrm{id}_V)$ for the eigenspace of H for the eigenvalue μ and $\overline{V}_\mu := \bigcup_{n\geq 0} \ker(H-\mu\mathrm{id}_V)^n$ for the corresponding generalized eigenspace. We refer to V_μ as the ***μ*-weight space** of V. Now we observe

$$E(H-\mu\mathrm{id}) = (H-(\mu+2)\mathrm{id})E, \quad F(H-\mu\mathrm{id}) = (H-(\mu-2)\mathrm{id})F$$

which implies $E(V_\mu) \subset V_{\mu+2}$ and $F(V_\mu) \subset V_{\mu-2}$ and similarly $E(\overline{V}_\mu) \subset \overline{V}_{\mu+2}$ and $F(\overline{V}_\mu) \subset \overline{V}_{\mu-2}$.

Examples 2.1.1 (1) The trivial representation $V = \mathbb{F}$ of $\mathfrak{sl}(2,\mathbb{F})$ coincides with its zero-weight space $V = V_0$ and the action of E, H, and F can be illustrated as:

$$\begin{tikzcd} 0 & \mathbb{F} \arrow[l, bend left, "F"] \arrow[r, bend left, "E"] \arrow[loop above, "H=0"] & 0 \end{tikzcd}$$

(2) The vector representation $V \cong \mathbb{F}^2$ is the direct sum of its nonzero weight spaces $V = V_{-1} \oplus V_1$, each of which is 1-dimensional and spanned by a standard basis vector in $\mathbb{F}^2$. The action of E, H, and F can be illustrated as:

$$\begin{tikzcd} 0 & \mathbb{F}\begin{pmatrix}0\\1\end{pmatrix} \arrow[l, bend left, "F"] \arrow[r, bend left, "E"] \arrow[loop above, "H=-\mathrm{id}"] & \mathbb{F}\begin{pmatrix}1\\0\end{pmatrix} \arrow[l, bend left, "F"] \arrow[r, bend left, "E"] \arrow[loop above, "H=\mathrm{id}"] & 0 \end{tikzcd}$$

(3) By the **adjoint representation**, the Lie algebra $\mathfrak{sl}(2,\mathbb{F})$ acts on itself via the assignment $x \cdot v := [x,v]$ for $x \in \mathfrak{sl}(2,\mathbb{F})$ and $v \in V := \mathfrak{sl}(2,\mathbb{F})$. By the above formulas, we have $V = V_{-2} \oplus V_0 \oplus V_2$ and the action of E, H, and F are illustrated by:

$$\begin{tikzcd} 0 & \mathbb{F}f \arrow[l, bend left, "F"] \arrow[r, bend left, "E"] \arrow[loop above, "H=-2\mathrm{id}"] & \mathbb{F}h \arrow[l, bend left, "F"] \arrow[r, bend left, "E"] \arrow[loop above, "H=0"] & \mathbb{F}e \arrow[l, bend left, "F"] \arrow[r, bend left, "E"] \arrow[loop above, "H=2\mathrm{id}"] & 0 \end{tikzcd}$$

More precisely, $E(h) = e \cdot h = -2e$ and $F(h) = f \cdot h = 2f$, so the associated arrows also scale the corresponding basis elements.

These three examples form the start of a sequence of representations of $\mathfrak{sl}(2, \mathbb{F})$, one of each positive dimension, which represent all isomorphism classes of finite-dimensional simple representations. L4 End

Theorem 2.1.2 Let $\mathbb{F}$ be a field of characteristic $\mathrm{char}(\mathbb{F}) = 0$ and let $\{e, h, f\}$ be a basis of $\mathfrak{sl}(2, \mathbb{F})$ satisfying $[h, e] = 2e$, $[h, f] = -2f$ and $[e, f] = h$.

(1) For every positive $n \in \mathbb{N}$, there exists a simple $\mathfrak{sl}(2, \mathbb{F})$-representation L of dimension n and it is unique up to isomorphism.

(2) Every simple $\mathfrak{sl}(2, \mathbb{F})$-representation L of dimension $m + 1$ decomposes into 1-dimensional eigenspaces for the action of h:

$$L = L_{-m} \oplus L_{2-m} \oplus \cdots \oplus L_{m-2} \oplus L_m$$

with integral eigenvalues $-m, 2 - m, \ldots, m - 2, m$. Moreover, if $L_j \neq 0 \neq L_{j+2}$, then the actions of e and f restrict to isomorphisms $e\colon L_j \xrightarrow{\cong} L_{j+2}$ and $f\colon L_{j+2} \xrightarrow{\cong} L_j$.

In other words, every finite-dimensional simple $\mathfrak{sl}(2, \mathbb{F})$-representation is of the form:

$$\begin{tikzcd}
0 & \mathbb{F} \arrow[l, bend left, "F"] \arrow[r, bend left, "E"] \arrow[loop above, "H=-m\mathrm{id}"] & \mathbb{F} \arrow[l, bend left, "F"] \arrow[r, bend left, "E"] \arrow[loop above, "H=(2-m)\mathrm{id}"] & \cdots \arrow[l, bend left, "F"] \arrow[r, bend left, "E"] & \mathbb{F} \arrow[l, bend left, "F"] \arrow[r, bend left, "E"] \arrow[loop above, "H=(m-2)\mathrm{id}"] & \mathbb{F} \arrow[l, bend left, "F"] \arrow[r, bend left, "E"] \arrow[loop above, "H=m\mathrm{id}"] & 0
\end{tikzcd}$$

Proof. We only consider the case of an algebraically closed field $\mathbb{F}$. We first prove the existence part of (1). For this we first construct an infinite-dimensional representation of $\mathfrak{sl}(2, \mathbb{F})$, namely on the polynomial ring $\mathbb{F}[x, y]$ in two variables. Specifically we define $\rho\colon \mathfrak{sl}(2, \mathbb{F}) \to \mathfrak{gl}(\mathbb{F}[x, y])$ on the basis e, h, f by:

$$\begin{aligned}
E &= \rho(e) = x\partial_y \\
H &= \rho(h) = x\partial_x - y\partial_y \\
F &= \rho(f) = y\partial_x
\end{aligned}$$

Here ∂_x, ∂_y denote partial differentiation with respect to x and y respectively, and x, y are the linear maps $\mathbb{F}[x, y] \to \mathbb{F}[x, y]$ that multiply by x and y respectively. To verify that ρ defines a representation, we have to check for all $p \in \mathbb{F}[x, y]$:

$$\begin{aligned}
(HE - EH)(p) &= (x\partial_x - y\partial_y)(x\partial_y)(p) - (x\partial_y)(x\partial_x - y\partial_y)(p) \\
&= (2x\partial_y)(p) = 2E(p) \\
(HF - FH)(p) &= (x\partial_x - y\partial_y)(y\partial_x)(p) - (y\partial_x)(x\partial_x - y\partial_y)(p) \\
&= (-2y\partial_x)(p) = -2F(p) \\
(EF - FE)(p) &= (x\partial_y)(y\partial_x)(p) - (y\partial_x)(x\partial_y)(p) = (x\partial_x - y\partial_y)(p) = H(p)
\end{aligned}$$

For $m \in \mathbb{Z}_{\geq 0}$, let $V(m) \subset \mathbb{F}[x, y]$ denote the subspace of polynomials of total degree m, which has as basis the monomials $\{v_i = y^i x^{m-i}\}_{0 \leq i \leq m}$. Note that the maps E, H, F preserve the total degree of polynomials in $\mathbb{F}[x, y]$, so $V(m)$ is a subrepresentation. We compute the action of these maps on the basis:

$$\begin{aligned}
E(v_i) &= (x\partial_y)(y^i x^{m-i}) = i(y^{i-1} x^{m-i+1}) = i v_{i-1} \\
H(v_i) &= (m - 2i) v_i \\
F(v_i) &= (m - i) v_{i+1}
\end{aligned}$$

where we set $v_{-1} = v_{m+1} = 0$ for convenience. Now we claim that $V(m)$ is simple. Any nonzero subrepresentation $U \subset V(n)$ must contain an eigenvector for H, i.e. one of the v_i. But then it contains also all images under repeated application of E and F, so all the v_i and we deduce $U = V(m)$. Thus for every $n \geq 1$ we have constructed a simple $\mathfrak{sl}(2, \mathbb{F})$-representation $V(n - 1)$ of dimension $\dim_{\mathbb{F}}(V(n - 1)) = n$.

Let V now be any finite-dimensional representation of $\mathfrak{sl}(2, \mathbb{F})$ by operators E, H, F with $E(v) = e \cdot v, H(v) = h \cdot v, F(v) = f \cdot v$ as above. If $V \neq 0$, then there exists at least one $\lambda \in \mathbb{F}$ such that the weight space $V_\lambda \neq 0$. By finite-dimensionality, we may choose λ such that $V_{\lambda+2} = 0$. For every $v \in V_\lambda$ we have $E(v) = 0$ and $H(v) = \lambda v$. Furthermore, $HF^n(v) = (\lambda - 2n)F^n \cdot v$ since $F^n(V_\lambda) \subset V_{\lambda-2n}$. Now one inductively shows

$$EF^n(v) = n(\lambda - n + 1)F^{n-1} \cdot v$$

for all $n \geq 0$. This implies that $U = \mathrm{span}_{\mathbb{F}}\{v_i = F^i(v) \mid i \geq 0\}$ is a subrepresentation of V. If V was simple, then we get $U = V$. Let $d \geq 0$ be minimal such that $F^d(v) = 0$, then $v, F(v), \ldots, F^{d-1}(v)$ are linearly independent, and thus a basis of V, and thus $\dim_{\mathbb{F}}(V) = d$. We also deduce $0 = EF^d(v) = d(\lambda - d + 1)F^{d-1}(v)$, which implies $\lambda = d - 1$. This verifies the claims of (2) and the uniqueness stated in (1), since the action of E, H, F in terms of the basis $\{v, F(v), \ldots, F^{d-1}(v)\}$ only depends on d. In particular, V is isomorphic to the simple subrepresentation $V(d-1)$ of $\mathbb{F}[x, y]$. □

Remark 2.1.3 For $m \geq 0$ we let $L(m)$ denote the $(m+1)$-dimensional simple $\mathfrak{sl}(2, \mathbb{F})$-representation, which is unique up to isomorphism and has highest weight m. Note that $L(0)$ is the trivial representation, $L(1)$ is the vector representation, and $L(2)$ is the adjoint representation. It is convenient to relabel the basis vectors $w_{m-2i} := v_i$ by their weights $\{m, m-2, \ldots, 2-m, -m\}$. Then one has:

$$\begin{aligned} E(w_j) &= e \cdot w_j = \frac{m-j}{2} w_{j+2} \\ H(w_j) &= h \cdot w_j = j w_j \\ F(w_j) &= f \cdot w_j = \frac{m+j}{2} w_{j-2} \end{aligned}$$

Proposition 2.1.4 Let $\mathbb{F}$ be a field of characteristic $\mathrm{char}(\mathbb{F}) = 0$. Every finite-dimensional representation of $\mathfrak{sl}(2, \mathbb{F})$ decomposes into a direct sum of simple subrepresentations.

Proof. This will be proved in greater generality in Theorem 4.2.5. An elementary proof for $\mathfrak{sl}(2, \mathbb{C})$ appears in Exercise 23. □

Consequences 2.1.5 Let V be a finite-dimensional representation of $\mathfrak{sl}(2, \mathbb{F})$ with $\mathrm{char}(\mathbb{F}) = 0$, then the action of $h \in \mathfrak{sl}(2, \mathbb{F})$ on V is diagonalizable and $\overline{V}_\lambda = V_\lambda$ for every $\lambda \in \mathbb{C}$, and so $V = \bigoplus_\lambda V_\lambda$. Moreover, we have $V_\lambda \neq 0 \Rightarrow \lambda \in \mathbb{Z}$. Let q denote an invertible formal variable and define the **character** of V:

$$\mathrm{Ch}(V) := \sum_{\lambda \in \mathbb{Z}} \dim_{\mathbb{F}}(V_\lambda) q^\lambda \in \mathbb{Z}[q, q^{-1}]$$

Note that $\mathrm{Ch}(V) \mapsto \dim_{\mathbb{F}}(V)$ under the specialization $q \mapsto 1$ and $\mathrm{Ch}(V \oplus W) = \mathrm{Ch}(V) + \mathrm{Ch}(W)$.

We record a number of important consequences of Theorem 2.1.2 and Proposition 2.1.4 in terms of characters.

(1) We have $V \cong L(m)$ if and only if $\mathrm{Ch}(V) = q^m + q^{m-2} + \cdots q^{2-m} + q^{-m} =: [m+1]$. The Laurent polynomial $[m+1]$ is called the **quantum integer** $m+1$. Note $[m+1] \mapsto m+1$ when $q \mapsto 1$.

(2) A Laurent polynomial $P = \sum_{i \in \mathbb{Z}} a_i q^i \in \mathbb{Z}[q, q^{-1}]$ is the character of an $\mathfrak{sl}(2, \mathbb{F})$-representation V if and only if its coefficients are non-negative ($a_i \geq 0$), **symmetric** ($a_{-i} = a_i$) and the even and odd subsequences are **unimodal**, i.e.

$$\begin{gathered} \cdots a_{-2i-2} \leq a_{-2i} \leq \cdots a_{-2} \leq a_0 \geq a_2 \geq \cdots \geq a_{2i} \geq a_{2i+2} \geq \cdots \\ \cdots a_{-2i-1} \leq a_{-2i+1} \leq \cdots \leq a_{-1} = a_1 \geq \cdots \geq a_{2i-1} \geq a_{2i+1} \geq \cdots . \end{gathered}$$

Moreover, in this case V is uniquely determined up to isomorphism, namely $V \cong \bigoplus_{i \in \mathbb{Z}_{\geq 0}} L(i)^{\oplus(a_i - a_{i+2})}$.

(3) If an $\mathfrak{sl}(2, \mathbb{F})$-representation V has weight spaces $V_0 = V_1 = 0$, then $V = 0$.

(4) If V, W are finite-dimensional $\mathfrak{sl}(2, \mathbb{C})$-representations, then:

$$\mathrm{Ch}(V^*) = \mathrm{Ch}(V) \quad \text{and} \quad \mathrm{Ch}(V \otimes W) = \mathrm{Ch}(V)\,\mathrm{Ch}(W)$$

(Details in Exercise 24.) In particular, $V \cong V^*$, i.e. V is self-dual and $\mathrm{Hom}(V, W) \cong V \otimes W$.

(5) For $m, n \geq 0$, the **Clebsch–Gordan rule** holds

$$L(m) \otimes L(n) \cong \mathrm{Hom}(L(m), L(n)) \cong L(m+n) \oplus L(m+n-2) \oplus \cdots \oplus L(|m-n|)$$

as well as (a special case of) **Schur's Lemma**:

$$\mathrm{Hom}_{\mathfrak{sl}(2,\mathbb{F})}(L(m),L(n)) = \begin{cases} \mathbb{F}\mathrm{id}_{L(m)} & \text{if } m=n \\ 0 & \text{if } m \neq n \end{cases}$$

(Details in Exercise 24.) L5 End

Examples 2.1.6 We give two examples that illustrate how Proposition 2.1.4 fails to generalize.

(1) For $\lambda \in \mathbb{C}$ let $\Delta_\lambda = \mathrm{span}_{\mathbb{C}}\{v_{\lambda-2k} \mid k \in \mathbb{Z}_{\geq 0}\}$ be the infinite-dimensional $\mathfrak{sl}(2,\mathbb{C})$-representation defined by:

$$F(v_\mu) = v_{\mu-2}$$
$$H(v_\mu) = \mu v_\mu$$
$$E(v_{\lambda-2k}) = k(\lambda-k+1)v_{\lambda-2k+2}$$

where $\mu \in \mathbb{C}, k \in \mathbb{Z}_{\geq 0}$. (Compare with the uniqueness part of the proof of Theorem 2.1.2.) Then for $\lambda \in \mathbb{Z}_{\geq 0}$ we have $E(v_{-\lambda-2}) = 0$ and so we have found a subrepresentation $\Delta_{-\lambda-2} \subset \Delta_\lambda$ such that $\Delta_\lambda/\Delta_{-\lambda-2} \cong L(\lambda)$, but we do not have a direct sum decomposition. If $\lambda \notin \mathbb{Z}_{\geq 0}$, then Δ_λ is simple.

(2) The adjoint representation of $\mathfrak{sl}(2,\mathbb{F}_2)$ has a simple subrepresentation $L(0) \cong \mathbb{F}h$ with quotient isomorphic to $L(0) \oplus L(0)$, but this is not a direct sum.

Exercise 21 Consider $\mathfrak{sl}(2,\mathbb{F})$ over an arbitrary field $\mathbb{F}$ with standard basis e,h,f satisfying $[h,e]=2e$, $[h,f]=-2f$, $[e,f]=h$. Let $\rho\colon \mathfrak{sl}(2,\mathbb{F}) \to \mathfrak{gl}(V)$ be a finite-dimensional representation and set $E=\rho(e), H=\rho(h), F=\rho(f)$. Show that the **Casimir operator**

$$C = 4FE + H(H+2\mathrm{id}) \in \mathrm{End}_{\mathbb{F}}(V)$$

is an endomorphism that commutes with E,H,F, i.e. $C \in \mathrm{End}_{\mathfrak{sl}(2,\mathbb{F})}(V)$. For $\mathbb{F}=\mathbb{C}$ show that C acts as scalar multiplication by $m(m+2)$ on $L(m)$ (you may use the formulas from Remark 2.1.3). Casimir operators will later be considered in greater generality, see Construction 4.2.6. End Week 4 Exerc.

Exercise 22 (may split) Let V be a finite-dimensional $\mathfrak{sl}(2,\mathbb{C})$-representation. Consider the eigenvalues of H on V and pick one with largest real part, call it λ.

(1) For $v \in \overline{V}_\lambda$ show that $Ev=0$ and $F^N v = 0$ for some $N>0$.

(2) Show that $(H-m)F^n = F^n(H-m-2n)$ for $n \geq 0$ and $m \in \mathbb{C}$ and then show $EF^n v = F^{n-1}n(H-n+1)v$ for $n \geq 0$ and $v \in V$ with $Ev=0$.

(3) For $v \in V$ such that $Ev=0$ and $k>0$, find a polynomial $P_k(x)$ of degree k, such that $E^kF^kv = P_k(H)v$. Hint: aim to prove a more general statement using (2) and induction.

(4) Show that H acts diagonalizably on the generalized eigenspace $\overline{V}_\lambda$, i.e. that $\overline{V}_\lambda = V_\lambda$. (Use that $P_k(x)$ does not have multiple roots.)

Exercise 23 (may split) Prove that every finite-dimensional representation V of $\mathfrak{sl}(2,\mathbb{C})$ decomposes into a direct sum of simple subrepresentations. Assume, one the contrary, that V is an indecomposable, non-simple $\mathfrak{sl}(2,\mathbb{C})$-representation of smallest possible finite dimension.

(1) Show that the Casimir operator from Exercise 21 has only one eigenvalue on V, say $m(m+2)$ for some $m \in \mathbb{Z}_{\geq 0}$.

(2) Show that V has a subrepresentation $L(m)$ such that $V/L(m) \cong L(m)^{\oplus n}$ for some $n \in \mathbb{Z}_{>0}$.

(3) Prove that the eigenspace V_m of H has dimension $n+1$. Pick a basis $v_1,\dots,v_{n+1}$ and show that the $F^j(v_i)$ for $1 \leq i \leq n+1$ and $0 \leq j \leq m$ form a basis of V. (Hint: if $F(v)=0$ and $H(v)=\mu v$, then $C(v)=\mu(\mu-2)v$ and so $\mu=-m$.)

(4) Set $W_i = \mathrm{span}_{\mathbb{C}}\{v_i,\dots,F^m(v_i)\}$ and show that the W_i are subrepresentations of V and deduce a contradiction to the assumption that V is indecomposable.

Exercise 24 Prove the statements of Consequences 2.1.5.(4) by computing the dimensions of the weight spaces of V^* and $V \otimes W$ in terms of the dimensions of the weight spaces of V and W. Then deduce Consequences 2.1.5.(5).

Exercise 25 Consider the polynomial ring $\mathbb{Z}[X]$. The polynomials $\{S_i\}_{i\in\mathbb{Z}_{\geq 0}} \subset \mathbb{Z}[X]$ that are recursively determined by

$$S_0 = 1, \quad S_1 = X, \quad S_{n+1} = XS_n - S_{n-1}$$

are called **Chebyshev polynomials of the second kind**. Note that they form a basis of the free $\mathbb{Z}$-module $\mathbb{Z}[X]$. Now show this basis is **positive** in the sense that the product of two such Chebyshev polynomials expands as a linear combination of Chebyshev polynomials with *nonnegative* coefficients:

$$S_i S_j = \sum_k c_{i,j}^k S_k \quad \text{with } c_{i,j}^k \in \mathbb{Z}_{\geq 0}$$

Hint: consider the ring homomorphism $\mathbb{Z}[X] \to \mathbb{Z}[q, q^{-1}]$ defined by $X \mapsto q + q^{-1}$ and use Consequences 2.1.5. (This is a basic example of a proof by **categorification**.) Is the analogous statement true for the **Chebyshev polynomials of the first kind**, which are determined by

$$T_0 = 2, \quad T_1 = X, \quad T_{n+1} = XT_n - T_{n-1}?$$

Exercise 26 Consider the vector representation $V := L(1)$ of $\mathfrak{sl}(2,\mathbb{C})$ and recall the exterior and symmetric power representations from Exercise 19. Now prove for $k \geq 0$:

$$\textstyle\bigwedge^k V \cong \begin{cases} L(0) & \text{if } k = 0, 2 \\ L(1) & \text{if } k = 1 \\ 0 & \text{if } k \geq 3 \end{cases}, \qquad S^k(V) \cong L(k)$$

2.2 Temperley–Lieb calculus

In this section we let $V := L(1) = \operatorname{span}_{\mathbb{C}}\{w_1, w_{-1}\}$ denote the vector representation of $\mathfrak{sl}(2,\mathbb{C})$. We fix the isomorphism $\phi\colon V \xrightarrow{\cong} V^*$ sending $\phi(w_1) = -w_{-1}^*$ and $\phi(w_{-1}) = w_1^*$.

Definition 2.2.1 Let $V := L(1)$ denote the vector representation of $\mathfrak{sl}(2,\mathbb{C})$. Then we define two morphisms of $\mathfrak{sl}(2,\mathbb{C})$-representations:

- $\cup\colon \mathbb{C} \to V \otimes V$ is defined by $1 \mapsto w_1 \otimes w_{-1} - w_{-1} \otimes w_1$ as the composition of the natural maps $\mathbb{C} \to \operatorname{End}_{\mathfrak{sl}(2,\mathbb{C})}(V) \to V \otimes V^*$ that send $1 \mapsto \mathrm{id}_V \mapsto w_1 \otimes w_1^* + w_{-1} \otimes w_{-1}^*$ with the isomorphism $V \otimes V^* \xrightarrow{\mathrm{id}\otimes\phi^{-1}} V \otimes V$.
- $\cap\colon V \otimes V \to \mathbb{C}$ is defined as the composition of the isomorphism $V \otimes V \xrightarrow{\phi\otimes\mathrm{id}} V^* \otimes V$ and the natural pairing $V^* \otimes V \to \mathbb{C}$. In formulas: $\cap(w_{\mp1} \otimes w_{\pm1}) = \pm 1$ and $\cap(w_{\pm1} \otimes w_{\pm1}) = 0$.

Remarks 2.2.2 (1) The morphisms $\cup$ and $\cap$ span their respective spaces of $\mathfrak{sl}(2,\mathbb{C})$-intertwiners, namely $\operatorname{Hom}_{\mathfrak{sl}(2,\mathbb{C})}(\mathbb{C}, V \otimes V)$ and $\operatorname{Hom}_{\mathfrak{sl}(2,\mathbb{C})}(V \otimes V, \mathbb{C})$, which are 1-dimensional by Consequences 2.1.5.(5) since $V \otimes V \cong L(2) \oplus \mathbb{C}$.

(2) By induction, one can show from the Clebsch–Gordan rule that

$$V^{\otimes 2n} = V \otimes \cdots \otimes V \cong \mathbb{C}^{C_n} \oplus \bigoplus_{i \geq 1} L(2i)^{l_{n+i,n-i}}$$

where C_n is the nth Catalan number and $l_{n+i,n-i} \in \mathbb{Z}_{\geq 0}$, see Exercise 27. In particular $\operatorname{Hom}_{\mathfrak{sl}(2,\mathbb{C})}(\mathbb{C}, V^{\otimes 2n}) \cong (V^{\otimes 2n})^{\mathfrak{sl}(2,\mathbb{C})} \cong \mathbb{C}^{C_n}$ and we will describe a basis for this morphism space in terms of tensor products of n copies of the map $\cup$.

(3) The Catalan number C_n can be described as the count of **Dyck paths** of length $2n$. A Dyck path of length $2n$ is a tuple $p = (p_1, \ldots, p_{2n}) \in \{+,-\}^{2n}$ containing exactly n symbols $+$ and n symbols $-$, such that any initial subtuple $(p_1, \ldots, p_k)$ for $1 \leq k \leq 2n$ contains at least as many $+$ as $-$. Let P_n denote the set of all Dyck paths of length $2n$. For example $C_3 = 5$ and

$$P_3 = \{ +++--- \,,\ ++-+-- \,,\ ++--+- \,,\ +-++-- \,,\ +-+-+- \}.$$

Side note: Dyck paths are in bijection with valid placements of n pairs of parentheses by simply replacing the symbol $+$ by (and $-$ by).

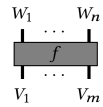
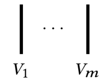
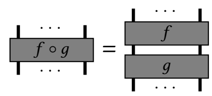
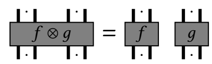
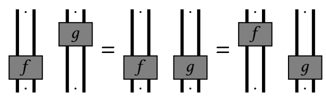
Construction 2.2.3 We will now inductively construct one $\mathfrak{sl}(2,\mathbb{C})$-intertwiner $\cup_p\colon \mathbb{C} \to V^{\otimes 2n}$ for each $p \in P_n$ and then show that these form a basis of $\mathrm{Hom}_{\mathfrak{sl}(2,\mathbb{C})}(\mathbb{C}, V^{\otimes 2n})$. For $n = 1$ and $p = +-$, we set $\cup_p = \cup$. Now suppose that $n \geq 1$ and the maps $\cup_{p'}$ have already been constructed for $p' \in P_{n-1}$. Given $p \in P_n$, we find the first occurrence of a the substring $+-$ in p, i.e. where a $-$ immediately follows a $+$. Let their positions in the Dyck path be $a+1$ and $a+2$. Furthermore, let $p' \in P_{n-1}$ be the Dyck path obtained by removing this pair $+-$ from p. Then we define

$$\cup_p := (\mathrm{id}_{V^{\otimes a}} \otimes \cup \otimes \mathrm{id}_{V^{\otimes 2n-2-a}}) \circ \cup_{p'}\colon \mathbb{C} \to V^{\otimes 2n}. \tag{3}$$

For example:

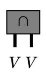
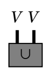
$$\begin{aligned}\cup_{++-+--} &= (\mathrm{id}_V \otimes \cup \otimes \mathrm{id}_{V^{\otimes 3}}) \circ \cup_{++--}\\ &= (\mathrm{id}_V \otimes \cup \otimes \mathrm{id}_{V^{\otimes 3}}) \circ (\mathrm{id}_V \otimes \cup \otimes \mathrm{id}_V) \circ \cup_{+-}\\ &= (\mathrm{id}_V \otimes \cup \otimes \mathrm{id}_{V^{\otimes 3}}) \circ (\mathrm{id}_V \otimes \cup \otimes \mathrm{id}_V) \circ \cup\end{aligned}$$

Remark 2.2.4 (String diagrams) In a monoidal category C, i.e. a category with a tensor product, there are two different ways in which morphisms can be "composed": the categorical composition and the tensor product. In this context, it is sometimes convenient to use the 2-dimensional notation of **string diagrams** instead of the usual 1-dimensional notation (sequences of symbols) that is common in algebra. Here a few pictures say more than a thousand words:

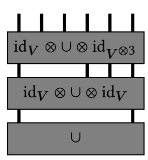
	usual notation	string diagram
morphisms	$f \in \mathrm{Hom}_C(V_1 \otimes \cdots \otimes V_m, W_1 \otimes \cdots \otimes W_n)$	
identity morphisms	$\mathrm{id}_{V_1 \otimes \cdots \otimes V_m}$	
composition	$f \circ g$	
tensor product	$f \otimes g$	
interchange law	$(\mathrm{id} \otimes g)(f \otimes \mathrm{id}) = f \otimes g = (f \otimes \mathrm{id})(\mathrm{id} \otimes g)$	

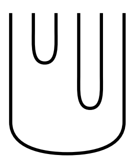
For the composition we require that the target of g is the source of f. For the string diagrams this means that the labels on the strings have to agree to glue them.

For maps between representations of $\mathfrak{sl}(2,\mathbb{C})$ we will use the following convenient shorthand for string diagrams of $\cup$ and $\cap$:

$\cap :=$  , $\cup :=$ 

Here we use the (common) convention of not drawing strings that are labelled with the tensor unit. The maps $\cup_p$ for $p \in P_n$ can now be given a very simple description. We again consider the example $\cup_{++-+--}$ from above:

$\cup_{++-+--} = (\mathrm{id}_V \otimes \cup \otimes \mathrm{id}_{V^{\otimes 3}}) \circ (\mathrm{id}_V \otimes \cup \otimes \mathrm{id}_V) \circ \cup \to$ 

$=$ 

The picture on the right is called a **cup diagram**. More generally, if $f \in \mathrm{Hom}_{\mathfrak{sl}}(2,\mathbb{C})(\mathbb{C}, V^{\otimes 2n})$ is constructed as a composition of morphisms of the form $(\mathrm{id}_{V^{\otimes a}} \otimes \cup \otimes \mathrm{id}_{V^{\otimes b}})$, then the associated string diagram is called a cup diagram. By the graphical interpretation of the interchange law, the relative height of two un-nested cups in a cup diagram can swapped. Thus, the only relevant information contained in a cup diagram is captured by the following notion.

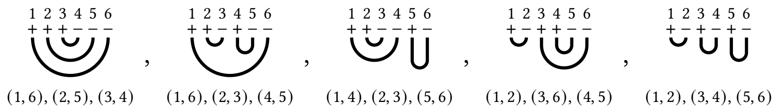
Definition 2.2.5 Let $n \in \mathbb{Z}_{\geq 0}$. A **crossingless** matching of $2n$ points is a partition of $\{1, \dots, 2n\}$ into n pairs (i, j) with $i < j$ such that there is no quadruple $0 \leq i < j < k < l \leq 2n$ such that (i, k) and (j, l) are paired. The set of crossingless matchings on $2n$ points will be denoted M_n.

Lemma 2.2.6 For $n \in \mathbb{Z}_{\geq 0}$ we have bijections:

$$P_n \xrightarrow{\cong} \{\text{cup diagrams with } 2n \text{ points}\} \xrightarrow{\cong} M_n$$

Proof sketch. The first map sends $p \in P_n$ to the string diagram of $\cup_p$. The second map sends a cup diagram to the crossingless matching in which $1 \leq i < j \leq 2n$ are paired if and only if there exists a cup whose ends are the ith and jth strand at the top. Finally, there is a map $M_n \to P_n$ which sends a crossingless matching to the Dyck path obtained by starting with the tuple $(1, \dots, 2n)$ and replacing i by $+$ and j by $-$ whenever (i, j) are paired. We leave it to the reader to verify that these maps are well-defined and that any cyclic composition of these three maps is the identity. □

Example 2.2.7 The identifications between Dyck paths, cup diagrams and crossingless matchings for $n = 3$:



Remarks 2.2.8 (1) Let $p, q \in \{+, -\}^n$, $p = (p_1, \dots, p_n)$ and $q = (q_1, \dots, q_2)$. We write $p \geq q$ if $\sum_{i=1}^{k} p_i \geq \sum_{i=1}^{k} q_i$ for all $1 \leq k \leq n$. This defines a partial order on $\{+, -\}^n$.

(2) Define $GP_n := \{p = (p_1, \dots, p_{2n}) \in \{+, -\}^{2n} \mid \sum_{i=1}^{2n} p_i = 0\}$ and note $P_n \subset GP_n$. The partial order from (1) restricts to a partial order on GP_n and we (arbitrarily) choose a refinement to a total order GP_n, again denoted by $\geq$. L6 End

(3) For $p = (p_1, \dots, p_{2n}) \in \{+, -\}^{2n}$ we define $w_p := w_{p_1} \otimes \cdots \otimes w_{p_{2n}} \in V^{\otimes 2n}$ where $w_+ := w_1$ and $w_- := w_{-1}$ are the standard basis vectors of V. Note that $\{w_p \mid p \in \{+, -\}^{2n}\}$ is a basis of $V^{\otimes 2n}$ and $\{w_p \mid p \in GP_n\}$ is an ordered basis of the degree zero weight space $(V^{\otimes 2n})_0$.

Proposition 2.2.9 Let $n \in \mathbb{Z}_{\geq 0}$

(1) For $p \in P_n$ the map $\cup_p \colon \mathbb{C} \to V^{\otimes 2n}$ sends

$$\mathbb{C} \ni 1 \mapsto w_p + \sum_{\substack{q \in GP_n \\ p > q}} c_{p,q} w_q$$

for some coefficients $c_{p,q} \in \{1, 0, -1\} \in \mathbb{C}$.

(2) The set $\{\cup_p \mid p \in P_n\}$ is a basis for $\mathrm{Hom}_{\mathfrak{sl}(2,\mathbb{C})}(\mathbb{C}, V^{\otimes 2n})$.

Proof. (1) Let M be the crossingless matching corresponding to p. From the definition of the map $\cup$, one deduces that the summands w_q that appear with non-zero coefficients in the expansion of $\cup_p(1)$ are exactly those where $q = (q_1, \dots, q_{2n}) \in \{+, -\}^{2n}$ satisfies $q_i = -q_j$ whenever $(i, j) \in M$ are paired. This implies $q \in GP_n$ and $c_{p,q} = \prod_{(i,j) \in M} q_i \in \{+1, -1\}$, and w_p appears with coefficient 1. Moreover, any $q \neq p$ with nonzero $c_{p,q}$ can be obtained from p by successively swapping paired entries $+$ and $-$. More precisely, if $(i, j) \in M$ and $q_i = +$ and $q_j = -$, then one can consider $q' \in GP_n$ with entries $q'_i = -$, $q'_j = +$ and all other entries as in q. For this q' we have $c_{p,q'} = -c_{p,q}$ and $q > q'$. This shows (1).

(2) We have the isomorphism $\mathrm{Hom}_{\mathfrak{sl}(2,\mathbb{C})}(\mathbb{C}, V^{\otimes 2n}) \xrightarrow{\cong} (V^{\otimes 2n})^{\mathfrak{sl}(2,\mathbb{C})} \subset (V^{\otimes 2n})_0$ given by the map $f \mapsto f(1)$ that evaluates a morphism on $1 \in \mathbb{C}$. A set of such morphisms f is linearly independent if and only if its images $f(1)$ are linearly independent. For the set $\{\cup_p \mid p \in P_n\}$ this now follows from (1). Moreover, since $|P_n| = C_n = \dim_{\mathbb{C}}(\mathrm{Hom}_{\mathfrak{sl}(2,\mathbb{C})}(\mathbb{C}, V^{\otimes 2n}))$ we have found a basis. □

Construction 2.2.10 Let R be a commutative ring and $\delta \in R$. The **Temperley–Lieb category** is the monoidal R-linear category $TL^R(\delta)$ described as follows:

- The set of objects is $\mathrm{Ob}(TL^R(\delta)) = \mathbb{Z}_{\geq 0} = \{0, 1, 2, 3, \dots\}$

- For $m, n \in \mathrm{Ob}(TL^R(\delta))$, the morphisms from m to n are

$$\mathrm{Hom}_{TL^R(\delta)}(m, n) = \mathrm{span}_R\{\text{crossingless matchings of } m+n \text{ points}\},$$

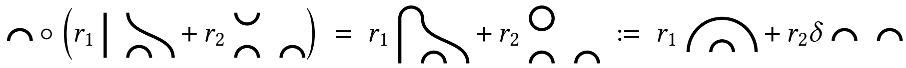
  the free R-module spanned by crossingless matchings of $m + n$ points. In particular, if $m + n$ is odd, then $\mathrm{Hom}_{TL^R(\delta)}(m, n) = 0$. Conversely, if $m + n$ is even, then a morphism from m to n is a R-linear combination of crossingless matchings that we illustrate as string diagrams from m points to n points, e.g. with $r_1, r_2 \in R$:

  Such string diagrams (without boxes) are called **Temperley–Lieb diagrams**. The exists a unique crossingless matching on 0 points, whose corresponding Temperley–Lieb diagram $\emptyset$ is called the **empty diagram**. In particular

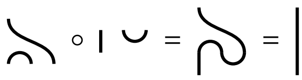
$$\mathrm{Hom}_{TL^R(\delta)}(0, 0) = \mathrm{span}_R\{\emptyset\} \cong R$$

- The tensor product is defined on objects by $m \otimes n := m + n$ and on morphisms by (the R-bilinear extension of) putting string diagrams side-by-side, see Remark 2.2.4. For example, if also $r_3, r_4 \in R$:

- The composition of morphisms is defined analogously, namely as the R-bilinear extension of stacking diagrams, see Remark 2.2.4. The only subtlety here is that the stacking of two crossingless matchings does not always produce a crossingless matching. This is exactly the case when the resulting string diagram contains closed components, i.e. circles. In this case we remove the circle from the diagram and multiply the remaining diagram by the scalar $\delta \in R$. For example:



  We also emphasize here that the morphism encoded by a diagram depends only depends on the crossingless matching represented by the diagram, not on how exactly the diagram is drawn (e.g. compare the first two summands with coefficient r_1 above). This means we consider string diagrams up to **planar isotopy relative to the boundary**. Another example is:



For $n \in \mathbb{Z}_{\geq 0}$ the **nth Temperley–Lieb algebra** is defined to be

$$\mathrm{TL}_n^R(\delta) := \mathrm{End}_{\mathrm{TL}^R(\delta)}(n).$$

Remark 2.2.11 Abstractly, $\mathrm{TL}_n^R(\delta)$ can be defined as the unital associative R-algebra given by the presentation with generators $U_1, \ldots, U_{n-1}$ and relations:

$$U_iU_i = \delta U_i, \quad U_iU_{i\pm1}U_i = U_i, \quad U_iU_j = U_jU_i \quad \text{if } |i-j| > 1$$

The generators U_i in this abstract description can be interpreted in terms of string diagrams as

where the cup and cap involve the ith and $(i + 1)$st strand.

Theorem 2.2.12 There exists a fully faithful monoidal $\mathbb{C}$-linear functor

$$F\colon \mathrm{TL}^{\mathbb{C}}(-2) \to \mathrm{Rep}(\mathfrak{sl}(2, \mathbb{C}))$$

(here $\mathrm{Rep}(\mathfrak{sl}(2, \mathbb{C}))$ denotes the monoidal, $\mathbb{C}$-linear category of representations of $\mathfrak{sl}(2, \mathbb{C})$) that satisfies

$$F(n) = V^{\otimes n}, \quad F(\smile) = \cup, \quad F(\frown) = \cap$$

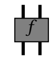
Proof sketch. If such a functor F exists, then it is already completely determined by the required assignments. On objects it is determined by sending $n \mapsto V^{\otimes n}$. Since every morphism in TL is a linear combination of compositions of tensor products of either $\smile$ or $\frown$ with identity morphisms, F has to send this morphism to the corresponding linear combination of compositions of tensor products of either $\cup$ or $\cap$ with identity morphisms.

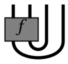
To see that F is well-defined we need to check that all relations between (linear combinations of) such compositions of tensor products in TL are also satisfied between their images under F in $\mathrm{Rep}(\mathfrak{sl}(2,\mathbb{C}))$. One can prove (but we will not do so) that the following three relations are sufficient:

$$F(\ ) \circ F(\ ) = (\mathrm{id}_V \otimes \cap) \circ (\cup \otimes \mathrm{id}_V) = \mathrm{id}_V = F\left(\,|\,\right) = F\left(\ \right),$$
$$F(\ ) \circ F(\ ) = (\cap \otimes \mathrm{id}_V) \circ (\mathrm{id}_V \otimes \cup) = \mathrm{id}_V = F\left(\,|\,\right) = F\left(\ \right),$$
$$F(\frown) \circ F(\smile) = \cap \circ \cup = -2\mathrm{id}_{\mathbb{C}} = -2F(\emptyset) = F(\bigcirc)$$

The second equation in each line is easily verified using the definition of $\cup$ and $\cap$.

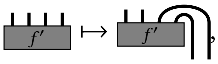
Next we need to show that F induces isomorphisms on the level of morphism spaces $F\colon \mathrm{Hom}_{\mathrm{TL}}(m,n) \xrightarrow{\cong} \mathrm{Hom}_{\mathfrak{sl}(2,\mathbb{C})}(V^{\otimes m}, V^{\otimes n})$. In fact, it suffices to show this for $m = 0$ because of the **bending trick** captured by the following commutative diagram:





$$\begin{tikzcd}
\in & \mathrm{Hom}_{\mathrm{TL}}(m,n) \arrow[r, "F"] \arrow[d, "\cong"] & \mathrm{Hom}_{\mathfrak{sl}(2,\mathbb{C})}(V^{\otimes m}, V^{\otimes n}) \arrow[d, "\cong"] & \in & g \arrow[d, mapsto] \\
\in & \mathrm{Hom}_{\mathrm{TL}}(0,n+m) \arrow[r, "F"] & \mathrm{Hom}_{\mathfrak{sl}(2,\mathbb{C})}(\mathbb{C}, V^{\otimes(n+m)}) & \in & (g \otimes \mathrm{id}_{V^{\otimes m}}) \circ \cup_{+\dots+-\dots-}
\end{tikzcd}$$

The two vertical maps are seen to be isomorphisms by explicitly specifying the inverses



$$g' \mapsto (\mathrm{id}_{V^{\otimes n}} \otimes \cap_{+\dots+-\dots-}) \circ (g' \otimes \mathrm{id}_{V^{\otimes m}})$$

where $\cap_{+\dots+-\dots-}$ denotes the morphism defined analogously to $\cup_{+\dots+-\dots-}$ from (3), but using $\cap$ instead of $\cup$. The diagram commutes because F is a monoidal functor and $\cap_{+\dots+-\dots-}$ and $\cup_{+\dots+-\dots-}$ are exactly the images under F of configurations of nested caps and cups.

The top horizontal map induced by F is an isomorphism if and only if the bottom horizontal map is an isomorphism. By definition, $\mathrm{Hom}_{\mathrm{TL}}(0,n+m)$ has a basis given by cup diagrams with $m+n$ endpoints. In Proposition 2.2.9 we have seen that their images under F, namely the $\cup_p$ for $p \in P_{(m+n)/2}$ form a basis of $\mathrm{Hom}_{\mathfrak{sl}(2,\mathbb{C})}(\mathbb{C}, V^{\otimes(n+m)})$. Thus F induces isomorphisms between morphism spaces, i.e. it is fully faithful. $\square$ L7 End

Corollary 2.2.13 Let $n \in \mathbb{Z}_{\geq 0}$. Then we have an isomorphism of associative $\mathbb{C}$-algebras:

$$\mathrm{TL}_n^{\mathbb{C}}(-2) \cong \mathrm{End}_{\mathfrak{sl}(2,\mathbb{C})}(V^{\otimes n})$$

Remark 2.2.14 Theorem 2.2.12 has a long and complicated history dating back to (at least) the 1930s[1]. The theorem actually holds in much greater generality, which makes the Tempereley–Lieb category useful for studying situations, in which the Clebsch–Gordan rule does not hold as stated in Consequences 2.1.5.(5).

Exercise 27 This completes Remarks 2.2.2.(1). Show that the multiplicity of $\mathbb{C} = L(0)$ in $V^{\otimes 2n}$ is given by the number of Dyck paths of length $2n$. Describe a generalization of Dyck paths that gives a combinatorial interpretation of the multiplicities $l_{\frac{n+m}{2},\frac{n-m}{2}}$ in

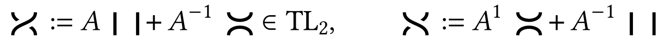
$$V^{\otimes n} = V \otimes \cdots \otimes V \cong \bigoplus_{m \geq 0} L(m)^{l_{\frac{n+m}{2},\frac{n-m}{2}}}$$

These are sometimes called **Lobb numbers**. Hint: deduce the first statement from the more general second statement; the latter has an easier proof (induction in n).

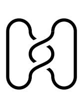
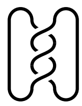
Exercise 28 (Kauffman bracket, may split) (1) Compute the image of the morphism $\ := |\ | + \ \in \mathrm{TL}_2^{\mathbb{C}}(-2)$ under the functor F from Theorem 2.2.12.

[1] Weyl, H., Rumer, G., and Teller, E.. "Eine für die Valenztheorie geeignete Basis der binären Vektorinvarianten." Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse 1932 (1932): 499-504.

(2) Let $R = \mathbb{Z}[A, A^{-1}]$ denote the ring of Laurent polynomials in a variable A with integer coefficients. Set $\delta = -A^2 - A^{-2}$ and abbreviate $\mathrm{TL} := \mathrm{TL}^R(\delta)$. Now we define

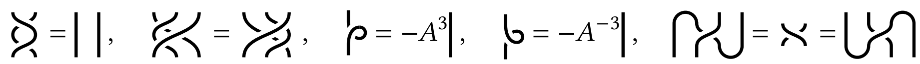


Compute the $P, Q \in R$ such that:

 $= P\,\emptyset \in \mathrm{TL}_0$ and  $= Q\,\emptyset$ in TL_0

More generally, for every link diagram L one obtains an element $\langle L\rangle \in R$ called **the Kauffman bracket** of L. It is a close relative of the **Jones polynomial**, an important invariant of knots and links.

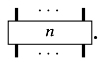
(3) The Kauffman bracket is an invariant of framed links. This means that the element $\langle L\rangle$ depends not on the diagram L, but only on the framed link represented by the diagram. The proof (that we will not complete) starts with the following verifications:



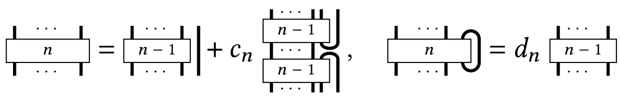
The first two relations are called the **second** and **third Reidemeister move** respectively.

Exercise 29 (Jones–Wenzl projectors, may split into three exercises) Set $\mathrm{TL} := \mathrm{TL}^{\mathbb{C}}(-2)$. The goal of this exercise is to prove that for every $n \in \mathbb{Z}_{\geq 0}$ there exists a unique element $\mathrm{JW}_n \in \mathrm{TL}_n$, called the **nth Jones–Wenzl projector**, satisfying the following properties:

- $(\mathrm{JW}_n)^2 = \mathrm{JW}_n$.
- $\mathrm{JW}_n \in \mathrm{id}_n + \langle U_1, \ldots, U_{n-1}\rangle \subset \mathrm{TL}_n$, where $\langle U_1, \ldots, U_{n-1}\rangle$ denotes the ideal generated by the U_i.
- $U_i\,\mathrm{JW}_n = \mathrm{JW}_n U_i = 0$ for all $1 \leq i < n$.

Jones–Wenzl projectors are commonly illustrated by boxes $\mathrm{JW}_n =$ .

(1) Start by translating the desired properties into the graphical calculus using boxes. Then make the ansatz:



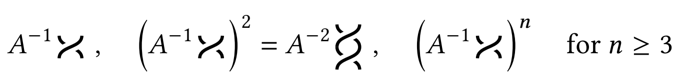
for some scalars $c_n, d_n \in \mathbb{C}$. Use induction to compute the scalars c_n, d_n under the assumption that the boxes satisfy the desired properties.

Now *define* the boxes recursively starting from $\mathrm{JW}_1 = \mathrm{id}_1$ and using the relation involving c_n. Finally check that the thus defined boxes satisfy the three desired properties.

(2) Use the Clebsch–Gordan rule to prove that the decomposition of $V^{\otimes n}$ into simples contains a unique direct summand $L(n)$. Then use Consequences 2.1.5 and Theorem 2.2.12 to show that $F(\mathrm{JW}_n) \in \mathrm{End}_{\mathfrak{sl}(2,\mathbb{C})}(V^{\otimes n})$ is exactly the projection-inclusion $V^{\otimes n} \twoheadrightarrow L(n) \hookrightarrow V^{\otimes n}$. In particular, JW_n is a symmetrization operator, see Exercise 26.

(3) Can JW_n be defined in $\mathrm{TL}^{\mathbb{F}}(-2)$ when $\mathrm{char}(\mathbb{F}) = p > 0$?

(4) Now let $R = \mathbb{Q}(A)$, i.e. rational functions in A, and $R' = \mathbb{Q}[A, A^{-1}]]$, i.e. formal Laurent series in A^{-1}. In both cases we consider $\delta = -A^2 - A^{-2}$. First, find $\mathrm{JW}_2 \in \mathrm{TL}_2^R(\delta)$ (satisfying the three properties listed above). Second, rewrite this as an element of $\mathrm{TL}_2^{R'}(\delta)$ by expanding the coefficients into Laurent series. Third, use induction to compute the following elements of $\mathrm{TL}_2^{R'}(\delta)$ and compare them with $\mathrm{JW}_\in \mathrm{TL}_2^{R'}(\delta)$:



End
Week 5
Exerc.

3 Nilpotent and solvable Lie algebras

3.1 A roadmap towards classification

Now that we've seen several examples of Lie algebras, one immediate question is that of **classification**. Put very naively: "how many different Lie algebras do exist?". This is a bit like asking "How many different molecules do exist?". First of all, we are not actually interested in the number. We are only interested in understanding Lie algebras up to isomorphism. Second, we have already seen that some Lie algebras can constructed as an extension of one Lie algebra by another one (Construction 1.3.9). A special case is that of a split extension, i.e. a semi-direct product (Construction 1.3.1). Our task thus boils down to understanding simple Lie algebras (roughly "atoms") and how they fit together. We will focus on the ground field $\mathbb{C}$. Lie algebras over $\mathbb{C}$ will be called **complex Lie algebras**.

The big milestones ahead of us are the following results. First, there is a notion of **solvability** for Lie algebras, which is analogous to that of groups. This will be the subject of Section 3.2. Every finite-dimensional Lie algebra $\mathfrak{g}$ has a maximal solvable ideal, called its radical $\operatorname{rad}\mathfrak{g}$. Then one can prove:

Theorem 3.1.1 [Levi's theorem] For every finite-dimensional complex Lie algebra $\mathfrak{g}$, the short exact sequence $\operatorname{rad}\mathfrak{g} \hookrightarrow \mathfrak{g} \twoheadrightarrow \mathfrak{g}/\operatorname{rad}\mathfrak{g}$ splits.

Thus $\mathfrak{g}$ is a semi-direct product of the solvable Lie algebra $\operatorname{rad}\mathfrak{g}$ and the Lie algebra $\mathfrak{g}/\operatorname{rad}\mathfrak{g}$. In Section 4 we will see that the latter is a direct product of simple Lie algebras.

Understanding and classifying finite-dimensional complex Lie algebras thus splits into three problems:

- Understanding finite-dimensional solvable complex Lie algebras, towards we see some progress in Section 3.2.
- Classifying finite-dimensional complex simple Lie algebras, see Theorem 3.1.2.
- Understanding representations of semisimple Lie algebras by derivations on solvable Lie algebras (compare with Example 1.5.6).

Theorem 3.1.2 [Killing classification] Every finite-dimensional simple complex Lie algebra is isomorphic to exactly one from the following list

$$\begin{array}{ll} \mathfrak{sl}(n+1,\mathbb{C}) & n \geq 1 \\ \mathfrak{so}(2n+1,\mathbb{C}) & n \geq 2 \\ \mathfrak{sp}(2n,\mathbb{C}) & n \geq 3 \\ \mathfrak{so}(2n,\mathbb{C}) & n \geq 4 \end{array}$$

(see Section 1.2) or one of the exceptional Lie algebras $\mathfrak{e}_6$, $\mathfrak{e}_7$, $\mathfrak{e}_8$, $\mathfrak{f}_4$, $\mathfrak{g}_2$.

We will return to this theorem at a later stage.

3.2 Nilpotent and solvable Lie algebras

In this section we encounter the notions of nilpotent and solvable Lie algebras. However, before we get to this Definition 3.2.5, we first study a certain case.

Lemma 3.2.1 Let V be a vector space. If $x \in \mathfrak{gl}(V)$ is nilpotent, then $\operatorname{ad}_x \in \mathfrak{gl}(\mathfrak{gl}(V))$ is also nilpotent.

Proof. For $m \in \mathbb{N}$ and any $y \in \mathfrak{gl}(V)$ an inductive argument shows

$$(\operatorname{ad}_x)^m(y) = \sum_{i=0}^{m} \binom{m}{i} (-1)^{m-i} x^i \circ y \circ x^{m-i}.$$

Note that $\max(i, m-i) \geq \lceil \frac{m}{2} \rceil$, thus $(\operatorname{ad}_x)^{2n} = 0$ provided $x^n = 0$. □

Theorem 3.2.2 [on Lie algebras of nilpotent endomorphisms] Let V be a finite-dimensional vector space and $\mathfrak{g} \subset \mathfrak{gl}(V)$ a Lie subalgebra, such that every $x \in \mathfrak{g}$ is nilpotent as an endomorphism of V. (Recall this means there exists an $n \in \mathbb{N}$ such that $x^n := x \circ \cdots \circ x \colon V \to V$ is the zero map.) Then the following hold:

(1) If $V \neq 0$, then there exists $v \in V$, $v \neq 0$ such that $x(v) = 0$ for every $x \in \mathfrak{g}$ (for this we will use the shorthand $\mathfrak{g}v = 0$).

(2) There exists a chain of subspaces $0 = V_0 \subset V_1 \subset \cdots \subset V_d = V$ with $\dim V_i = i$ and $x(v_i) \in V_{i-1}$ for any $x \in \mathfrak{g}$ and $v_i \in V_i$. (For this we write $\mathfrak{g}V_i \subset V_{i-1}$.)

(3) There exists a basis of V, with respect to which all elements of $\mathfrak{g}$ are represented by strictly upper triangular matrices.

A chain of subspaces as in Theorem 3.2.2.(2) is called a **(complete) flag** of V. Any ordered basis $\{v_1, \ldots, v_d\}$ of V determines a flag with $V_i := \mathrm{span}\{v_1, \ldots, v_i\}$ and conversely, given a flag, one can find a corresponding ordered basis (although usually not a unique one). Such a basis is called **adapted** to the flag.

Proof of Theorem 3.2.2. (1) The proof proceeds by induction on $\dim(\mathfrak{g})$. If $\dim(\mathfrak{g}) = 1$, then $\mathfrak{g} = \mathrm{span}\{x\}$ for a single nilpotent endomorphism of V. The statement then follows as we can find a vector $v \neq 0$ in the kernel $\ker(x) \neq 0$, which is nonzero since x is nilpotent. For the induction step, we consider $\mathfrak{g} \subset \mathfrak{gl}(V)$ with $\dim(\mathfrak{g}) = d \geq 2$ and assume that the statement has been proven for all smaller dimensions. More precisely, we will use:

Induction hypothesis: if $U \neq 0$ is a finite-dimensional vector space and $\mathfrak{g}' \subset \mathfrak{gl}(U)$ a Lie algebra with $\dim(\mathfrak{g}') < \dim(\mathfrak{g})$, then there exists $u \in U$, $u \neq 0$, such that $\mathfrak{g}'u = 0$.

Our goal is to prove the analogous statement for $\mathfrak{g}$: to find a vector $v \in V$, $v \neq 0$, such that $\mathfrak{g}v = 0$. To make use of the induction hypothesis, we need to produce a smaller Lie algebra. We start by choosing a maximal proper Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$, i.e. a Lie subalgebra with $\mathfrak{h} \neq \mathfrak{g}$ that is maximal under inclusion with this property. To this we can apply the induction hypothesis, and we will do so later. But first we study the relationship between $\mathfrak{h}$ and $\mathfrak{g}$.

Note that the composition

$$\mathrm{ad}_{|\mathfrak{h}} \colon \mathfrak{h} \hookrightarrow \mathfrak{g} \xrightarrow{\mathrm{ad}} \mathfrak{gl}(\mathfrak{g})$$

defines a representation of $\mathfrak{h}$ on $\mathfrak{g}$ that has $\mathfrak{h} \xrightarrow{\mathrm{ad}} \mathfrak{gl}(\mathfrak{h})$ as a subrepresentation. Now we form the quotient representation

$$\overline{\mathrm{ad}} \colon \mathfrak{h} \to \mathfrak{gl}(\mathfrak{g}/\mathfrak{h})$$

where we note that $\mathfrak{g}/\mathfrak{h}$ refers to the quotient vector space of $\mathfrak{g}$ by $\mathfrak{h}$. From Lemma 3.2.1 we deduce that ad acts by nilpotent endomorphisms, i.e. $\mathrm{ad}_x := \mathrm{ad}(x)$ is a nilpotent endomorphism of $\mathfrak{g}$ for every $x \in \mathfrak{g}$. This property is inherited by $\mathrm{ad}_{|\mathfrak{h}}$ and $\overline{\mathrm{ad}}$: for every $y \in \mathfrak{h}$, the endomorphism $\mathrm{ad}_{|\mathfrak{h}}(y)$ of $\mathfrak{g}$ and the endomorphism $\overline{\mathrm{ad}}(y)$ of $\mathfrak{g}/\mathfrak{h}$ are both nilpotent.

By the induction hypothesis (for $U = \mathfrak{g}/\mathfrak{h}$ and $\mathfrak{g}' = \overline{\mathrm{ad}}(\mathfrak{h})$, noting $\mathfrak{g}/\mathfrak{h} \neq 0$ and $\dim(\overline{\mathrm{ad}}(\mathfrak{h})) \leq \dim(\mathfrak{h}) < \dim(\mathfrak{g})$), we can find a vector $\bar{l} \in \mathfrak{g}/\mathfrak{h}$, $\bar{l} \neq 0$, such that $\overline{\mathrm{ad}}(\mathfrak{h})\bar{l} = 0$. Let $l \in \mathfrak{g}$ be a representative of $\bar{l}$ (i.e. $l + \mathfrak{h} = \bar{l}$), then $\bar{l} \neq 0$ implies $l \notin \mathfrak{h}$ and we rewrite

$$\begin{aligned} \overline{\mathrm{ad}}(\mathfrak{h})\bar{l} = 0 &\iff \forall \bar{y} \in \overline{\mathrm{ad}}(\mathfrak{h}) : \bar{y}(\bar{l}) = 0 \\ &\iff \forall y \in \mathfrak{h} : \overline{\mathrm{ad}}_y(\bar{l}) = 0 \\ &\iff \forall y \in \mathfrak{h} : \mathrm{ad}_y(l + \mathfrak{h}) = \mathfrak{h} \\ &\iff [\mathfrak{h}, l] \subset \mathfrak{h} \end{aligned}$$

The two properties $l \notin \mathfrak{h}$ and $[\mathfrak{h}, l] \subset \mathfrak{h}$ imply that $\mathfrak{h} + \mathrm{span}\{l\}$ is a Lie subalgebra of $\mathfrak{g}$ that properly contains $\mathfrak{h}$. However, since $\mathfrak{h}$ was chosen maximal, we must have

$$\mathfrak{h} + \mathrm{span}\{l\} = \mathfrak{g}.$$

Now we turn to locating a vector $v \in V$ with $v \neq 0$ such that $\mathfrak{g}v = 0$. As space of candidates we use the subspace $W = \{v \in V \mid \mathfrak{h}v = 0\}$. By the induction hypothesis (for $U = V$ and $\mathfrak{g}' = \mathfrak{h}$, noting $V \neq 0$ by assumption and $\dim(\mathfrak{h}) < \dim(\mathfrak{g})$) we have that $W \neq 0$.

Now make the **claim** that $l(W) \subset W$. To see this we let $w \in W$ and $y \in \mathfrak{h}$ and compute

$$y(l(w)) = l(y(w)) + [y, l](w) = 0 + 0 = 0$$

where we have used $[\mathfrak{h}, l] \subset \mathfrak{h}$ in the last step. Now $y(l(w)) = 0$ for all $y \in \mathfrak{h}$ implies $l(w) \in W$ and the claim is verified

By assumption $l \in \mathfrak{g}$ is a nilpotent endomorphism of V, so now we deduce that $l|_W$ is a nilpotent endomorphism of W. This allows us to choose $v \in \ker l|_W$ with $v \neq 0$. Now we have $l(v) = 0$ and $\mathfrak{h}v = 0$ since $v \in W$. Since $\mathfrak{h} + \mathrm{span}\{l\} = \mathfrak{g}$, we deduce $\mathfrak{g}v = 0$.

(2) Let $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ be a finite-dimensional representation by nilpotent endomorphisms (a more general setting than in the statement). We will prove by induction on the dimension d of V that it admits a flag $0 = V_0 \subset V_1 \subset \cdots \subset V_d = V$ with $\mathfrak{g}V_i \subset V_{i-1}$ for $1 \le i \le d$. First we use (1) to find a non-zero vector $v \in V$ such that $\mathfrak{g}v = 0$ and set $V_1 = \operatorname{span}\{v_1\}$. If $\dim(V) = d = 1$ we are done. Otherwise we note that V_1 defines a subrepresentation and we can form the quotient representation $V' := V/V_1$ and denote by $q\colon V \twoheadrightarrow V/V_1$ the canonical projection. By the induction hypothesis, V' admits a flag $0 = V'_0 \subset V'_1 \subset \cdots \subset V'_{d-1} = V'$ with $\mathfrak{g}V'_i \subset V'_{i-1}$ for $1 \le i \le d-1$. Now we set $V_i = q^{-1}(V'_{i-1})$ for $i \ge 1$ to complete the desired flag for V.

(3) is equivalent to (2), as already observed. □ L8 End

In Lemma 3.2.1 we have seen that ad_x is nilpotent whenever $x \in \mathfrak{gl}(V)$ is nilpotent. For arbitrary Lie algebras, the appropriate notion is the following.

Definition 3.2.3 An element x of a Lie algebra $\mathfrak{g}$ is **ad-nilpotent** if $\mathrm{ad}_x \in \mathfrak{gl}(\mathfrak{g})$ is nilpotent.

Example 3.2.4 Let V be a nonzero vector space and $x \in \mathfrak{gl}(V)$ nonzero. Let $\mathfrak{g} = \operatorname{span}\{x\} \subset \mathfrak{gl}(V)$ be the 1-dimensional abelian Lie subalgebra spanned by x. Then $\mathrm{ad}_x \in \mathfrak{gl}(\mathfrak{g})$ is nilpotent, and thus x is ad-nilpotent in $\mathfrak{g}$. But this holds regardless of whether x is nilpotent in $\mathfrak{gl}(V)$.

In Theorem 3.2.2 we have obtained a description of Lie algebras of nilpotent endomorphisms. The next theorem concerns (abstract) Lie algebras whose elements are all ad-nilpotent. To state it, we need some notions.

In Example 1.3.7 we have introduced the derived Lie algebra $[\mathfrak{g}, \mathfrak{g}]$, which is an ideal in $\mathfrak{g}$. This can be extended to two chains of nested ideals in $\mathfrak{g}$.

Definition 3.2.5 Let $\mathfrak{g}$ be a Lie algebra.

(1) The **derived series** of $\mathfrak{g}$ is defined as:

$$\mathfrak{g}^{(0)} = \mathfrak{g}, \quad \mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}], \quad \mathfrak{g}^{(i+1)} = [\mathfrak{g}^{(i)}, \mathfrak{g}^{(i)}] \text{ for } i \ge 0$$

(2) $\mathfrak{g}$ is called **solvable** (or **soluble**) if there exists an $m \in \mathbb{N}$ such that $\mathfrak{g}^{(m)} = 0$ (and then $\mathfrak{g}^{(n)} = 0$ for all $n \ge m$).

(3) The **lower central series** of $\mathfrak{g}$ is defined as:

$$\mathfrak{g}^0 = \mathfrak{g}, \quad \mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}], \quad \mathfrak{g}^{i+1} = [\mathfrak{g}, \mathfrak{g}^i] \text{ for } i \ge 0$$

(4) $\mathfrak{g}$ is called **nilpotent** if there exists an $m \in \mathbb{N}$ such that $\mathfrak{g}^m = 0$ (and then $\mathfrak{g}^n = 0$ for all $n \ge m$).

Remarks 3.2.6 (1) Using Remark 1.3.8 one checks inductively that all $\mathfrak{g}^{(i)}$ and $\mathfrak{g}^i$ are ideals in $\mathfrak{g}$ and $\mathfrak{g}^{(i)} \subset \mathfrak{g}^i$. Thus, any nilpotent Lie algebra is solvable.

(2) If $\mathfrak{g}$ is simple, then $\mathfrak{g}^{(i)} = \mathfrak{g}^i = \mathfrak{g}$ for all $i \ge 0$. (Otherwise $\mathfrak{g}^{(1)} = \mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}]$ would be an ideal not equal to $\mathfrak{g}$. By simplicity the only option is the zero ideal, but this would mean $\mathfrak{g}$ is abelian, a contradiction to simplicity.) Thus, simple Lie algebras are neither solvable nor nilpotent.

(3) The upper triangular matrices $\mathfrak{t}(n, \mathbb{F})$ are solvable and the strictly upper triangular matrices $\mathfrak{n}(n, \mathbb{F})$ are nilpotent.

(4) Every quotient and every Lie subalgebra of a nilpotent Lie algebra is again nilpotent. Every quotient and every Lie subalgebra of a solvable Lie algebra is again solvable. (Compare their lower central series and derived series.)

(5) In the solvable case, a stronger statement holds. For a short exact sequence $\mathfrak{g}_1 \hookrightarrow \mathfrak{g} \twoheadrightarrow \mathfrak{g}_2$ of Lie algebras, $\mathfrak{g}$ is solvable if and only if $\mathfrak{g}_1$ and $\mathfrak{g}_2$ are solvable.

(6) A Lie algebra $\mathfrak{g}$ is solvable if and only if there is a sequence

$$\mathfrak{g} = I_0 \supset I_1 \supset \cdots \supset I_m = 0$$

such that I_{i+1} is an ideal in I_i and I_i/I_{i+1} is abelian for all $0 \le i < m$.

(7) The sum of two solvable ideals I, J in $\mathfrak{g}$ is again solvable. This follows from the short exact sequence:

$$I \hookrightarrow I + J \twoheadrightarrow I + J/I \cong J/(I \cap J)$$

and Remarks 3.2.6.(5). A finite-dimensional Lie algebra $\mathfrak{g}$ thus always has a maximal solvable ideal $\operatorname{rad}\mathfrak{g}$, the **radical** of $\mathfrak{g}$, namely the sum of all solvable ideals.

Corollary 3.2.7 Let V be a finite-dimensiona vector space and $\mathfrak{g} \subset \mathfrak{gl}(V)$ a Lie subalgebra of nilpotent endomorphisms of V as in Theorem 3.2.2. Then $\mathfrak{g}$ is nilpotent in the sense of Definition 3.2.5.(4)

Proof. Theorem 3.2.2 implies that $\mathfrak{g}$ is isomorphic to a Lie subalgebra of strictly upper triangular matrices. By Remarks 3.2.6.(3-4) $\mathfrak{g}$ is also nilpotent. □

Theorem 3.2.8 [Engel's theorem] Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. Then the following are equivalent:

(1) $\mathfrak{g}$ is nilpotent (in the sense of Definition 3.2.5.(4))

(2) Every element of $\mathfrak{g}$ is ad-nilpotent.

Proof. (1) $\Rightarrow$ (2): Let $x \in \mathfrak{g}$. Note that $\mathrm{ad}_x \colon \mathfrak{g}^i \longrightarrow \mathfrak{g}^{i+1}$ for $i \geq 0$, thus if $\mathfrak{g}^n = 0$, then $(\mathrm{ad}_x)^n = 0$ and x is ad-nilpotent.

(2) $\Rightarrow$ (1): $\mathrm{ad}(\mathfrak{g})$ is a Lie subalgebra of nilpotent endomorphisms in $\mathfrak{gl}(\mathfrak{g})$ and thus nilpotent by Corollary 3.2.7. Let $i \in \mathbb{N}$ be sufficiently large, such that $(\mathrm{ad}(\mathfrak{g}))^i = 0$. Since $f(\mathfrak{g}^i) = f(\mathfrak{g})^i$ for every Lie algebra morphism with source $\mathfrak{g}$, we have $\mathrm{ad}(\mathfrak{g}^i) = (\mathrm{ad}(\mathfrak{g}))^i = 0$, so $\mathfrak{g}^i \subset \ker(\mathrm{ad}) = Z(\mathfrak{g})$, which implies $\mathfrak{g}^{i+1} = 0$, so $\mathfrak{g}$ is nilpotent. □ L9 End

Next we turn our attention to solvable Lie algebras, first again in the linear setting. We work over $\mathbb{C}$, but everything in this subsection also works for an algebraically closed field of characteristic zero.

Theorem 3.2.9 [Lie's theorem] Let V be a non-zero, finite-dimensional vector space over $\mathbb{C}$ and $\mathfrak{g}$ a solvable Lie subalgebra of $\mathfrak{gl}(V)$. Then there exists a simultaneous eigenvector v for all $x \in \mathfrak{g}$. I.e. there exists $v \in V$ with $v \neq 0$ and $\mathfrak{g}v \subset \mathbb{C}v$.

Proof. Clearly $\mathfrak{g}$ is finite-dimensional and we proceed by induction in $\dim(\mathfrak{g})$, with the case $\dim(\mathfrak{g}) = 0$ being immediate. For $\dim(\mathfrak{g}) > 0$ we claim that there exists an ideal $I \subset \mathfrak{g}$ of codimension 1 (i.e. with $\dim(\mathfrak{g}/I) = 1$). To see this, consider the Lie algebra $\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ and note that it is abelian, which implies that every linear subspace is an ideal. As $\mathfrak{g}$ is solvable, we must have $\mathfrak{g} \neq \mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}]$, and so $\dim(\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]) \geq 1$. Thus we can find a subspace $I' \subset \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$ of codimension 1 and set $I = q^{-1}(I')$. As preimage of an ideal under a Lie algebra morphism, I is an ideal.

Now we can apply the induction hypothesis to I, since it is still a Lie subalgebra of $\mathfrak{gl}(V)$ and solvable by Remarks 3.2.6.(4). Thus we find $v' \in V$. $v' \neq 0$ and $Iv' \subset \mathbb{C}v'$. This determines a linear functional $\lambda \colon I \longrightarrow \mathbb{C}$ that satisfies $x \cdot v' = \lambda(x)v'$ for $x \in I$. We now consider the simultaneous eigenspace

$$V_\lambda = \{w \in V \mid x \cdot w = \lambda(x)w \text{ for all } x \in I\}$$

and note $v' \in V_\lambda$. In Lemma 3.2.10 we will see that $\mathfrak{g}V_\lambda \subset V_\lambda$. Now we choose $y \in \mathfrak{g} \setminus I$ and deduce that y restricts to an endomorphism on V_λ. As we are working over $\mathbb{C}$, we can find an eigenvector $v \in V_\lambda$ of y. Since $\mathfrak{g} = I + \mathbb{C}y$, this v is a simultaneous eigenvector for all endomorphisms in $\mathfrak{g}$. □

We have used the following lemma.

Lemma 3.2.10 Let V be a finite-dimensional representation of a Lie algebra $\mathfrak{g}$, both over a field with $\mathrm{char}(\mathbb{F}) = 0$, $I \subset \mathfrak{g}$ an ideal, and $\lambda \colon I \longrightarrow \mathbb{F}$ a linear functional. Then the simultaneous eigenspace $V_\lambda = \{w \in V \mid x \cdot w = \lambda(x)w \text{ for all } x \in I\}$ is a subrepresentation of V.

Proof. For any $y \in \mathfrak{g}$ and $w \in V_\lambda$ we have to show that $y \cdot w \in V_\lambda$, i.e. that for every $x \in I$ we have $x \cdot (y \cdot w) = \lambda(x)(y \cdot w)$. We compute the difference as

$$\begin{aligned} x \cdot (y \cdot w) - \lambda(x)(y \cdot w) &= x \cdot (y \cdot w) - y \cdot (\lambda(x)w) \\ &= x \cdot (y \cdot w) - y \cdot (x \cdot w) \\ &= [x, y] \cdot w \\ &= \lambda([x, y])w \qquad (I \text{ is an ideal}) \end{aligned}$$

If $V_\lambda = 0$, then the statement of the lemma follows trivially. If $V_\lambda \neq 0$, we have to show that $\lambda([x, y]) = 0$ for all $x \in I, y \in \mathfrak{g}$. Fix a $y \in \mathfrak{g}$ and $w \in V_\lambda$, $w \neq 0$ and define a nested sequence of subspaces $W_0 \subset W_1 \subset W_2 \subset \cdots \subset V$ by:

$$W_0 := \operatorname{span}\{w\}, \quad W_i := \operatorname{span}\{w, y \cdot w, \ldots, y^i \cdot w\}$$

(Strictly speaking, $y^i \cdot w$ is an abuse of notation, albeit a convenient one. Rigorously we should denote this as $(\rho(y))^i(w)$ if the representation is given by $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$, or via the awkward expression $y \cdot (y \cdot (\cdots (y \cdot w)) \cdots)$.) Since V is finite-dimensional, the sequence stabilizes $W_0 \subset \cdots W_{n-1} \subset W_n = W_{n+1} = \cdots$. Let $n \in \mathbb{N}$ denote this this index and set $W := W_n$, which has $\{w, y \cdot w, \ldots, y^n \cdot w\}$ as basis and is stable under y.

Let $x \in I$. Using the equation $x \cdot (y^i \cdot w) = y \cdot (x \cdot (y^{i-1} \cdot w)) + [x, y] \cdot (y^{i-1} \cdot w)$ and $w \in V_\lambda$, one shows inductively in i that W_i is stable under x. A second induction shows

$$x \cdot (y^i \cdot w) \in y^i \cdot (x \cdot w) + W_{i-1} = \lambda(x) y^i \cdot w + W_{i-1}$$

Thus x acts on W as an endomorphism, represented by an upper triangular matrix with $\lambda(x)$ on the diagonal, hence has trace $\operatorname{tr}(x|_W) = (n+1)\lambda(x)$. Replacing x by $[x, y]$ and using that traces of the commutator of two endomorphisms of W vanish, we get $(n+1)\lambda([x, y]) = \operatorname{tr}([x, y]|_W) = 0$ and hence $\lambda([x, y]) = 0$. □

Remark 3.2.11 The proof works equally well over an arbitrary field $\mathbb{F}$ with $\operatorname{char}(\mathbb{F}) > \dim(V)$.

Corollary 3.2.12 [**representations of solvable Lie algebras**] Let V be a finite-dimensional representation of a complex solvable Lie algebra $\mathfrak{g}$. Then the following hold:

(1) V admits a flag $0 = V_0 \subset V_1 \subset \cdots \subset V_d = V$ of subrepresentations with $\dim(V_i) = i$. Equivalently, V has a basis, for which the action of $\mathfrak{g}$ is represented by upper triangular matrices.

(2) If V is simple, then $\dim(V) = 1$.

Proof. Analogous to the proof of Theorem 3.2.2.(2-3), based on Lie's Theorem 3.2.9 instead of Theorem 3.2.2.(1). □

Corollary 3.2.13 If $\mathfrak{g}$ is a finite-dimensional, solvable Lie algebra over $\mathbb{C}$, then $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent. Conversely, any Lie algebra $\mathfrak{g}$ with nilpotent $[\mathfrak{g}, \mathfrak{g}]$ is solvable.

Proof. For the first statement, we use Corollary 3.2.12.(1) to see that $\operatorname{ad}(\mathfrak{g}) \subset \mathfrak{gl}(\mathfrak{g})$ consists of upper triangular matrices (for some chosen basis). Thus $[\operatorname{ad}(\mathfrak{g}), \operatorname{ad}(\mathfrak{g})] = \operatorname{ad}([\mathfrak{g}, \mathfrak{g}])$ consists of strictly upper triangular matrices. This implies $(\operatorname{ad}([\mathfrak{g}, \mathfrak{g}]))^i = 0$ for $i \gg 0$ and thus $[\mathfrak{g}, \mathfrak{g}]^i \subset Z([\mathfrak{g}, \mathfrak{g}])$ (as in the proof of Theorem 3.2.8) and then $[\mathfrak{g}, \mathfrak{g}]^{i+1} = [[\mathfrak{g}, \mathfrak{g}], [\mathfrak{g}, \mathfrak{g}]^i] = 0$, so $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent.

Conversely, suppose $\mathfrak{h} = [\mathfrak{g}, \mathfrak{g}]$ is nilpotent. Then $\mathfrak{g}^{(i+1)} \subset \mathfrak{h}^i$ for all $i \geq 0$. For $i = 0$, this is immediate from the definition. For $i > 0$ we compute inductively:

$$\mathfrak{g}^{(i+1)} = [\mathfrak{g}^{(i)}, \mathfrak{g}^{(i)}] \subset [\mathfrak{h}^{i-1}, \mathfrak{h}^{i-1}] \subset [\mathfrak{h}, \mathfrak{h}^{i-1}] = \mathfrak{h}^i$$

Thus nilpotency for $\mathfrak{h}$ implies the solvability of $\mathfrak{g}$. □ L10 End

---

Exercise 30 Verify Remark 3.2.6.(1).

Exercise 31 Verify Remark 3.2.6.(4).

Exercise 32 Verify Remark 3.2.6.(5).

Exercise 33 Verify Remark 3.2.6.(6).

Exercise 34 Let $\mathbb{F}$ be a field of characteristic $\operatorname{char}(\mathbb{F}) = p \neq 0$. Consider the $p \times p$-matrices x and y with $x_{i,j} = \delta_{i+1,j} + \delta_{i,p}\delta_{j,1}$ and $y_{i,j} = \delta_{i,j}(i-1)$ for $1 \leq i, j \leq p$. Compute the commutator $[x, y]$ and deduce that these matrices span a 2-dimensional solvable Lie subalgebra of $\mathfrak{gl}(p, \mathbb{F})$. Show that x, y have no common eigenvector and discuss this from the perspective of Lie's theorem.

End Week 6 Exerc.
End Week 7 Exerc.

3.3 Cartan's criterion of solvability

Here we work towards a criterion for solvability in Theorem 3.3.6 whose proof requires some preparation.

Definition 3.3.1 An endomorphism $\varphi \in \mathrm{End}_{\mathbb{F}}(V)$ of a finite-dimensional vector space over $\mathbb{F}$ is called **semisimple** if it is diagonalizable over the algebraic closure $\overline{\mathbb{F}}$ of $\mathbb{F}$, i.e. if

$$\begin{aligned} \overline{\varphi}\colon \overline{\mathbb{F}} \otimes_{\mathbb{F}} V &\longrightarrow \overline{\mathbb{F}} \otimes_{\mathbb{F}} V \\ \overline{\lambda} \otimes v &\mapsto \overline{\lambda} \otimes \varphi(v) \end{aligned}$$

is diagonalizable.

We recall an important result from linear algebra.

Lemma 3.3.2 [Jordan decomposition] Let V be a finite-dimensional vector space over $\mathbb{C}$ and $x \in \mathrm{End}_{\mathbb{C}}(V)$. Then there exists a unique decomposition $x = x_s + x_n$. into a diagonalizable (semisimple) part $x_s \in \mathrm{End}_{\mathbb{C}}(V)$ and a nilpotent part $x_n \in \mathrm{End}_{\mathbb{C}}(V)$ such that $x_s x_n = x_n x_s$.

Remarks 3.3.3 Let V, W be finite-dimensional vector spaces over $\mathbb{C}$ and $x \in \mathrm{End}_{\mathbb{C}}(V)$.

(1) Choose a basis of V, such that x is represented by a matrix in Jordan normal form. Then x_s is represented by the diagonal part and x_n by the strictly upper triangular part of the matrix for x. E.g. for a single Jordan block of size 3, the splitting is:

$$x = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} \quad \Longrightarrow \quad x_s = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix}, \quad x_n = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

More intrinsically, one defines x_s by declaring the **generalized eigenspaces** of x to be eigenspaces of x_s with corresponding eigenvalue. I.e. for any $\lambda \in \mathbb{C}$, the endomorphism x_s will act by the scalar λ on the subspace

$$\overline{V_\lambda}(x) := \bigcup_{n \geq 0} \ker(x - \lambda \mathrm{id}_V)^n$$

The nilpotent part is then defined as the difference $x_n = x - x_s$.

(2) The Jordan decomposition is **functorial** in the following sense. If a linear map $\varphi\colon V \longrightarrow W$ intertwines two endomorphisms $x \in \mathrm{End}(V)$ and $y \in \mathrm{End}(W)$, then it also intertwines their semisimple and nilpotent parts, respectively. I.e. if the first square commutes, then so do the other two:

$$\begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "x"'] & W \arrow[d, "y"] \\ V \arrow[r, "\varphi"] & W \end{tikzcd} \qquad \begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "x_s"'] & W \arrow[d, "y_s"] \\ V \arrow[r, "\varphi"] & W \end{tikzcd} \qquad \begin{tikzcd} V \arrow[r, "\varphi"] \arrow[d, "x_n"'] & W \arrow[d, "y_n"] \\ V \arrow[r, "\varphi"] & W \end{tikzcd}$$

This is a consequence of $\varphi(\overline{V_\lambda}(x)) \subset \overline{W_\lambda}(y)$. It follows that an endomorphism $y \in \mathrm{End}_{\mathbb{C}}(V)$ commutes with x if and only if y commutes with x_s and x_n. (The "only if" direction uses functoriality; the "if" direction is elementary.) Another consequence is that if $x(B) \subset A$ for subspaces $A \subset B \subset V$, then also $x_s(B) \subset A$ and $x_n(B) \subset A$.

Lemma 3.3.4 Let V be a finite-dimensional vector space over $\mathbb{C}$. If $x \in \mathfrak{gl}(V)$ has Jordan decomposition $x = x_s + x_n$, then

$$\mathrm{ad}_x = \mathrm{ad}_{x_s} + \mathrm{ad}_{x_n}$$

is the Jordan decomposition of $\mathrm{ad}_x \in \mathrm{End}(\mathfrak{gl}(V))$. I.e. we have $(\mathrm{ad}_x)_s = \mathrm{ad}_{x_s}$ and $(\mathrm{ad}_x)_n = \mathrm{ad}_{x_n}$.

Proof. Since ad is a Lie algebra morphism we have $[\mathrm{ad}_{x_s}, \mathrm{ad}_{x_n}] = \mathrm{ad}_{[x_s, x_n]} = 0$, i.e. ad_{x_s} and ad_{x_n} commute. In Lemma 3.2.1 we have seen that ad_{x_n} is nilpotent. We thus only need to show that ad_{x_s} is diagonalizable to conclude the stated result using the uniqueness of the Jordan decomposition.

To this end, let $\{v_i\}_{1 \leq i \leq \dim(V)}$ be a basis of V consisting of eigenvectors of x_s, i.e. $x_s v_i = \lambda_i v_i$ for some $\lambda_i \in \mathbb{C}$. Then $\mathfrak{gl}(V) \cong V \otimes V^*$ has a basis consisting of the endomorphisms

$$v_j \otimes v_i^* \colon v \mapsto v_i^*(v) v_j$$

and we claim that these are eigenvectors of ad_{x_s}. To see this, we compute for all v_k:

$$\begin{aligned}\mathrm{ad}_{x_s}(v_j \otimes v_i^*)(v_k) &= (x_s \circ (v_j \otimes v_i^*))(v_k) - ((v_j \otimes v_i^*) \circ x_s)(v_k)\\ &= \delta_{i,k} x_s(v_j) - \lambda_k (v_j \otimes v_i^*)(v_k)\\ &= \delta_{i,k}(\lambda_j - \lambda_k)(v_j)\\ &= (\lambda_j - \lambda_i)(v_j \otimes v_i^*)(v_k)\end{aligned}$$

Thus $\mathrm{ad}_{x_s}(v_j \otimes v_i^*) = (\lambda_j - \lambda_i)(v_j \otimes v_i^*)$. □

The following somewhat contrived lemma serves as a tool to detect nilpotent endomorphisms.

Lemma 3.3.5 Let V be a finite-dimensional vector space over $\mathbb{C}$. Let $A \subset B \subset \mathrm{End}_{\mathbb{C}}(V)$ be subspaces and $x \in \mathrm{End}_{\mathbb{C}}(V)$ with $[x, B] \subset A$. Suppose $\mathrm{tr}(x \circ z) = 0$ for all $z \in \mathrm{End}_{\mathbb{C}}(V)$ with $[z, B] \subset A$, then x is nilpotent.

Proof. By Lemma 3.3.4 we have the compatible Jordan decompositions $x = x_s + x_n$ and $\mathrm{ad}_x = \mathrm{ad}_{x_s} + \mathrm{ad}_{x_n}$. Then Remarks 3.3.3.(2) implies:

$$\mathrm{ad}_{x_s}(B) \subset A\,, \quad \mathrm{ad}_{x_n}(B) \subset A.$$

In particular, the eigenspaces of $\mathrm{ad}_{x_s} \colon B \longrightarrow B$ for non-zero eigenvalues are contained in A. Choose a basis of V consisting of eigenvectors v_i for x_s and then define a semisimple endomorphism $z \in \mathrm{End}_{\mathbb{C}}(V)$ by the matrix (with respect to the chosen basis) given by the complex conjugate of the (diagonal) matrix of x_s. A computation as in the proof of Lemma 3.3.4 shows that $\{v_j \otimes v_i^*\}$ is an eigenbasis of $V \otimes V^* \cong \mathrm{End}_{\mathbb{C}}(V)$ for ad_{x_s} and for ad_z with eigenvalues $\lambda_j - \lambda_i$ and $\overline{\lambda_j} - \overline{\lambda_i}$ respectively. The eigenspace for ad_{x_s} for eigenvalue $\lambda \in \mathbb{C}$ thus equals the eigenspace of ad_z for $\overline{\lambda}$. This implies $[z, B] \subset A$. By assumption we have $0 = \mathrm{tr}(x_s \circ z)$, but $x_s \circ z$ is represented by a diagonal matrix with entries $|\lambda_i|^2$, so all $\lambda_i = 0$, thus $x_s = 0$ and $x = x_n$ is nilpotent. □

Theorem 3.3.6 [**Cartan's criterion of solvability**] Let V be a finite-dimensional vector space over $\mathbb{C}$ and $\mathfrak{g}$ a Lie subalgebra of $\mathfrak{gl}(V)$. Then $\mathfrak{g}$ is solvable if and only if $\mathrm{tr}(x \circ y) = 0$ for all $x \in [\mathfrak{g}, \mathfrak{g}]$ and $y \in \mathfrak{g}$.

Proof. Suppose that $\mathfrak{g}$ is solvable, then Corollary 3.2.12 says that all $y \in \mathfrak{g}$ act on V by upper triangular matrices with respect to some fixed basis. Consequently, any $x \in [\mathfrak{g}, \mathfrak{g}]$ acts by a strictly upper triangular matrix, and so does $x \circ y$, but these have trace zero.

Conversely, suppose that $\mathrm{tr}(x \circ y) = 0$ for all $x \in [\mathfrak{g}, \mathfrak{g}]$ and $y \in \mathfrak{g}$. We prove that $\mathfrak{g}$ is solvable by showing that $[\mathfrak{g}, \mathfrak{g}]$ is nilpotent (Corollary 3.2.13). It is enough to show that all $x \in [\mathfrak{g}, \mathfrak{g}]$ act by nilpotent endomorphisms of V (Corollary 3.2.7). For this we would like to use the technical Lemma 3.3.5 with $A = [\mathfrak{g}, \mathfrak{g}]$ and $B = \mathfrak{g}$. To do so, let $z \in \mathrm{End}_{\mathbb{C}}(V)$ with $[z, \mathfrak{g}] \subset [\mathfrak{g}, \mathfrak{g}]$ and expand $x = \sum_i [c_i, d_i]$ as a sum of Lie brackets (note that $x \in [\mathfrak{g}, \mathfrak{g}]$ need not be a single Lie bracket!). Then we compute:

$$\begin{aligned}\mathrm{tr}(x \circ z) &= \sum_i \mathrm{tr}([c_i, d_i] \circ z)\\ &= \sum_i \mathrm{tr}(c_i \circ d_i \circ z - d_i \circ c_i \circ z)\\ &= \sum_i \mathrm{tr}(c_i \circ d_i \circ z - c_i \circ z \circ d_i)\\ &= \sum_i \mathrm{tr}(c_i \circ [d_i, z]) = 0\end{aligned}$$

where we have used $\mathrm{tr}(y \circ x) = \mathrm{tr}(x \circ y) = 0$ for all $x \in [\mathfrak{g}, \mathfrak{g}]$ and $y \in \mathfrak{g}$ in the last line. Now the assumptions of Lemma 3.3.5 are satisfied, its conclusion completes the proof. □

Definition 3.3.7 Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over $\mathbb{F}$. The **Killing form** of $\mathfrak{g}$ is the bilinear form:

$$\begin{aligned}\kappa = \kappa_{\mathfrak{g}} \colon \mathfrak{g} \times \mathfrak{g} &\longrightarrow \mathbb{F},\\ (x, y) &\mapsto \mathrm{tr}(\mathrm{ad}_x \circ \mathrm{ad}_y).\end{aligned}$$

A bilinear form $b \colon \mathfrak{g} \times \mathfrak{g} \to \mathbb{F}$ is called **invariant** if $b([x, y], z) = b(x, [y, z])$ for all $x, y, z \in \mathfrak{g}$.

Note that the Killing form is symmetric and invariant (a computation analogous to the end of the proof of Theorem 3.3.6).

Corollary 3.3.8 Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over $\mathbb{C}$. Then $\mathfrak{g}$ is solvable if and only if $\mathfrak{g} \perp [\mathfrak{g}, \mathfrak{g}]$ with respect to the Killing form, i.e. $\kappa(x, [y, z]) = 0$ for all $x, y, z \in \mathfrak{g}$.

Proof. Note that $\kappa(x, [y, z]) = \operatorname{tr}(\operatorname{ad}_x \circ \operatorname{ad}_{[y,z]}) = \operatorname{tr}(\operatorname{ad}_x \circ [\operatorname{ad}_y, \operatorname{ad}_z]) = \operatorname{tr}([\operatorname{ad}_y, \operatorname{ad}_z] \circ \operatorname{ad}_x)$. By Cartan's criterion Theorem 3.3.6 the second condition is equivalent to $\operatorname{ad}(\mathfrak{g})$ being solvable. But this holds if and only if $\mathfrak{g}$ is solvable, as seen from the short exact sequence $Z(\mathfrak{g}) \hookrightarrow \mathfrak{g} \twoheadrightarrow \operatorname{ad}(\mathfrak{g})$ and Remark 3.2.6.(5). □ L11

End

In particular, if the Killing form vanishes identically, $\mathfrak{g}$ must be solvable.

Remark 3.3.9 Using a slighly more careful proof, Lemma 3.3.5 and then also Theorem 3.3.6 and Corollary 3.3.8 extend to an arbitrary field of characteristic zero.

Lemma 3.3.10 Let $\mathfrak{g}$ be a finite-dimensional Lie algebra and $I \subset \mathfrak{g}$ and ideal. Then the Killing form of I is the restriction of the Killing form of $\mathfrak{g}$, i.e. $\kappa_I = \kappa_{\mathfrak{g}}|_{I \times I}$.

Proof. More generally, if $I \subset \mathfrak{g}$ are finite-dimensional vector spaces and $a \in \operatorname{End}_{\mathbb{F}}(\mathfrak{g})$ with $a(\mathfrak{g}) \subset I$, then $\operatorname{tr}(a) = \operatorname{tr}(a|_I)$ for the restriction $a|_I \colon I \longrightarrow I$. This we apply to $a = \operatorname{ad}_x \circ \operatorname{ad}_y$ for $x, y \in I$. □

Exercise 35 Let $x \in \operatorname{End}_{\mathbb{C}}(V)$ be an endomorphism of a finite-dimensional vector space over $\mathbb{C}$. Show that there exist polynomials $P, Q \in \mathbb{C}[X]$ without constant term (i.e. $P(0) = Q(0) = 0$) such that $x_s = P(x)$ and $x_n = Q(x)$. (Hint: use the Chinese Remainder Theorem.) Now deduce the statements of Remarks 3.3.3.(2).

4 Complex semisimple Lie algebras

In this section we work over a field $\mathbb{F}$ of characteristic zero.

4.1 Characterization

Definition 4.1.1 A Lie algebra is called **semisimple** if it is isomorphic to a finite product of finite-dimensional simple Lie algebras. A Lie algebra is called **reductive** if it is isomorphic to a product of a semisimple and a finite-dimensional abelian Lie algebra.

Remarks 4.1.2 (1) Any semisimple Lie algebra is also reductive.

(2) The Lie algebra $\mathfrak{g} = 0$ is semisimple. Abelian Lie algebras $\mathfrak{g} \neq 0$ are not semisimple but reductive.

(3) If $\mathfrak{g}$ is finite-dimensional and simple, then $\mathfrak{g}$ is semisimple. However, there are also infinite-dimensional simple Lie algebras, so semisimplicity is not a generalization of simplicity.

Theorem 4.1.3 [Characterization of semisimple Lie algebras] Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over $\mathbb{F}$ of characteristic zero. Then the following are equivalent:

(1) $\mathfrak{g}$ is a direct sum of its simple ideals;

(2) $\mathfrak{g}$ is semisimple;

(3) $\mathfrak{g}$ has no abelian ideal except 0;

(4) $\mathfrak{g}$ has no solvable ideal except 0 (i.e. $\mathrm{rad}(\mathfrak{g}) = 0$);

(5) $\mathfrak{g}$ has a non-degenerate Killing form.

If $\mathfrak{g}$ is a Lie algebra and $\mathfrak{g}_1, \dots, \mathfrak{g}_n$ Lie subalgebras, then we say $\mathfrak{g}$ decomposes into the product of the $\mathfrak{g}_i$ and write $\mathfrak{g} = \mathfrak{g}_1 \times \cdots \times \mathfrak{g}_n$ if the map $\mathfrak{g}_1 \times \cdots \times \mathfrak{g}_n \longrightarrow \mathfrak{g}$ given by addition is an isomorphism of Lie algebras. This is equivalent to the $\mathfrak{g}_i$ being ideals of $\mathfrak{g}$ and the map $\mathfrak{g}_1 \times \cdots \times \mathfrak{g}_n \longrightarrow \mathfrak{g}$ an isomorphism of vector spaces.

Proof. (1) $\Rightarrow$ (2): directly from the definition. (2) $\Rightarrow$ (3): a nontrivial abelian ideal would have nontrivial intersection with at least one simple factor of $\mathfrak{g}$, in contradiction to simplicity. (3) $\Rightarrow$ (4): Any nontrivial solvable ideal contains a nontrivial abelian ideal at the end of of its derived series (all entries of this derived series are ideals **of** $\mathfrak{g}$ because of Remark 1.3.8). (4) $\Rightarrow$ (3): an nontrivial abelian ideal would also be solvable.

(4) $\Rightarrow$ (5): We construct a candidate solvable ideal, namely the radical of the Killing form $\kappa = \kappa_{\mathfrak{g}}$:

$$\mathrm{rad}_\kappa := \{x \in \mathfrak{g} \mid \kappa(x, y) = 0 \text{ for all } y \in \mathfrak{g}\}$$

One easily checks that this is an ideal because the Killing form is invariant. The restriction of κ to rad_κ, i.e. the Killing form *of* rad_κ (Lemma 3.3.10) is zero, thus rad_κ is solvable by Corollary 3.3.8. Now (4) implies that $\mathrm{rad}_\kappa = 0$, i.e. that κ is non-degenerate, which is (5).

(5) $\Rightarrow$ (3): Let $I \subset \mathfrak{g}$ be an abelian ideal. We claim that $I \subset \mathrm{rad}_\kappa$. If the latter is zero by (5), then also the former (3). To verify the claim, we need to check that for any $x \in \mathfrak{g}$ and $y \in I$ we have $\kappa(x, y) = \mathrm{tr}(\mathrm{ad}_x \circ \mathrm{ad}_y) = 0$. The latter holds because $\mathrm{ad}_x \circ \mathrm{ad}_y$ is nilpotent, in fact $(\mathrm{ad}_x \circ \mathrm{ad}_y)^2 = 0$. To see this, note that for $z \in \mathfrak{g}$ we have $(\mathrm{ad}_x \circ \mathrm{ad}_y)^2(z) = [x, [y, [x, [y, z]]]]$. Now $[x, [y, z]] \in I$ since I is an ideal and then $[y, [x, [y, z]]] = 0$ since I is abelian.

(4) $\Rightarrow$ (1): If $I \subset \mathfrak{g}$ is an ideal, then so is $I^\perp := \{y \in \mathfrak{g} \mid \kappa(y, I) = 0\}$ because κ is invariant. Furthermore, κ vanishes on the ideal $I \cap I^\perp$, which is thus solvable by Corollary 3.3.8 (and Lemma 3.3.10). Now assume that (4) holds. This implies $I \cap I^\perp = 0$ and then $[I, I^\perp] = 0$ (both are ideals). Comparing dimensions we see $\mathfrak{g} = I \oplus I^\perp$ and thus $\mathfrak{g} = I \times I^\perp$. Any ideal J of I or $I^\perp$ would also be an ideal of $\mathfrak{g}$, thus (4) applies to I and $I^\perp$ and we can iterate the argument. If in each step we choose a proper, nontrivial ideal, then this process will terminate when we have written $\mathfrak{g} = I_1 \times \cdots \times I_r$ for simple ideals I_j. This decomposition is unique, since for any other simple ideal J, one has $J = [J, \mathfrak{g}] = [J, I_1] \times \cdots \times [J, I_r]$, so $J = [J, I_j] = I_j$ for some j. □

Remarks 4.1.4 (1) Every ideal in a complex semisimple Lie algebra is a product of simple ideals. Every quotient and every homomorphic image of a complex semisimple Lie algebra is semisimple.

(2) A reductive Lie algebra $\mathfrak{g}$ decomposes uniquely as $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \times Z(\mathfrak{g})$ into semisimple and abelian parts.

(3) If $\mathfrak{g}$ is semisimple, then $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$. Such Lie algebras are called **perfect**. There are perfect Lie algebras that are not semisimple and not reductive.

Now we can supply one result that was missing during the discussion of classification problems in Section 3.1.

Lemma 4.1.5 Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. If $\mathfrak{g}$ is solvable, then $\mathfrak{g}/\operatorname{rad}\mathfrak{g} = 0$. Conversely, if $\mathfrak{g}$ is not solvable, then $\mathfrak{g}/\operatorname{rad}\mathfrak{g} \neq 0$ is semisimple.

Proof. The first statement is immediate since $\operatorname{rad}\mathfrak{g} = \mathfrak{g}$ for solvable $\mathfrak{g}$. For the second, suppose $I \neq 0$ is a solvable ideal in $\mathfrak{g}/\mathrm{rad}$. Then the short exact sequence

$$\operatorname{rad}\mathfrak{g} \hookrightarrow \mathfrak{g} \twoheadrightarrow \mathfrak{g}/\operatorname{rad}\mathfrak{g}$$

contains the short exact sequence of ideals

$$\operatorname{rad}\mathfrak{g} \hookrightarrow q^{-1}(I) \twoheadrightarrow I$$

where q again denotes the quotient morphism. By Remarks 3.2.6.(5) $q^{-1}(I)$ is again solvable but it properly contains $\operatorname{rad}\mathfrak{g}$, a contradiction to maximality. □ L12 End

---

Exercise 36 Prove Remarks 4.1.4.(1).

Exercise 37 Prove Remarks 4.1.4.(2) and deduce that $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$ for semisimple $\mathfrak{g}$, as stated in Remarks 4.1.4.(3). End Week 8 Exerc.

4.2 Weyl's theorem

Recall the notation $\operatorname{Hom}_{\mathfrak{g}}(V, W)$ and $\operatorname{End}_{\mathfrak{g}}(V)$ from Remarks 1.4.5 for spaces of intertwiners of $\mathfrak{g}$-representations.

Lemma 4.2.1 [Schur's Lemma] If $\mathfrak{g}$ is a Lie algebra over $\mathbb{C}$ and L a finite-dimensional simple representation, then $\operatorname{End}_{\mathfrak{g}}(L) = \mathbb{C}\mathrm{id}_L$.

Proof. Any $\phi \in \operatorname{End}_{\mathbb{C}}(L)$ has at least one eigenvalue λ since $L \neq 0$. If furthermore $\phi \in \operatorname{End}_{\mathfrak{g}}(L)$, then the eigenspace L_λ of ϕ for the eigenvalue λ is a nontrivial subrepresentation of L. To see this, note that for $x \in \mathfrak{g}$ and $v \in L_\lambda$:

$$\phi(v) = \lambda v \implies \phi(x \cdot v) = x \cdot (\phi(v)) = \lambda(x \cdot v)$$

Since L is simple, we have $L = L_\lambda$ and so $\phi = \lambda \mathrm{id}_L$. □

Remark 4.2.2 Slightly more general, we get the following: let V, W be representations of a Lie algebra over $\mathbb{F}$, not necessarily algebraically closed, and $\phi \in \operatorname{Hom}_{\mathfrak{g}}(V, W)$.

(1) If V is simple, then ϕ is injective or zero, since $\ker(\phi)$ is a subrepresentation of V.

(2) If W is simple, then ϕ is surjective or zero, since $\operatorname{im}(\phi)$ is a subrepresentation of W.

(3) $\operatorname{End}_{\mathfrak{g}}(V)$ is a division algebra over $\mathbb{F}$ since any $\phi \in \operatorname{End}_{\mathfrak{g}}(V)$ is either zero or an isomorphism.

Definition 4.2.3 A Lie algebra representation V is called **semisimple** if it is a direct sum of simple subrepresentations, $V = \bigoplus_i V_i$ with $V_i \subset V$ simple. The zero representation $V = 0$ is semisimple.

Remarks 4.2.4 (1) A finite-dimensional Lie algebra is reductive if and only if its adjoint representation is semisimple in the sense of Definition 4.2.3, i.e. a direct sum of its simple representations. This follows directly from Remarks 1.5.2.

(2) Every subrepresentation and quotient of a semisimple representation is again semisimple.

(3) For a representation V the following are equivalent:

- V is semisimple,
- V is a sum (not necessarily a direct sum) of simple subrepresentations,

- every subrepresentation of V has a complement that is a representation.

Theorem 4.2.5 [Weyl's Theorem] Every finite-dimensional representation of a complex semisimple Lie algebra is semisimple.

The proof appears below and requires a bit of preparation.

Construction 4.2.6 Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over $\mathbb{F}$ and $b\colon \mathfrak{g}\times\mathfrak{g}\to\mathbb{F}$ a non-degenerate invariant bilinear form. For any $\mathfrak{g}$-representation V we define a linear map

$$C_b = C_b^V : V \to V$$

as follows. Choose a basis $\{x_1,\dots,x_n\}$ of $\mathfrak{g}$ and let $\{x^1,\dots,x^n\}$ the basis dual with respect to b, i.e. the basis determined by $b(x_i,x^j)=\delta_{i,j}$. Now set

$$C_b(v) = \sum_{i=1}^{n} x_i\cdot(x^i\cdot v).$$

(In fact, this does not depend on the choice of basis.)

Example 4.2.7 Consider $\mathfrak{g}=\mathfrak{sl}(2,\mathbb{C})$ and $b=8\kappa$ the scaled Killing form, then one can check that C_b is the Casimir operator from Exercise 21.

Lemma 4.2.8 The map C_b commutes with the action of $\mathfrak{g}$, i.e. $C_b\in\mathrm{End}_{\mathfrak{g}}(V)$.

Proof. Let $y\in\mathfrak{g}$ and expand $[x_i,y]=\sum_{j=1}^n a_{i,j}x_j$ and $[y,x^j]=\sum_{i=1}^n b_{j,i}x^i$, then the invariance of the bilinear form $b([x_i,y],x^j)=b(x_i,[y,x^j])$ implies $a_{i,j}=b_{j,i}$. Thus for $v\in V$ we have

$$\begin{aligned} y\cdot C_b(v)-C_b(y\cdot v) &= \sum_i [y,x_i]\cdot(x^i\cdot v)+\sum_i x_i\cdot([y,x^i]\cdot v)\\ &= \sum_{i,j} -a_{i,j}x_j\cdot(x^i\cdot v)+\sum_{i,j} b_{i,j}x_i\cdot(x^j\cdot v)=0 \end{aligned}$$ □

Remark 4.2.9 The Casimir operator also admits a basis-independent definition as a composition of $\mathfrak{g}$-intertwiners

$$V\to\mathfrak{g}\otimes\mathfrak{g}^*\otimes V\to\mathfrak{g}\otimes\mathfrak{g}\otimes V\to V,$$

where the first map is induced by the image of $\mathrm{id}_{\mathfrak{g}}\in\mathrm{End}_{\mathbb{F}}(\mathfrak{g})\cong\mathfrak{g}\otimes\mathfrak{g}^*$, namely $\sum_i x_i\otimes x_i^*$ (where x_i^* denotes the elements of the dual basis), the second map is built using the inverse to the map $\mathfrak{g}\to\mathfrak{g}^*, x^i\mapsto x_i^*$ induced by the non-degenerate bilinear form b, and the third map is given by using the action twice. We leave the details to the reader.

Lemma 4.2.10 Let V be a finite-dimensional vector space over $\mathbb{F}$ of $\mathrm{char}(\mathbb{F})=0$ and $\mathfrak{g}\subset\mathfrak{gl}(V)$ a semisimple Lie subalgebra. Then:

(1) $(x,y)\mapsto\mathrm{tr}(x\circ y)$ is a nondegenerate, invariant, symmetric bilinear form $b=b_V$ on $\mathfrak{g}$.

(2) For $C=C_b^V$ we have $\mathrm{tr}(C)=\dim(\mathfrak{g})$.

Proof. The bilinear form is clearly symmetric and also invariant, as a short computation (see end of proof of Theorem 3.3.6) shows. In particular, rad_b is an ideal, that is solvable by Theorem 3.3.6 (see 3.3.9) and thus zero by Theorem 4.1.3 since $\mathfrak{g}$ is semisimple. Now $\mathrm{rad}_b=0$ just says that b is non-degenerate. For the second assertion, consider a basis $\{x_1,\dots,x_n\}$ of $\mathfrak{g}$ and the basis $\{x^1,\dots,x^n\}$ that is dual with respect to the trace pairing on V, i.e. $\mathrm{tr}(x_i\circ x^j)=\delta_{i,j}$. Since $\mathfrak{g}\subset\mathfrak{gl}(V)$, we have the simple formula $C_b^V=\sum_{i=1}^n x_i\circ x^i\colon V\to V$, and by linearity of the trace

$$\mathrm{tr}(C_b^V)=\sum_{i=1}^n\mathrm{tr}(x_i\circ x^i)=n=\dim(\mathfrak{g})$$ □

Lemma 4.2.11 Every finite-dimensional representation V of a complex semisimple Lie algebra decomposes as $V\cong V^{\mathfrak{g}}\oplus\mathfrak{g}\cdot V$. In particular, one can identify the invariants $V^{\mathfrak{g}}$ with the coinvariants $V/\mathfrak{g}\cdot V$.

Here $\mathfrak{g}\cdot V$ denotes the subspace of V spanned by all $x\cdot v$ for $x\in\mathfrak{g}, v\in V$, which is clearly a subrepresentation.

Proof. We proceed by induction on the dimension of V, with the cases $\dim(V) = 0, 1$ being straightforward. If $V^{\mathfrak{g}} = V$, then we are done as $\mathfrak{g} \cdot V = 0$. Thus suppose that $V^{\mathfrak{g}} \neq V$. This implies that the Lie algebra morphism $\rho\colon \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ defining the representation is nonzero, giving us a nontrivial Lie subalgebra $\rho(\mathfrak{g}) \subset \mathfrak{gl}(V)$ that is again semisimple by Remarks 4.1.4.(1). Let $C \in \mathrm{End}_{\rho(\mathfrak{g})}(V)$ be the associated intertwiner from Lemma 4.2.10. By Lemma 1.4.6 V decomposes as the direct sum of the generalized eigenspaces of C. If C has more than one eigenvalue, then we could write $V = V_1 \oplus V_2$ for subrepresentations with $\dim(V_1) < \dim(V) > \dim(V_2)$ and then use $(V_1 \oplus V_2)^{\mathfrak{g}} = V_1^{\mathfrak{g}} \oplus V_2^{\mathfrak{g}}$ and $\mathfrak{g} \cdot (V_1 \oplus V_2) = \mathfrak{g} \cdot V_1 \oplus \mathfrak{g} \cdot V_2$ to conclude the statement from the induction hypothesis. Thus suppose that C has a single eigenvalue, which has to be nonzero since $\mathrm{tr}(C) = \dim(\rho(\mathfrak{g})) \neq 0$ by Lemma 4.2.10. Thus we have $V = \mathrm{im}(C)$ and $V^{\mathfrak{g}} \subset \ker(C) = 0$, and then $V = \mathfrak{g} \cdot V = V^{\mathfrak{g}} \oplus \mathfrak{g} \cdot V$. □ L13 End

Remark 4.2.12 As mentioned in the proof, the statement of the lemma is straightforward for 1-dimensional V, since either all of $\mathfrak{g}$ acts trivially on a spanning vector of V, and then $V = V^{\mathfrak{g}}$, or not, in which case $V = \mathfrak{g} \cdot V$. In fact, for semisimple $\mathfrak{g}$ only the former occurs, since by Example 1.4.7 the 1-dimensional representations are classified, up to isomorphism, by $(\mathfrak{g}/[\mathfrak{g}, \mathfrak{g}])^*$, which is trivial since $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$.

Proof of Weyl's Theorem 4.2.5. We need to show that every finite-dimensional representation V of a complex semisimple Lie algebra is semisimple. If $U \subset V$ is a subrepresentation, then restriction of linear maps provides a surjective morphism $\mathrm{Hom}_{\mathbb{C}}(V, U) \twoheadrightarrow \mathrm{Hom}_{\mathbb{C}}(U, U)$ of $\mathfrak{g}$-representations. By Lemma 4.2.11 and since $\mathfrak{g} \cdot \mathrm{Hom}_{\mathbb{C}}(V, U)$ is mapped to $\mathfrak{g} \cdot \mathrm{Hom}_{\mathbb{C}}(U, U)$, this induces a surjection $\mathrm{Hom}_{\mathbb{C}}(V, U)^{\mathfrak{g}} \twoheadrightarrow \mathrm{Hom}_{\mathbb{C}}(U, U)^{\mathfrak{g}}$. After choosing a preimage $f \in \mathrm{Hom}_{\mathfrak{g}}(V, U) = \mathrm{Hom}_{\mathbb{C}}(V, U)^{\mathfrak{g}}$ of $\mathrm{id}_U \in \mathrm{Hom}_{\mathbb{C}}(U, U)^{\mathfrak{g}}$ we decompose the representation V as $V \cong U \oplus \ker(f)$. Induction completes the proof. □

Proposition 4.2.13 A finite-dimensional complex Lie algebra is reductive if and only if every solvable ideal is contained in the center.

Proof. By Remarks 4.2.4.(1) a finite-dimensional Lie algebra $\mathfrak{g}$ is reductive if and only if its adjoint representation is semisimple. The latter implies that every ideal of $\mathfrak{g}$ is a direct sum of simple or abelian ideals and every ideal has a vector space complement that is again an ideal. Then every 1-dimensional ideal is contained in $Z(\mathfrak{g})$ and every solvable ideal is a sum of 1-dimensional ideals, thus also central. Conversely, if every solvable ideal is central, then $\mathrm{ad}(\mathfrak{g}) \subset \mathfrak{gl}(\mathfrak{g})$ is semisimple (because $\mathrm{rad}(\mathfrak{g}) = Z(\mathfrak{g})$ and Lemma 4.1.5) and by Theorem 4.2.5 $\mathfrak{g}$ is a semisimple representation of $\mathrm{ad}(\mathfrak{g})$. In other words, the adjoint representation of $\mathfrak{g}$ is semisimple, and thus $\mathfrak{g}$ is reductive. □

Theorem 4.2.14 [Sufficient condition for semisimplicity] Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over $\mathbb{C}$.

(1) If $\mathfrak{g}$ admits a faithful, simple, finite-dimensional representation V, then $\mathfrak{g}$ is reductive and $\dim(Z(\mathfrak{g})) \leq 1$.

(2) If $\mathfrak{g}$ acts on this V by endomorphisms of trace zero, then $\mathfrak{g}$ is semisimple.

Proof. Using Proposition 4.2.13 it suffices to prove that every solvable ideal $I \subset \mathfrak{g}$ is central. By Lie's Theorem 3.2.9, there exists a $v \in V$, $v \neq 0$ such that $I \cdot v \subset \mathbb{C}v$. This determines a linear form $\lambda \in I^*$ with $x \cdot v = \lambda(x)v$ for all $x \in I$. By Lemma 3.2.10 the corresponding simultaneous eigenspace $V_\lambda = \{w \in V \mid x \cdot w = \lambda(x)w \text{ for all } x \in I\}$ is a subrepresentation of V, but since $0 \neq v \in V$ and V was simple we deduce $V = V_\lambda$. This means a solvable ideal $I \subset \mathfrak{g}$ acts on V by scalar multiples of the identity map. If the representation is faithful, then $\dim(I) \leq 1$ and $[I, \mathfrak{g}] = 0$, i.e. I is central. Since $Z(\mathfrak{g})$ is an abelian (and thus solvable) ideal in $\mathfrak{g}$, the dimension constraint follows. If the action is by trace zero matrices, then any solvable I acts by the zero multiple of the identity matrix, hence $I = 0$ by faithfulness, and $\mathfrak{g}$ is semisimple. □

Example 4.2.15 We deduce that $\mathfrak{gl}(n, \mathbb{C})$ is reductive and $\mathfrak{sl}(n, \mathbb{C})$ is semisimple.

Exercise 38 Verify Remarks 4.2.4.(2)-(3). End Week 9 Exerc.

4.3 Jordan decomposition in semisimple Lie algebras

Theorem 4.3.1 [Jordan decomposition in semisimple Lie algebras] Let $\mathfrak{g}$ be a complex semisimple Lie algebra.

(1) Every $x \in \mathfrak{g}$ admits a unique decomposition $x = s + n$ with ad_s diagonalizable, ad_n nilpotent, and $[s, n] = 0$. This is called the **absolute Jordan decomposition of x in** $\mathfrak{g}$.

(2) If $\rho\colon \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ is a finite-dimensional representation and $x = s + n$ is the absolute Jordan decomposition of x in $\mathfrak{g}$, then $\rho(x) = \rho(s) + \rho(n)$ is the Jordan decomposition of $\rho(x) \in \mathrm{End}(V)$ in the sense of Lemma 3.3.2.

(3) If $\phi\colon \mathfrak{g} \to \mathfrak{g}'$ is a morphism of Lie algebras with $\mathfrak{g}'$ semisimple and $x = s + n$ the absolute Jordan decomposition of x in $\mathfrak{g}$, then $\phi(x) = \phi(s) + \phi(n)$ is the absolute Jordan decomposition of $\phi(x)$ in $\mathfrak{g}'$.

The proof appears below after two preparatory results.

Lemma 4.3.2 Every semisimple ideal I of a finite-dimensional Lie algebra $\mathfrak{g}$ over $\mathbb{C}$ has a vector space complement that is itself an ideal.

An ideal is called semisimple if it is semisimple as Lie algebra.

Proof. The proof is reminiscent of the part (4) $\Rightarrow$ (1) in Theorem 4.1.3. As candidate for the complement we consider the orthogonal complement of I with respect to the Killing form of $\mathfrak{g}$. In formulas, $I^\perp = \{x \in \mathfrak{g} \mid \kappa_{\mathfrak{g}}(x, y) = 0 \text{ for all } y \in I\}$. By invariance of the Killing form, $I^\perp$ is an ideal of $\mathfrak{g}$. Since $I \cap I^\perp$ is an ideal of I, it is semisimple, but its Killing form vanishes, forcing $I \cap I^\perp = 0$ by Theorem 4.1.3. Comparing dimension, using the non-degeneracy of the Killing form on I, we see $\mathfrak{g} \cong I \times I^\perp$. □

Lemma 4.3.3 Let V be a finite-dimensional complex vector space and $\mathfrak{g} \subset \mathfrak{gl}(V)$ a semisimple Lie subalgebra. If $x \in \mathfrak{g}$ has the Jordan decomposition $x = x_s + x_n$, then $x_s, x_n \in \mathfrak{g}$.

Proof. The proof relies on a characterization of $\mathfrak{g}$ as Lie subalgebra of $\mathfrak{gl}(V)$, namely by identifying it with:

$$N := N_{\mathfrak{gl}(V)}(\mathfrak{g}) \cap \bigcap_{\substack{W \subset V \\ \mathfrak{g}\text{-subrep}}} SN_{\mathfrak{gl}(V)}(W)$$

where

$$N_{\mathfrak{gl}(V)}(\mathfrak{g}) := \{y \in \mathfrak{gl}(V) | [y, \mathfrak{g}] \subset \mathfrak{g}\}$$
$$SN_{\mathfrak{gl}(V)}(W) := \{y \in \mathfrak{gl}(V) | y \cdot W \subset W \text{ and } \operatorname{tr}(y|_W) = 0\}.$$

The inclusion $\mathfrak{g} \subset N$ follows from the general fact $\mathfrak{g} \subset N_{\mathfrak{gl}(V)}(\mathfrak{g})$ together with the observations that each $y \in \mathfrak{g}$ maps $y \cdot W \subset W$ if $W \subset V$ is a $\mathfrak{g}$-subrepresentation and $\operatorname{tr}(y|_W) = 0$ since $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}]$ and the traces of (linear combinations of) commutators vanish. Conversely, $N \subset N_{\mathfrak{gl}(V)}(\mathfrak{g})$ now implies that N contains $\mathfrak{g}$ as ideal. By Lemma 4.3.2 this has a complement $\mathfrak{g}^\perp$ with $[\mathfrak{g}, \mathfrak{g}^\perp] = 0$. In particular, every $y \in \mathfrak{g}^\perp$ gives rise to an intertwiner $y|_W$ of any $\mathfrak{g}$-subrepresentation W of V. If W is simple, then $y|_W$ acts as a scalar by Schur's Lemma 4.2.1. But since $\operatorname{tr}(y|_W) = 0$ we deduce $y|_W = 0$. Now Theorem 4.2.5 implies that V is a direct sum of simple $\mathfrak{g}$-representations and so $y = 0$ and hence $\mathfrak{g}^\perp = 0$ and $N = \mathfrak{g}$.

Next we show the implication $x \in N_{\mathfrak{gl}(V)}(\mathfrak{g}) \Rightarrow x_s, x_n \in N_{\mathfrak{gl}(V)}(\mathfrak{g})$. In fact $x \in N_{\mathfrak{gl}(V)}(\mathfrak{g})$ is equivalent to $\operatorname{ad}_x(\mathfrak{g}) \subset \mathfrak{g}$, and then Lemma 3.3.4 and Remarks 3.3.3.(2) imply $(\operatorname{ad}_{x_s})(\mathfrak{g}) = (\operatorname{ad}_x)_s(\mathfrak{g}) \subset \mathfrak{g}$, which again is equivalent to $x_s \in N_{\mathfrak{gl}(V)}(\mathfrak{g})$. The argument for x_n is parallel.

Finally we need to prove $x \in SN_{\mathfrak{gl}(V)}(W) \Rightarrow x_s, x_n \in SN_{\mathfrak{gl}(V)}(W)$ for every $\mathfrak{g}$-subrepresentation $W \subset V$. Recall from Exercise 35 that x_s and x_n can be written as polynomials without constant term, evaluated on x. Thus $x \cdot W \subset W$ implies $x_s \cdot W \subset W$ and $x_n \cdot W \subset W$. Furthermore, by nilpotency $\operatorname{tr}(x_n|_W) = 0$. If $\operatorname{tr}(x|_W) = 0$, then also $\operatorname{tr}(x_s|_W) = \operatorname{tr}(x|_W) - \operatorname{tr}(x_n|_W) = 0$.

This finishes the proof since we have shown the implication $x \in N \Rightarrow x_s, x_n \in N$ and $N = \mathfrak{g}$. □

Proof of Theorem 4.3.1. (1) Consider the Jordan decomposition $\operatorname{ad}_x = (\operatorname{ad}_x)_s + (\operatorname{ad}_x)_n$ of ad_x in $\mathfrak{gl}(\mathfrak{g})$. By Lemma 4.3.3, applied to $\operatorname{ad}(\mathfrak{g}) \subset \mathfrak{gl}(\mathfrak{g})$, there exist $s, n \in \mathfrak{g}$, such that $\operatorname{ad}_s = (\operatorname{ad}_x)_s$ and $\operatorname{ad}_n = (\operatorname{ad}_x)_n$. Now $x = s + n$ since $x - s - n \in \ker(\operatorname{ad}) = Z(\mathfrak{g}) = 0$ as $\mathfrak{g}$ is semisimple. This shows the existence of an absolute Jordan decomposition. The uniqueness of the absolute Jordan decomposition is also a consequence of the faithfulness of the adjoint representation, because the concrete Jordan decomposition $\operatorname{ad}_x = \operatorname{ad}_s + \operatorname{ad}_n$ is unique.

(2) Let $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$ be a finite-dimensional representation. Consider the commutative diagram

$$\begin{tikzcd}
\mathfrak{g} \arrow[r, two heads] \arrow[d, "\operatorname{ad}_x^{\mathfrak{g}}"'] & \rho(\mathfrak{g}) \arrow[r, hook] \arrow[d, "\operatorname{ad}_{\rho(x)}^{\rho(\mathfrak{g})}"] & \mathfrak{gl}(V) \arrow[d, "\operatorname{ad}_{\rho(x)}^{\mathfrak{gl}(V)}"] \\
\mathfrak{g} \arrow[r, two heads] & \rho(\mathfrak{g}) \arrow[r, hook] & \mathfrak{gl}(V)
\end{tikzcd}$$

where we have placed superscripts to distinguish the different adjoint actions. By Remarks 3.3.3.(2) we obtain analogous commutative diagrams for the semisimple parts. By definition of s we have $(\mathrm{ad}^{\mathfrak{g}}_x)_s = \mathrm{ad}^{\mathfrak{g}}_s$ and Lemma 3.3.4 implies $(\mathrm{ad}^{\mathfrak{gl}(V)}_{\rho(x)})_s = \mathrm{ad}^{\mathfrak{gl}(V)}_{\rho(x)_s}$. The commutative diagram of semisimple parts thus takes the form

$$\begin{tikzcd}
\mathfrak{g} \arrow[r, two heads] \arrow[d, "\mathrm{ad}^{\mathfrak{g}}_s"'] & \rho(\mathfrak{g}) \arrow[r, hook] \arrow[d] & \mathfrak{gl}(V) \arrow[d, "\mathrm{ad}^{\mathfrak{gl}(V)}_{\rho(x)_s}"] \\
\mathfrak{g} \arrow[r, two heads] & \rho(\mathfrak{g}) \arrow[r, hook] & \mathfrak{gl}(V)
\end{tikzcd}$$

and the middle vertical arrow is determined as $\mathrm{ad}^{\rho(\mathfrak{g})}_{\rho(s)} = \mathrm{ad}^{\rho(\mathfrak{g})}_{\rho(x)_s}$. Since $\mathrm{ad}^{\rho(\mathfrak{g})} \colon \rho(\mathfrak{g}) \to \mathfrak{gl}(\rho(\mathfrak{g}))$ is injective, we deduce $\rho(s) = \rho(x)_s$. Similarly one proves $\rho(n) = \rho(x)_n$.

(3) Let $\phi \colon \mathfrak{g} \to \mathfrak{g}'$ be a morphism between semisimple Lie algebras and let $x \in \mathfrak{g}$ with absolute Jordan decomposition $x = s + n$. Consider the adjoint representation $\mathrm{ad}^{\mathfrak{g}'} \colon \mathfrak{g}' \to \mathfrak{gl}(\mathfrak{g}')$ and apply (2) to $\rho := \mathrm{ad}^{\mathfrak{g}'} \circ \phi$. This implies that $\mathrm{ad}^{\mathfrak{g}'}(\phi(s))$ is semisimple and $\mathrm{ad}^{\mathfrak{g}'}(\phi(n))$ is nilpotent. Clearly the other conditions $\phi(x) = \phi(s) + \phi(n)$ and $[\phi(s), \phi(n)] = 0$ for the absolute Jordan decomposition of $\phi(x)$ in $\mathfrak{g}'$ are also satisfied by $\phi(s)$ and $\phi(n)$. □

Definition 4.3.4 Let $\mathfrak{g}$ be a Lie algebra. An element $x \in \mathfrak{g}$ is **ad-semisimple** resp. **ad-nilpotent** if $\mathrm{ad}_x \in \mathrm{End}(\mathfrak{g})$ is semisimple resp. nilpotent. If $\mathfrak{g}$ is semisimple, then these get abbreviated to **semisimple** resp. **nilpotent**. For the Jordan decomposition $x = s + n$ in a semisimple Lie algebra, s is called the **semisimple part** and n the **nilpotent part** of x. L14 End

4.4 Root space decomposition

Lemma 4.4.1 Let V be a vector space and $T \subset \mathrm{End}(V)$ a finite-dimensional subspace of diagonalizable and pairwise commuting endomorphisms. Then V decomposes into simultaneous eigenspaces:

$$V = \bigoplus_{\lambda \in T^*} V_\lambda \quad \text{where } V_\lambda = \{v \in V \mid x(v) = \lambda(x)v \text{ for all } x \in T\}$$

Proof. The proof proceeds by induction on $\dim(T) = n$, with the case $n = 0$ being vacuous. For $n \geq 1$, let $\{x_1, \ldots, x_n\}$ be a basis of T. Since x_1 is diagonalizable, V decomposes into eigenspaces for x_1. Since $x_2, \ldots, x_n$ commute with x_1, they stabilize these eigenspaces, which we then decompose using the induction hypothesis. □

In the situation of the lemma, the set $P(V) := \{\lambda \in T^* \mid V_\lambda \neq 0\} \subset T^*$ is called the set of **weights** of V and V_λ is called the **weight space** for λ.

Example 4.4.2 For $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{F})$ consider the abelian Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ of diagonal matrices. Then the image of $\mathfrak{h} \subset \mathfrak{g}$ under $\mathrm{ad}^{\mathfrak{g}}$ is an abelian Lie subalgebra of $\mathfrak{gl}(\mathfrak{g})$, i.e. a subspace of pairwise commuting endomorphisms of $\mathfrak{g}$, which are also diagonalizable (compare with the proof of Lemma 3.3.4). To see this concretely, let $E_{i,j} \in \mathfrak{g}$ denote the standard matrix with a single nonzero entry 1 in the ith row and jth column. Further, let $h = \mathrm{diag}(h_1, \ldots, h_n) \in \mathfrak{h}$ be the diagonal matrix with entries $h_1, \ldots, h_n \in \mathbb{F}$. Then we compute:

$$\mathrm{ad}_h(E_{i,j}) = [h, E_{i,j}] = (h_i - h_j)E_{i,j}.$$

So ad_h is diagonal in the standard basis for $\mathfrak{g}$. Lemma 4.4.1 now yields a decomposition:

$$\mathfrak{g} = \bigoplus_{\lambda \in \mathfrak{h}^*} \mathfrak{g}_\lambda \quad \text{where } \mathfrak{g}_\lambda = \{x \in \mathfrak{g} \mid [h, x] = \lambda(h)x \text{ for all } h \in \mathfrak{h}\}$$

Let $\epsilon_i \in \mathfrak{h}^*$ denote the linear form which extracts the ith diagonal entry from a matrix in $\mathfrak{h}$. Then we have

$$[h, E_{i,j}] = ((\epsilon_i - \epsilon_j)(h))E_{i,j}$$

and so $P(\mathfrak{g}) = \{\epsilon_i - \epsilon_j \mid 1 \leq i, j \leq n\}$ is the set of weights of $\mathfrak{g}$. As weight spaces we have $\mathfrak{g}_0 = \mathfrak{h}$ and for $i \neq j$ we have $\mathfrak{g}_{\epsilon_i - \epsilon_j} = \mathbb{F}E_{i,j}$ if $\mathrm{char}(\mathbb{F}) \neq 2$ and $\mathfrak{g}_{\epsilon_i - \epsilon_j} = \mathbb{F}E_{i,j} \oplus \mathbb{F}E_{j,i}$ if $\mathrm{char}(\mathbb{F}) = 2$.

Definition 4.4.3 A Lie subalgebra $\mathfrak{h} \subset \mathfrak{g}$ of a semisimple Lie algebra over $\mathbb{C}$ is a **Cartan subalgebra** if it is abelian, contains only semisimple elements of $\mathfrak{g}$, and is maximal under inclusion of Lie subalgebras with respect to the first two properties.

Example 4.4.4 In $\mathfrak{sl}(n,\mathbb{C})$ the diagonal matrices of trace zero form a Cartan subalgebra.

Remarks 4.4.5 (1) The action of elements of a Cartan subalgebra act by diagonalizable matrices in any representation, see Theorem 4.3.1. If the elements are observables of a quantum theory, then a Cartan subalgebra chooses a maximal number of commuting observables, for which one can find eigenbases in every representation.

(2) In general Lie algebras one defines the Cartan subalgebras to be the nilpotent, self-normalizing subalgebras. One can show that this agrees with the definition given here in the complex semisimple case.

(3) In Definition 4.4.3 we required a Cartan algebra to be abelian. This is superfluous, as it already follows from the other conditions. To see this, suppose we have a Lie algebra $\mathfrak{h}$ whose elements are ad-semisimple, but which is not abelian. In this case, we could find $x \in \mathfrak{h}$ with $\mathrm{ad}_x \neq 0$, so there would be a $y \in \mathfrak{h}$ and $\lambda \in \mathbb{F}$ with $\mathrm{ad}_x(y) = \lambda y \neq 0$. Then:

$$\mathrm{ad}_y(x) = [y,x] = -[x,y] = -\mathrm{ad}_x(y) = -\lambda y \neq 0, \quad (\mathrm{ad}_y)^2(x) = -\lambda[y,y] = 0$$

But then y would not be ad-semisimple, a contradiction.

Using Cartan subalgebras, we now obtain decompositions analogous to those from Example 4.4.2 for $\mathfrak{gl}(n,\mathbb{F})$.

Definition 4.4.6 (Root space decomposition) Let $\mathfrak{g}$ be a complex semisimple Lie algebra and $\mathfrak{h} \subset \mathfrak{g}$ a Cartan subalgebra. The pairing of $\lambda \in \mathfrak{h}^*$ with $h \in \mathfrak{h}$ will be denoted $\langle \lambda, h\rangle := \lambda(h)$. By Lemma 4.4.1 we obtain a decomposition:

$$\mathfrak{g} = \bigoplus_{\lambda \in \mathfrak{h}^*} \mathfrak{g}_\lambda \quad \text{where } \mathfrak{g}_\lambda = \{x \in \mathfrak{g} \mid [h,x] = \langle \lambda, h\rangle x \ \text{ for all } h \in \mathfrak{h}\}$$

and if we set $R(\mathfrak{g},\mathfrak{h}) := \{\alpha \in \mathfrak{h}^* \mid \alpha \neq 0 \text{ and } \mathfrak{g}_\alpha \neq 0\} = P(\mathfrak{g}) \setminus \{0\}$, then this is written as:

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in R(\mathfrak{g},\mathfrak{h})} \mathfrak{g}_\alpha$$

The finite set $R(\mathfrak{g},\mathfrak{h}) \subset \mathfrak{h}^*$ is called the **root system** of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Its elements are the **roots** and the simultaneous eigenspaces $\mathfrak{g}_\alpha$ are the **root spaces**. Note that $\mathfrak{g}_0 = C_{\mathfrak{g}}(\mathfrak{h})$ is the centralizer of the Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{g}$.

Remark 4.4.7 A word of warning. The root space decompositions from Definition 4.4.6 give isomorphisms of vector spaces and, more specifically, representations of $\mathrm{ad}^{\mathfrak{g}}(\mathfrak{h}) \subset \mathfrak{gl}(\mathfrak{g})$, but not (!) isomorphisms of Lie algebras. Root spaces $\mathfrak{g}_\alpha$ are not ideals in $\mathfrak{g}$.

Example 4.4.8 Let $\mathfrak{g} = \mathfrak{sl}(n,\mathbb{C})$ and denote by $\mathfrak{h} \subset \mathfrak{g}$ the Lie subalgebra of diagonal matrices of trace zero. As before $\epsilon_i\colon \mathfrak{h} \to \mathbb{C}$ is the linear form that extracts the ith entry of a diagonal matrix. We have $R(\mathfrak{g},\mathfrak{h}) = \{\epsilon_i - \epsilon_j \mid i \neq j\}$, $\mathfrak{g}_0 = \mathfrak{h}$, and $\mathfrak{g}_\alpha = \mathbb{C}E_{i,j}$ for $\alpha = \epsilon_i - \epsilon_j$. Note that the ϵ_i are not linearly dependent in $\mathfrak{h}^*$, since $\dim(\mathfrak{h}) = n-1$.

Theorem 4.4.9 [on root space decompositions] Let $\mathfrak{g}$ be a complex semisimple Lie algebra with Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and root system $R := R(\mathfrak{g},\mathfrak{h}) \subset \mathfrak{h}^*$. For any $\lambda \in \mathfrak{h}^*$, recall that $\mathfrak{g}_\lambda$ denotes the corresponding weight space. Then we have:

(1) The Cartan subalgebra is its own centralizer $\mathfrak{h} = \mathfrak{g}_0 = C_{\mathfrak{g}}(\mathfrak{h})$.

(2) For every $\alpha \in R$, the root space $\mathfrak{g}_\alpha$ is 1-dimensional and there exists an injective morphism of Lie algebras $\mathfrak{sl}(2,\mathbb{C}) \hookrightarrow \mathfrak{g}$ mapping

$$\mathbb{C}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \xrightarrow{\cong} \mathfrak{g}_\alpha, \quad \mathbb{C}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \xrightarrow{\cong} \mathfrak{g}_{-\alpha}, \quad \mathbb{C}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \xrightarrow{\cong} [\mathfrak{g}_\alpha, -\mathfrak{g}_\alpha] \subset \mathfrak{h}$$

(3) The negative of a root is again a root, but not other scalar multiples of a root are roots. In formulas, if $\alpha \in R$ then $R \cap \mathbb{C}\alpha = \{\alpha, -\alpha\}$.

(4) If $\alpha, \beta \in R$ such that $\alpha + \beta \in R$, then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$.

The proof is spread over the following lemmas. We retain the notation throughout.

Lemma 4.4.10 (1) For $\lambda, \mu \in \mathfrak{h}^*$ we have $[\mathfrak{g}_\lambda, \mathfrak{g}_\mu] \subset \mathfrak{g}_{\lambda+\mu}$.

(2) The Killing form κ of $\mathfrak{g}$ satisfies $\kappa(\mathfrak{g}_\lambda, \mathfrak{g}_\mu) = 0$ if $\lambda \neq -\mu$.

(3) The restriction of the Killing form κ to $\mathfrak{g}_0$ is nondegenerate.

Proof. (1) If $x \in \mathfrak{g}_\lambda$ and $y \in \mathfrak{g}_\mu$ this means $[h, x] = \lambda(h)x$ and $[h, y] = \mu(h)y$ for all $h \in \mathfrak{h}$. Then the Jacobi identity and antisymmetry imply

$$[h, [x, y]] = [[h, x], y] + [x, [h, y]] = (\lambda(h) + \mu(h))[x, y] \quad \text{for all } h \in \mathfrak{h}$$

and thus $[x, y] \in \mathfrak{g}_{\lambda+\mu}$.

(2) With $\nu \in \mathfrak{h}^*$ part (1) further implies $(\mathrm{ad}_x \circ \mathrm{ad}_y)(\mathfrak{g}_\nu) \subset \mathfrak{g}_{\nu+\lambda+\mu}$. If $\lambda + \mu \neq 0$, then $\mathrm{ad}_x \circ \mathrm{ad}_y$ is nilpotent (already because only finitely weight spaces of $\mathfrak{g}$ are nonzero) and thus $\kappa(x, y) = \mathrm{tr}(\mathrm{ad}_x \circ \mathrm{ad}_y) = 0$.

(3) For $z \in \mathfrak{g}_0$ we have $\kappa(z, \mathfrak{g}_\alpha) = 0$ for all $\alpha \in R$ by (2). If additionally $\kappa(z, \mathfrak{g}_0) = 0$, then $\kappa(z, \mathfrak{g}) = 0$, and thus $z = 0$ by the nondegeneracy of κ on $\mathfrak{g}$, which holds since $\mathfrak{g}$ is semisimple, see Theorem 4.1.3. Thus κ is also nondegenerate when restricted to $\mathfrak{g}_0$. □ L15

End

Proof of Theorem 4.4.9.(1). We immediately have $\mathfrak{h} \subset \mathfrak{g}_0$ and need to show the opposite inclusion.

For $x \in \mathfrak{g}_0$ note that ad_x vanishes on $\mathfrak{h}$. Let $x = s + n$ be the Jordan decomposition, then $\mathrm{ad}_s = (\mathrm{ad}_x)_s$ and $\mathrm{ad}_n = (\mathrm{ad}_x)_n$ vanish on $\mathfrak{h}$ by Remarks 3.3.3.(2), and so $s, n \in \mathfrak{g}_0$ by definition of $\mathfrak{g}_0$. Now s is semisimple and $[s, \mathfrak{h}] = 0$, so by maximality of $\mathfrak{h}$ we must have $s \in \mathfrak{h}$. ($\mathbb{C}\,\mathrm{ad}_s + \mathrm{ad}(\mathfrak{h})$ is a subspace of commuting diagonalizable endomorphisms of $\mathfrak{g}$, thus simulatenously diagonalizable. So $\mathbb{C}s + \mathfrak{h}$ is an abelian subalgebra of semisimple elements, which must be $\mathfrak{h}$ by maximality.)

Continuing with $x = s + n$ from above, $s \in \mathfrak{h}$ implies $\mathrm{ad}_s^{\mathfrak{g}_0} = 0$ by definition of $\mathfrak{g}_0$, thus $\mathrm{ad}_x^{\mathfrak{g}_0} = \mathrm{ad}_n^{\mathfrak{g}_0}$ is nilpotent. As $x \in \mathfrak{g}_0$ was arbitrary, all elements of $\mathfrak{g}_0$ are ad-nilpotent and so Engel's Theorem 3.2.8 implies that $\mathfrak{g}_0$ is a nilpotent Lie algebra and, in particular, solvable. Now we apply Corollary 3.2.12 to the representation $\mathrm{ad}^{\mathfrak{g}}|_{\mathfrak{g}_0} : \mathfrak{g}_0 \to \mathfrak{gl}(\mathfrak{g})$ to find a basis of $\mathfrak{g}$, such that ad_x for $x \in \mathfrak{g}_0$ are represented by upper triangular matrices. If $z \in \mathfrak{g}_0$ is ad-nilpotent, then ad_z would be represented by a strictly upper triangular matrix, and thus $\kappa(z, \mathfrak{g}_0) = 0$ and finally $z = 0$ by Lemma 4.4.10. We conclude that $\mathfrak{g}_0$ contains only ad-semisimple elements, but these are all contained in $\mathfrak{h}$, as observed in the first part of the proof. □

Lemma 4.4.11 For every root $\alpha \in R$, we have $\dim([\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]) = 1$ and α does not vanish on the line $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subset \mathfrak{h}$.

Proof. For $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_{-\alpha}$, and $h \in \mathfrak{h}$ we compute

$$\kappa(h, [x, y]) = \kappa([h, x], y]) = \alpha(h)\kappa(x, y) \tag{4}$$

and so $\ker(\alpha) \subset [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]^\perp$, where the orthogonal complement is taken with respect to the restriction of the Killing form κ to $\mathfrak{h} = \mathfrak{g}_0$. Since this is nondegenerate by Lemma 4.4.10, $\dim(\ker(\alpha)) = \dim(\mathfrak{g}_0) - 1$ implies $\dim([\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]) \leq 1$. Equality holds if we can find $x \in \mathfrak{g}_\alpha$, $y \in \mathfrak{g}_{-\alpha}$ with $[x, y] \neq 0$. As the Killing form of $\mathfrak{g}$ in non-degenerate, for a given $0 \neq x \in \mathfrak{g}_\alpha$ we can find $y \in \mathfrak{g}$ such that $\kappa(x, y) \neq 0$. By Lemma 4.4.10, we must have $y \in \mathfrak{g}_{-\alpha}$. Now we choose $h \in \mathfrak{h}$ such that $\alpha(h) \neq 0$, Then the right-hand side of (4) is non-zero, so also the left-hand side, which implies $[x, y] \neq 0$ and hence $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ is 1-dimensional. We still need to check that α does not vanish on this space. Set $h := [x, y] \neq 0$ and suppose that $\alpha(h) = 0$. Then we have

$$[h, x] = \alpha(h)x = 0, \quad [h, y] = -\alpha(h)y = 0$$

which implies that the 3-dimensional Lie subalgebra of $\mathfrak{g}$ spanned by x, y, h is nilponent and thus solvable. In a suitable basis, $\mathrm{ad}_x^{\mathfrak{g}}$, $\mathrm{ad}_y^{\mathfrak{g}}$ and $\mathrm{ad}_h^{\mathfrak{g}}$ would be given by upper triangular matrices. Moreover, $\mathrm{ad}_h^{\mathfrak{g}} = \mathrm{ad}_{[x,y]}^{\mathfrak{g}} = [\mathrm{ad}_x^{\mathfrak{g}}, \mathrm{ad}_y^{\mathfrak{g}}]$ would be strictly upper triangular, hence nilpotent. Now $h \in \mathfrak{h}$ is nonzero and a nilpotent element of $\mathfrak{g}$, in contradiction to the requirement that the Cartan subalgebra consists of semisimple elements of $\mathfrak{g}$.

□

Definition 4.4.12 Let $\mathfrak{g}$ be a complex semisimple Lie algebra with Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$. For every root $\alpha \in R(\mathfrak{g}, \mathfrak{h})$ we define an element $\alpha^\vee \in \mathfrak{h}$ by the two conditions $\alpha^\vee \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}]$ and $\langle \alpha, \alpha^\vee \rangle = 2$.

Proof of Theorem 4.4.9.(2-3). Note that $(-\alpha)^\vee = -\alpha^\vee$ directly from the definition. Now we can find $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_{-\alpha}$ such that $[x, y] = \alpha^\vee$. Then we also have:

$$[\alpha^\vee, x] = \alpha(\alpha^\vee)x = 2x, \quad [\alpha^\vee, y] = (-\alpha)(\alpha^\vee)y = -2y$$

Thus $x, \alpha^\vee, y$ span a Lie subalgebra $\mathfrak{g}^\alpha \subset \mathfrak{g}$, which is isomorphic to $\mathfrak{sl}(2,\mathbb{C})$ via

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mapsto x, \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \mapsto y, \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \mapsto \alpha^\vee$$

Next we consider only roots α such that $\alpha/2$ is not a root. (If there is a root, then there is at least such a special root, since R is finite.) The subalgebra g^α acts on $\mathfrak{g}$ by the restriction of $\mathrm{ad}^{\mathfrak{g}}$. This contains the subrepresentation $U := \mathbb{C}\alpha^\vee \oplus \bigoplus_{t\neq 0} \mathfrak{g}_{t\alpha}$, which contains $\mathfrak{g}^\alpha$ and, by Weyl's theorem, its complement that we denote by V. Suppose $V \neq 0$, then the element $\alpha^\vee$ acts on V by an invertible map since $V \subset \bigoplus_{t\neq 0} \mathfrak{g}_{t\alpha}$. By our classification result Theorem 2.1.2, we deduce that V must be isomorphic to a direct sum of simple $\mathfrak{sl}(2,\mathbb{C})$-representations $L(m)$ with all m odd (for even m, we would have a non-trivial weight space $L(m)_0$ on which $\alpha^\vee = h$ does not act invertibly). In particular, if $V \neq 0$, then the weight space $V_1 \neq 0$. In other words, the eigenspace in V for $\alpha^\vee$ and the eigenvalue 1 is nonzero; thus $\mathfrak{g}_{\alpha/2} \neq 0$, a contradiction to our assumption. Thus we have $V = 0$ and (2) and (3) follow for our special choice of α. But by (3), this actually covers all roots. □

Definition 4.4.13 Let V be a vector space over $\mathbb{F}$ of $\mathrm{char}(\mathbb{F}) = 0$. A subset $R \subset V$ is an **abstract root system** if the following conditions are satisfied:

(1) The set R is finite, spans V, and $0 \notin R$.

(2) For every $\alpha \in R$ there exists a linear map $s_\alpha \colon V \to V$ satisfying:

- $s_\alpha(\alpha) = -\alpha$,
- $s_\alpha(R) \subset R$,
- $s_\alpha(\beta) - \beta \in \mathbb{Z}\alpha$ for all $\beta \in R$.

In this case the elements $\alpha \in R$ are called **roots**. An abstract root system is **reduced** if $\mathbb{F}\alpha \cap R = \{\alpha, -\alpha\}$, i.e. if R contains no nontrivial multiples of roots, except negatives.

The maps s_α are in fact uniquely determined by the required properties. To see this, let $t\colon V \to V$ denote another candidate. Then $(ts_\alpha - \mathrm{id})(\beta) = \big(t(s_\alpha(\beta)) - s_\alpha(\beta)\big) + \big(s_\alpha(\beta) - \beta\big) \in \mathbb{F}\alpha$, and so $\mathrm{im}(ts_\alpha - \mathrm{id}) \subset \mathbb{F}\alpha$ since R spans V. This implies $(ts_\alpha - \mathrm{id})^2 = 0$ and ts_α is invertible. For $n \geq 1$ we have $(ts_\alpha)^n = (\mathrm{id} + (ts_\alpha - \mathrm{id}))^n = \mathrm{id} + n(ts_\alpha - \mathrm{id})$ by the binomial theorem and ts_α has finite order, since it permutes the roots, so $ts_\alpha = \mathrm{id}$. In particular, we deduce $s_\alpha^2 = \mathrm{id}$ and thus $s_\alpha = t$. L16 End

Definition 4.4.14 We call s_α the **reflection** associated to α. The fixed point set of s_α is called the **reflecting hyperplane**, it can also be described as the kernel of the **coroot** $\alpha^\vee \in V^*$ for α, which is determined the equation $s_\alpha(v) = v - \langle v, \alpha^\vee\rangle\alpha$ for all $v \in V$ (here we write $\langle v, \alpha^\vee\rangle := \alpha^\vee(v)$). The subgroup $W \subset \mathrm{GL}(V)$ generated by the reflections s_α for $\alpha \in R$ is called the **Weyl group** of the root system.

Remark 4.4.15 We record two important observations: First, for roots $\alpha, \beta \in R$ we have $\langle \beta, \alpha^\vee\rangle \in \mathbb{Z}$ by the third condition in Definition 4.4.13.(2). Second, the Weyl group W is finite since its elements permute the finite spanning set R.

Theorem 4.4.16 Let $R \subset V$ be an abstract (reduced) root system.

(1) The set $R^\vee = \{\alpha^\vee \mid \alpha \in R\}$ forms an abstract (reduced) root system in V^* and the canonical isomorphism $V \to V^{**}$ maps $\alpha \mapsto (\alpha^\vee)^\vee$.

(2) If $\alpha_1, \ldots, \alpha_n \in R$ form a basis of V and $\beta \in R$, then $\beta \in \mathrm{span}_{\mathbb{Q}}\{\alpha_1, \ldots, \alpha_n\}$.

Proof. Suppose first that $\mathbb{F} = \mathbb{Q}$ and let W be the Weyl group of R. Next we construct a W-invariant inner product on V by first choosing an arbitrary inner product b and then averaging over W. Namely, for $v, w \in V$ we define:

$$(v, w) := \sum_{g \in W} b(g(v), g(w))$$

This inner product is W-invariant in the sense that it satisfies $(g(v), g(w)) = (v, w)$ for every $g \in W$. It follows that the reflecting hyperplane for s_α is perpendicular to α. To see this, suppose $s_\alpha(v) = v$, then $(\alpha, v) = (\alpha, s_\alpha(v)) = (s_\alpha(\alpha), v) = -(\alpha, v)$ and so $(\alpha, v) = 0$. In particular, the reflection s_α can now be expressed by the usual formula for orthogonal reflections:

$$s_\alpha(v) = v - 2\frac{(v,\alpha)}{(\alpha,\alpha)}\alpha \tag{5}$$

For all $v \in V$ we now have

$$\langle v, \alpha^\vee \rangle = \frac{2(v,\alpha)}{(\alpha,\alpha)} \tag{6}$$

since both linear forms (in v) agree on α and its orthogonal complement. The inner product defines the isomorphism

$$V \xrightarrow{\cong} V^*, \quad v \mapsto (-, v)$$

under which $2\alpha/(\alpha,\alpha) \mapsto \alpha^\vee$. This shows that V^* is generated by the coroots $R^\vee$. We also need to provide reflections and verify the properties from Definition 4.4.13.(2). We define the reflection $s_{\alpha^\vee} \colon V^* \to V^*$ by conjugating s_α by the above isomorphism. For $\beta \in R$ we have:

$$\begin{tikzcd}
\beta \arrow[r, mapsto] \arrow[d, mapsto, "s_\alpha"'] & \frac{(\beta,\beta)}{2}\beta^\vee \arrow[d, mapsto, "s_{\alpha^\vee}"] \\
s_\alpha(\beta) \arrow[r, mapsto] & \frac{(s_\alpha(\beta), s_\alpha(\beta))}{2}(s_\alpha(\beta))^\vee
\end{tikzcd}$$

and since $(\beta,\beta) = (s_\alpha(\beta), s_\alpha(\beta))$, we see that $s_{\alpha^\vee}$ must send $\beta^\vee \mapsto (s_\alpha(\beta))^\vee$. This immediately implies $s_{\alpha^\vee}(\alpha^\vee) = -\alpha^\vee$ and $s_{\alpha^\vee}(R^\vee) \subset R^\vee$. Moreover, we claim $s_{\alpha^\vee}(\beta^\vee) - \beta^\vee = -\langle \alpha, \beta^\vee \rangle \alpha^\vee \in \mathbb{Z}\alpha^\vee$. To see this, we pull back along the isomorphism and use (6) to obtain the equivalent equation:

$$\frac{2s_\alpha(\beta)}{(s_\alpha(\beta), s_\alpha(\beta))} - \frac{2\beta}{(\beta,\beta)} = -\frac{2(\alpha,\beta)}{(\beta,\beta)}\frac{2\alpha}{(\alpha,\alpha)}$$

This holds because of $(\beta,\beta) = (s_\alpha(\beta), s_\alpha(\beta))$ and (5). Thus we have shown (1) for $\mathbb{F} = \mathbb{Q}$.

Now let $\mathbb{F}$ be arbitrary (of $\operatorname{char}(\mathbb{F}) = 0$) and let $V_\mathbb{Q} := \operatorname{span}_\mathbb{Q} R$. Then $R \subset V_\mathbb{Q}$ is an abstract root system and the coroot $\alpha^\vee_\mathbb{Q} \in V^*_\mathbb{Q}$ is the restriction to $V_\mathbb{Q}$ of $\alpha^\vee \in V^*$. (This linear form is already determined by its (integral!) values on a generating set R of V, and thus by its values on $V_\mathbb{Q}$.) Now we have $\dim_\mathbb{Q}(V_\mathbb{Q}) \geq \dim_\mathbb{F}(V)$ since every $\mathbb{Q}$-basis of $V_\mathbb{Q}$ still spans V over $\mathbb{F}$. Now let $\{\alpha_1, \ldots, \alpha_n\} \subset R$ be a $\mathbb{Q}$-basis of $V_\mathbb{Q}$ and $\{\beta_1^\vee, \ldots, \beta_n^\vee\}$ a $\mathbb{Q}$-basis of $V^*_\mathbb{Q}$ consisting of (restrictions of) coroots.

The matrix with entries $\langle \alpha_i, \beta_j^\vee \rangle \in \mathbb{Z}$ for $1 \leq i, j \leq n$ is non-singular, hence invertible over $\mathbb{Q}$ and then also over $\mathbb{F}$. Thus the families $\{\alpha_1, \ldots, \alpha_n\}$ and $\{\beta_1^\vee, \ldots, \beta_n^\vee\}$ are both linearly independent over $\mathbb{F}$ as well and we deduce $\dim_\mathbb{Q}(V_\mathbb{Q}) = \dim_\mathbb{F}(V)$. In particular $\operatorname{span}_\mathbb{F}\{\beta_1^\vee, \ldots, \beta_n^\vee\} = V^*$, which completes the proof of (1). Finally, any $\mathbb{F}$-basis of V consisting of $\{\alpha_1, \ldots, \alpha_n\} \in R$ still spans over $\mathbb{Q}$ and hence is a basis of $V_\mathbb{Q}$. In particular, every $\beta \in R \subset V_\mathbb{Q}$ can then be written as a $\mathbb{Q}$-linear combination of the α_i. □

Remark 4.4.17 The zero vector is the only vector in V that is fixed by the Weyl group of a root system in V. To see this, note that such a vector must be fixed by all reflections, hence must lie in the intersection of all reflecting hyperplanes. However, since the coroots $R^\vee$ span V^*, the intersection of their kernels is the zero vector space.

Theorem 4.4.18 [root systems of a semisimple Lie algebra] Let $\mathfrak{g}$ be a complex semisimple Lie algebra with Cartan subalgebra $\mathfrak{h}$. Then $R(\mathfrak{g}, \mathfrak{h}) \subset \mathfrak{h}^*$ is a reduced abstract root system in the sense of Definition 4.4.13 and for every $\alpha \in R(\mathfrak{g}, \mathfrak{h})$, the $\alpha^\vee \in \mathfrak{h}$ from Definition 4.4.12 maps to the coroot $\alpha^\vee \in \mathfrak{h}^{**}$ from Definition 4.4.14 under the canonical isomorphism $\mathfrak{h} \to \mathfrak{h}^{**}$.

Proof. We first show that $R := R(\mathfrak{g}, \mathfrak{h})$ spans $\mathfrak{h}^*$. This follows if we prove $\bigcap_{\alpha \in R} \ker(\alpha) = 0$. So, given $h \in \mathfrak{h}$ with $\alpha(h) = 0$ for all $\alpha \in R$, we get $[h, \mathfrak{g}_\alpha] = 0$ for all $\alpha \in R$. We also have $[h, \mathfrak{h}] = 0$, and thus $h \in Z(\mathfrak{g}) = 0$ since $\mathfrak{g}$ is semisimple.

Next we fix $\alpha \in R$ and a linearly independent root $\beta \neq \pm\alpha$. Consider the subspace $T = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_{\beta + i\alpha} \subset \mathfrak{g}$ and note that this provides a $\mathfrak{g}^\alpha$-subrepresentation of $\mathfrak{g}$, the $\mathfrak{g}_{\beta+i\alpha}$ are eigenspaces of $\alpha^\vee$ with eigenvalue $\langle \beta, \alpha^\vee \rangle + 2i$, and they are at most 1-dimensional by Theorem 4.4.9.(2). By our classification of finite-dimensional $\mathfrak{sl}(2, \mathbb{C})$-representations (Consequences 2.1.5) we know that the eigenvalues of $\alpha^\vee$ on T must be integers and symmetric about zero. Thus $\langle \beta, \alpha^\vee \rangle \in \mathbb{Z}$ and $-\langle \beta, \alpha^\vee \rangle$ is another eigenvalue. Now set $s_\alpha(\lambda) := \lambda - \langle \lambda, \alpha^\vee \rangle \alpha$, then all requirements of an abstract root system are satisfied and $\alpha^\vee$ corresponds to the coroot of α under $\mathfrak{h} \cong \mathfrak{h}^{**}$. We already know that the root system is reduced from Theorem 4.4.9.(3). □ L17
End

Now we are ready to complete the proof of Theorem 4.4.9.

Proof of Theorem 4.4.9.(4). Retain notation from the proof of Theorem 4.4.18. Since all eigenspaces of $\alpha^\vee$ in T are 1-dimensional and since all eigenvalues are even or odd, Consequences 2.1.5.(2) implies that T must be a simple representation of $\mathfrak{g}^\alpha \cong \mathfrak{sl}(2, \mathbb{C})$. From the explicit description of such simple representations in Theorem 2.1.2 it follows that $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$ if $\alpha, \beta, \alpha + \beta \in R$. □

Proposition 4.4.19 In the setting of Theorem 4.4.18 we further have:

(1) The coroots $\alpha^\vee$ for $\alpha \in R$ span the Cartan subalgebra $\mathfrak{h}$.

(2) Consider the vector space
$$\mathfrak{h}_{\mathbb{Q}} = \mathrm{span}_{\mathbb{Q}}\{\alpha^\vee \mid \alpha \in R\}$$
over $\mathbb{Q}$. Then on $t_1, t_2 \in \mathfrak{h}_{\mathbb{Q}}$ the Killing form takes rational values $\kappa_{\mathfrak{g}}(t_1, t_2) \in \mathbb{Q}$. Moreover $\kappa|_{\mathfrak{h}_{\mathbb{Q}}}$ is positive definite. The analogous results also hold for $\mathbb{R}$ in place of $\mathbb{Q}$.

Proof. (1) Let $h \in \mathfrak{h}$ be in orthogonal complement of $\mathrm{span}_{\mathbb{C}}\{\alpha^\vee \mid \alpha \in R\}$ with respect to the Killing form. Then for $\alpha \in R$ and standard generators $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_{-\alpha}$ for $\mathfrak{g}^\alpha$ with $[x, y] = \alpha^\vee$ we have:
$$0 = \kappa(h, \alpha^\vee) = \kappa(h, [x, y]) = \kappa([h, x], y) = \alpha(h)\kappa(x, y).$$
Since κ is non-degenerate, we must have $\kappa(x, y) \neq 0$ by Lemma 4.4.10.(2), and thus $\alpha(h) = 0$. This holds for all $\alpha \in R$, so $h \in Z(\mathfrak{g}) = 0$.

(2) For $h, h' \in \mathfrak{h}$, we have $\kappa_{\mathfrak{g}}(h, h') = \mathrm{tr}(\mathrm{ad}_h \circ \mathrm{ad}_{h'}) = \sum_{\alpha \in R} \alpha(h)\alpha(h')$ which implies the claims. □

This result, together with Theorem 4.4.16 motivate the study of root systems over $\mathbb{Q}$ and, after extending to $\mathbb{R}$, in Euclidean vector spaces. We will do so in Section 4.5. Before moving on, we take a look ahead.

Definition 4.4.20 A **morphism of root systems** over a fixed field $\mathbb{F}$ is a linear map between the associated vector spaces that maps every root to a root or to zero. An isomorphism of root systems is thus an isomorphism of the associated vector spaces that maps roots to roots.

Theorem 4.4.21 [Classification of complex semisimple Lie algebras] Given a complex semisimple Lie algebra $\mathfrak{g}$, then the choice of a Cartan subalgebra $\mathfrak{h} \in \mathfrak{g}$ determines a complex root system $R(\mathfrak{g}, \mathfrak{h}) \subset \mathfrak{h}^*$. This map
$$\begin{array}{ccc} \left\{\begin{matrix}\text{complex semisimple}\\ \text{Lie algebras}\end{matrix}\right\} & \to & \left\{\begin{matrix}\text{complex reduced}\\ \text{root systems}\end{matrix}\right\} \\ \mathfrak{g} & \mapsto & R(\mathfrak{g}, \mathfrak{h}) \subset \mathfrak{h}^* \end{array}$$
induces a bijection between isomorphism classes of complex semisimple Lie algebras and isomorphism classes of complex reduced root systems.

We will discuss selected aspects of the proof of this theorem in the following subsections.

Exercise 39 Let $R \subset V$ be an abstract root system. For $\alpha, \beta \in R$ show that $s_\alpha s_\beta s_\alpha = s_{s_\alpha(\beta)}$ and then deduce $w s_\beta w^{-1} = s_{w(\beta)}$ for all $w \in W$.

Exercise 40 (root system of type D_n) Let $\mathfrak{g} = \mathfrak{so}(2n, \mathbb{C})$ as presented in Example 1.2.9 and consider the Cartan subalgebra $\mathfrak{h}$ of diagonal matrices of the form $\mathrm{diag}(h_1, \ldots, h_n, -h_1, \ldots, -h_n)$. Let $\epsilon_i : \mathfrak{h} \to \mathbb{C}$ denote the linear form extracting the ith diagonal entry. Compute the root system $R(\mathfrak{g}, \mathfrak{h}) \subset \mathfrak{h}^*$ in terms of the ϵ_i and describe the associated Weyl group. How does it compare to the Weyl group of $\mathfrak{sl}(2n, \mathbb{C})$?

End W.10 Exerc.

4.5 Finite reflection groups

Definition 4.5.1 Let V be a finite-dimensional vector space over a field $\mathbb{F}$ of characteristic $\mathrm{char}(\mathbb{F}) \neq 2$.

(1) A **reflection** is a linear map $s : V \to V$ whose set of fixed points
$$V^s := \{v \in V \mid s(v) = v\}$$
is a hyperplane, i.e. $\dim_{\mathbb{F}}(V^s) = \dim_{\mathbb{F}}(V) - 1$, and $s^2 = 1$. The hyperplane V^s is called the **reflecting hyperplane** of s.

(2) If furthermore $\mathbb{F}$ is ordered (and hence $\mathrm{char}(\mathbb{F}) = 0$), then a **finite reflection group** is a finite subgroup $W \subset \mathrm{GL}(V)$ that is generated by reflections.

Remarks 4.5.2 Retain the setting of Definition 4.5.1

(1) If two reflections in a finite reflection group W have the same hyperplane, they are equal. Hint: use that W is finite and $\mathrm{char}(\mathbb{F}) = 0$.

(2) The (-1)-eigenspace of s is 1-dimensional, spanned by some eigenvector $\alpha \in V$. We can find $\alpha^\vee \in V^*$ such that $\alpha^\vee(h) = 0$ for all $h \in V^s$ and $\langle \alpha, \alpha^\vee \rangle := \alpha^\vee(\alpha) = 2$. I.e. $\alpha^\vee$ is the *equation describing the hyperplane* V^s. Then we can express the reflection as:

$$s(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha$$

Conversely, every pair of $\alpha \in V$ and $\alpha^\vee \in V^*$ with $\langle \alpha, \alpha^\vee \rangle = 2$ defines a reflection by this formula. (Here we only use $\mathrm{char}(\mathbb{F}) \neq 2$)

(3) For any finite reflection group $W \subset GL(V)$ the vector space V can be equipped with a W-invariant inner product (positive definite, symmetric, bilinear form), as demonstrated in the proof of Theorem 4.4.16. In particular, all reflections $s \in W$ are then **orthogonal** with respect to this scalar product, in formulas $(s(v), s(w)) = (v, w)$. As consequences, the reflecting hyperplane V^s is exactly the orthogonal complement $\alpha^\perp = \{v \in V \mid (v, \alpha) = 0\}$ of α, for every $v \in V$ we have

$$\langle v, \alpha^\vee \rangle = \frac{2(v, \alpha)}{(\alpha, \alpha)}$$

as in (6) and the isomorphism

$$V \xrightarrow{\cong} V^*, \quad v \mapsto (., v)$$

induced by the inner product sends $2\alpha/(\alpha, \alpha) \mapsto \alpha^\vee$. If $\mathbb{F} = \mathbb{R}$ and V is equipped with such a W-invariant inner product, then we call $W \subset \mathrm{GL}(V)$ a **Euclidean** finite reflection group.

Definition 4.5.3 Let V be a vector space over an ordered field $\mathbb{F}$. For $v, w \in V$, we define the subsets

$$[v, w] := \{tv + (1-t)w \mid 0 \leq t \leq 1\}, \qquad (v, w] := \{tv + (1-t)w \mid 0 < t \leq 1\}.$$

Recall that a subset $A \subset V$ of a vector space V over an ordered field $\mathbb{F}$ is **convex** if $v, w \in S$ implies $[v, w] \subset A$. This means that for every two points in A, the entire line segment between v and w is also contained in A.

Definition 4.5.4 Let V be a finite-dimensional vector space over an ordered field $\mathbb{F}$ and $W \subset \mathrm{GL}(V)$ a finite reflection group. The maximal convex subsets of

$$V \setminus \bigcup_{\substack{s \in W \\ s \text{ is a reflection}}} V^s$$

are called **Weyl chambers** or **alcoves**. If $\mathbb{F} = \mathbb{R}$ and V is equipped with the usual topology, then the alcoves can also be described as the connected components of the complement of the hyperplanes. A hyperplane V^s is called a **wall** of an alcove A if there exists a $v \in V^s$ and $a \in A$, such that $(v, a] \subset A$ and v is not contained in any other reflecting hyperplane V^t for $t \neq s$.

Examples 4.5.5 (1) The Weyl group (Definition 4.4.14) associated to an abstract root system for V over an ordered field $\mathbb{F}$ is an example of a finite reflection group. The alcoves in $V_{\mathbb{Q}}$ are called **Weyl chambers**.

(2) If V is 1-dimensional, then there exists only one nontrivial finite reflection group, namely the one generated by the reflection in the origin. This group is isomorphic to $\mathbb{Z}_2$.

(3) If V is 2-dimensional vector space $\mathbb{R}^2$ and $r \geq 1$, then we consider r distinct lines through the origin, separated by angles $\pi n/r$ for $n \in \mathbb{Z}$. The reflections in these lines generate a finite reflection group D_r called a **dihedral group**, which also describes the symmetries of a regular r-gon: the r reflections in our r lines and the r rotations in an angle $2\pi n/r$ for $0 \leq n < r$.

(4) Let $n \geq 1$ and consider in $\mathbb{R}^n$ the hyperplanes $H_{i,j} := \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i = x_j\}$ for $1 \leq i < j \leq n$. The reflection in $H_{i,j}$ is the linear map which swaps the ith and jth coordinates of a vector. These reflection generate a finite reflection group that is isomorphic to the symmetric group S_n.

(5) The reflections in the coordinate hyperplanes $H_i = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i = 0\}$ for $1 \le i \le n$ generate a finite reflection group of order 2^n. It is isomorphic to $\mathbb{Z}_2 \times \cdots \times \mathbb{Z}_2$.

Theorem 4.5.6 Let V be a Euclidean vector space and $W \subset \mathrm{GL}(V)$ a finite reflection group. Consider the set of reflecting hyperplanes of W and the corresponding set of alcoves. Let A be an alcove.

(1) The reflections on the walls of A generate W.

(2) If $w = s_{i_1} s_{i_2} \cdots s_{i_r}$ is a shortest possible expression of $w \in W$ as product of reflections in walls of A, i.e. a **reduced expression** of w, then there exist exactly r reflecting hyperplanes that separate A from $w(A)$. We write $l(w) := r$ for the **length** of w.

(3) The group W acts freely and transitively on the set of alcoves.

(4) If $S \subset W$ is a set of reflections that generates W, then every reflection $\sigma \in W$ is conjugate to an element in S. I.e. there exist $w \in W$ and $s \in S$, such that $\sigma = wsw^{-1}$.

(5) The alcove A is the intersection of those half spaces defined by its walls, which contain A.

Proof. Omitted. □

Corollary 4.5.7 Let W be the Weyl group of an abstract root system R over $\mathbb{R}$, then every reflection $s \in W$ is of the form s_α for an $\alpha \in R$.

Proof. By Theorem 4.5.6.(4), there exists $w \in W$ and $\beta \in R$ with $s = ws_\beta w^{-1}$ and $ws_\beta w^{-1} = s_{w(\beta)}$ by Exercise 39, and $w(\beta) \in R$. □

Lemma 4.5.8 Let V be a Euclidean vector space and $\gamma \in V$ nonzero, then we denote by $\gamma^\perp$ the hyperplane perpendicular to γ. Suppose that $\alpha_1, \ldots, \alpha_n \in V$ satisfy $(\alpha_i, \gamma) > 0$ for all i and $(\alpha_i, \alpha_j) \le 0$ for $i \neq j$, then the $\alpha_1, \ldots, \alpha_n$ are linearly independent.

The two conditions express that the α_i all lie on one side of the hyperplane $\gamma^\perp$ and form obtuse angles among each other.

Proof. Suppose $0 = \sum_i c_i \alpha_i$ and set $I_+ := \{i \mid c_i \ge 0\}$ and $I_- := \{i \mid c_i < 0\}$, then we have:

$$x = \sum_{i \in I_+} c_i \alpha_i = \sum_{i \in I_-} (-c_i)\alpha_i$$

and

$$(x, x) = \sum_{i \in I_+, j \in I_-} c_i(-c_j)(\alpha_i, \alpha_j) \le 0$$

which implies $x = 0$ and thus

$$0 = (x, \gamma) = \sum_{i \in I_+} c_i(\alpha_i, \gamma)$$

so all $c_i = 0$ for all $i \in I_+$. Similarly one shows $I_- = \emptyset$. □

Theorem 4.5.9 Let V be a finite-dimensional vector space over an ordered field $\mathbb{F}$ and $W \subset \mathrm{GL}(V)$ a finite reflection group. Fix an an alcove A, and denote by $H_1, \ldots, H_n$ its walls and by $s_i \colon v \mapsto v - \langle v, \alpha_i^\vee \rangle \alpha_i$ the corresponding reflections. Then we have:

(1) The families $\{\alpha_i\} \subset V$ and $\{\alpha_i^\vee\} \subset V^*$ are linearly independent.

(2) If α_i is on the same side of H_i as A, then $\langle \alpha_i, \alpha_j^\vee \rangle \le 0$ if $i \neq j$.

L18 End

Proof. We equip V with an inner product as in Remarks 4.5.2.(3). We have

$$\langle \alpha_j, \alpha_i^\vee \rangle = \frac{2(\alpha_j, \alpha_i)}{(\alpha_i, \alpha_i)}. \tag{7}$$

Without loss of generality, we may assume that all α_i are chosen to lie on the same side of H_i as A (in formulas: $\langle A, \alpha_i^\vee \rangle \ge 0$), otherwise we replace $\alpha_i \mapsto -\alpha_i$ and $\alpha_i^\vee \mapsto -\alpha_i^\vee$.

Now we claim $(\alpha_i, \alpha_j) \leq 0$ if $i \neq j$. To prove the claim, let $i \neq j$. Consider the subgroup $W' \subset W$ generated by the reflections in H_i and H_j, and let A' denote the alcove for W' that contains A. Let $v \in H_i$ such that $(\alpha_j, v) < 0$. By invariance of the scalar product, this implies $(\alpha_j, s_j(v)) > 0$. Now suppose $(\alpha_i, \alpha_j) > 0$, then we would have

$$\begin{aligned}(\alpha_i, s_j(v)) &= (s_j(\alpha_i), v) = (\alpha_i, v) - \langle \alpha_i \alpha_j^\vee \rangle (\alpha_j, v)\\ &= -\langle \alpha_i, \alpha_j^\vee \rangle (\alpha_j, v) > 0.\end{aligned}$$

Thus we have $(\alpha_i, s_j(v)) > 0$ and $(\alpha_j, s_j(v)) > 0$, which means $s_j(v) \in A'$. On the other hand, $s_j(v)$ lies on the hyperplane $s_j(H_i)$, a contradiction and the claim is verified.

By Lemma 4.5.8 and choosing $\gamma \in A$, the claim implies the linear independence of the α_i and with (7) the statement of (2). The linear independence of the $\alpha_j^\vee$ also follows since the isomorphism $V^* \to V$ induced by the scalar product sends $\alpha_i^\vee \mapsto \frac{2}{(\alpha_i, \alpha_i)} \alpha_i$. ◻

Lemma 4.5.10 Let $R \subset V$ be an abstract root system and $\alpha, \beta \in R$ such that $\alpha \notin \mathbb{F}\beta$, then $\langle \alpha, \beta^\vee \rangle \langle \beta, \alpha^\vee \rangle \in \{0, 1, 2, 3\}$. If $\mathbb{F} = \mathbb{R}$ and V is equipped with an inner product invariant under the Weyl group of R, then the angle $\phi_{\alpha,\beta}$ between the roots α and β satisfies:

$$4\cos^2(\phi_{\alpha,\beta}) = \langle \alpha, \beta^\vee \rangle \langle \beta, \alpha^\vee \rangle$$

For $(\alpha, \beta) \neq 0$ we further have:

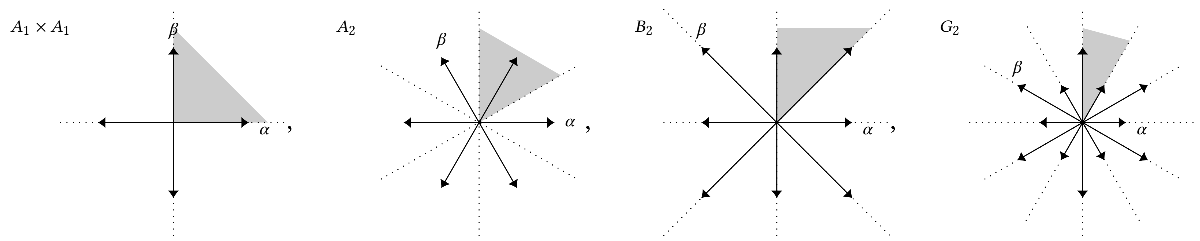
$$\frac{\|\alpha\|^2}{\|\beta\|^2} = \frac{\langle \alpha, \beta^\vee \rangle}{\langle \beta, \alpha^\vee \rangle}$$

Proof. We may consider the root system on $V_{\mathbb{Q}}$ and then extend to real coefficients, so without loss of generality $\mathbb{F} = \mathbb{R}$ and we are in the Euclidean situation. Recall $\langle \alpha, \beta^\vee \rangle = \frac{2(\alpha,\beta)}{(\beta,\beta)} \in \mathbb{Z}$, which immediately implies the two equations. Since $\cos^2$ takes values in the interval $[0, 1]$, the only possible values for $\langle \alpha, \beta^\vee \rangle \langle \beta, \alpha^\vee \rangle \in \mathbb{Z}$ are $\{0, 1, 2, 3, 4\}$, with value 4 if and only if $\phi_{\alpha,\beta} \in \{0, \pi\}$, so if and only if the roots are proportional. ◻

Example 4.5.11 Given roots $\alpha \notin \mathbb{R}\beta$ and suppose $\|\beta\| \geq \|\alpha\|$, then the following table lists all possibilities allowed by Lemma 4.5.10:

$\langle \alpha, \beta^\vee \rangle$	$\langle \beta, \alpha^\vee \rangle$	$\phi_{\alpha,\beta}$	$\|\beta\|^2 / \|\alpha\|^2$
0	0	$\pi/2$	?
1	1	$\pi/3$	1
-1	-1	$2\pi/3$	1
1	2	$\pi/4$	2
-1	-2	$3\pi/4$	2
1	3	$\pi/6$	3
-1	-3	$5\pi/6$	3

This gives four options of 2-dimensional root systems:



where we have only labeled the versions in which α and β form an obtuse angle, i.e. the versions with $\langle \beta, \alpha^\vee \rangle \leq 0$. Every arrow points to a root. The reflecting hyperplanes are drawn as dotted lines perpendicular to the roots. In each case, the grey shading indicates the unique alcove (Weyl chamber, when working over $\mathbb{Q}$) which lies in both of the positive half spaces specified by the labelled roots. The associated Weyl groups are the dihedral groups D_2, D_3, D_4, and D_6. The dihedral groups D_5 and D_r for $r \geq 7$ are finite reflection groups, but they do not come from root systems.

Exercise 41 Let R be a root system. For $\alpha, \beta \in R$ with $\alpha \neq \pm\beta$ show that $\langle \beta, \alpha^\vee \rangle > 0 \implies \beta - \alpha \in R$ and $\langle \beta, \alpha^\vee \rangle < 0 \implies \beta + \alpha \in R$. Hint: $\langle \beta, \alpha^\vee \rangle > 0$ and Lemma 4.5.10 implies that $\langle \beta, \alpha^\vee \rangle = 1$ or $\langle \alpha, \beta^\vee \rangle = 1$.

Exercise 42 Let R be a root system. For $\alpha, \beta \in R$ with $\alpha \neq \pm\beta$ show that $I = \{i \in \mathbb{Z} \mid \beta + i\alpha \in R\}$ is an interval in $\mathbb{Z}$.

Exercise 43 Classify up to isomorphism all groups generated by two elements s and t that square to the identity, i.e. $s^2 = e = t^2$. Which if of them arise as finite reflection groups?

4.6 Bases of root systems, positive roots, and alcove combinatorics

Here we use the convention that by a root system we mean an abstract reduced root system in the sense of Definition 4.4.13.

Definition 4.6.1 Let V be a vector space over $\mathbb{F}$ of $\mathrm{char}(\mathbb{F}) = 0$ and $R \subset V$ a root system. A subset $\Pi \subset R$ is a **basis** of the root system R if:

(1) Π is a basis for the vector space V.

(2) For any $\beta \in R$, the expansion $\beta = \sum_{\alpha\in\Pi} n_{\beta,\alpha}\alpha$ has all coefficients $n_{\beta,\alpha} \in \mathbb{Z}_{\geq 0}$ or all coefficients $n_{\beta,\alpha} \in \mathbb{Z}_{\leq 0}$.

The elements of Π are called **simple roots**.

Any basis Π of a root system R thus determines a partition $R = R^+(\Pi) \sqcup R^-(\Pi)$ into sets of **positive** and **negative roots**. The positive roots are $R^+(\Pi) := \{\beta \in R \mid \beta = \sum_{\alpha\in\Pi} n_{\beta,\alpha}\alpha \quad \text{with } n_{\beta,\alpha} \in \mathbb{Z}_{\geq 0}\}$, i.e. those which are linear combinations of simple roots with exclusively non-negative coefficients. This is formalized in the following definition.

Definition 4.6.2 A subset R^+ of a root system R is **system of positive roots** if the following conditions are satisfied:

(1) For every root $\alpha \in R$ we have $\alpha \in R^+$ if and only if $-\alpha \notin R^+$.

(2) If $\alpha, \beta \in R^+$ and $\alpha + \beta \in R$, then $\alpha + \beta \in R^+$.

An element $\gamma \in R^+$ is called indecomposable, if it cannot be written as $\gamma = \alpha + \beta$ for $\alpha, \beta \in R^+$.

L19
End

Remark 4.6.3 For two simple roots $\alpha, \beta \in \Pi$ we have $\langle \beta, \alpha^\vee \rangle \in \mathbb{Z}_{\leq 0}$. Otherwise the root $s_\alpha(\beta) = \beta - \langle \beta, \alpha^\vee \rangle \alpha$ would have an expansion with both positive and negative coefficients, in contradiction to the requirements of a basis. In this sense, α and β form an obtuse angle.

Lemma 4.6.4 Let R be a root system, $\Pi \subset R$ a basis, and $R^+ = R^+(\Pi)$ the set of positive roots. Then for every $\alpha \in \Pi$ we have:

$$s_\alpha(R^+) = (R^+ \setminus \{\alpha\}) \cup \{-\alpha\}.$$

Proof. Clearly $s_\alpha(\alpha) = -\alpha$. If $\beta \in R^+ \setminus \{\alpha\}$, then we claim that

$$s_\alpha(\beta) = \beta - \langle \beta, \alpha^\vee \rangle \alpha$$

is again positive. To see this, note that $s_\alpha(\beta)$ expands either positively or negatively in Π. But β contains at least one positive multiple of a simple root distinct from α, which then also contributes a positive coefficient in the expansion of $s_\alpha(\beta)$. □

Theorem 4.6.5 Let $R \subset V$ be a root system. Then we have a commutative diagram of bijections:

$$\begin{tikzcd}
\{\text{Weyl chambers in } V_{\mathbb{Q}}^*\} \arrow[r, "(6)"] \arrow[d, shift right, "(3)^\vee"'] & \{\text{Weyl chambers in } V_{\mathbb{Q}}\} \arrow[d, shift right, "(3)"'] \\
\{\text{Bases of } R\} \arrow[r, "(5)"] \arrow[u, shift right, "(4)^\vee"'] \arrow[d, shift right, "(1)"'] & \{\text{Bases of } R^\vee\} \arrow[u, shift right, "(4)"'] \arrow[d, shift right, "(1)^\vee"'] \\
\{\text{Systems of positive roots in } R\} \arrow[r, "(5)"] \arrow[u, shift right, "(2)"'] & \{\text{Systems of positive roots in } R^\vee\} \arrow[u, shift right, "(2)^\vee"']
\end{tikzcd}$$

with maps as follows:

(1) sends a basis $R \subset V$ to its set of positive roots $\Pi^+(R)$.

(2) sends a system of positive roots to its subset of indecomposable elements.

(3) sends a Weyl chamber $A \subset V_\mathbb{Q}$ to $\Pi(A) := \{\alpha^\vee \in R^\vee \mid \ker(\alpha^\vee) \text{ is a wall of } A \text{ and } \langle A, \alpha^\vee\rangle \subset \mathbb{Q}_{>0}\}$

(4) sends a basis Π of $R^\vee$ to $A(\Pi) := \{\lambda \in V_\mathbb{Q} \mid \langle \lambda, \alpha^\vee\rangle > 0 \text{ for all } \alpha^\vee \in \Pi\}$, the **dominant Weyl chamber**.

(5) sends a set of roots to the corresponding set of coroots.

(6) sends an alcove to its image under any isomorphism $V_\mathbb{Q}^* \xrightarrow{\cong} V_\mathbb{Q}$ induced by a W-invariant inner product.

Proof. For the maps (1) and (6) we know that they have the indicated target. Next we check this for (3), which will also prove that every root system has a basis. In Theorem 4.5.9 it was deduced from Lemma 4.5.8 that the set $\Pi(A)$ is linearly independent in $V_\mathbb{Q}^*$, and thus also in V^* by Theorem 4.4.16.(2). Theorem 4.5.6 shows that the reflections s_α for $\alpha^\vee \in \Pi(A)$ generate the Weyl group. As in Remark 4.4.17, this implies that the intersection of the kernels of the $\alpha^\vee \in \Pi(A)$ is zero, so they span V^* and hence form a basis.

Next we need to check for every $\beta^\vee \in R^\vee$ that $\beta^\vee = \sum_{\alpha^\vee \in \Pi(A)} n_{\alpha,\beta}\alpha^\vee$ has only nonnegative or only nonpositive integral coefficients. By Theorem 4.4.16.(2) the coefficients are rational $n_{\alpha,\beta} \in \mathbb{Q}$. First we claim that the $n_{\alpha,\beta}$ for fixed β all have the same sign. Consider the basis $\{\Lambda_\alpha\}_{\alpha^\vee \in \Pi(A)} \subset V_\mathbb{Q}$ dual to $\Pi(A)$, consisting of the so called **fundamental (dominant) weights**. I.e. the vectors determined by

$$\langle \Lambda_\alpha, \beta^\vee\rangle = \beta^\vee(\Lambda_\alpha) = \delta_{\alpha^\vee,\beta^\vee} \text{ for } \alpha^\vee, \beta^\vee \in \Pi(A).$$

The Weyl chamber A can now be described as

$$A = \sum_{\alpha^\vee \in \Pi(A)} \mathbb{Q}_{>0}\Lambda_\alpha\,, \quad \overline{A} := \sum_{\alpha^\vee \in \Pi(A)} \mathbb{Q}_{\geq 0}\Lambda_\alpha$$

and we call $\overline{A}$ its **closure**. The coroot $\langle -, \beta^\vee\rangle$ takes either positive or negative values on A, depending on whether A is contained in the positive or negative half space defined by the hyperplane $\ker(\beta^\vee)$. Thus $\langle -, \beta^\vee\rangle$ takes either non-negative or non-positive values on $\overline{A}$, and in particular on Λ_α for $\alpha^\vee \in \Pi(A)$. The claim follows from $n_{\alpha,\beta} = \langle \Lambda_\alpha, \beta^\vee\rangle$.

Next we need to show that the $n_{\alpha,\beta} \in \mathbb{Z}$. By Theorem 4.5.6 every Weyl chamber is conjugate to A under some element of W. Thus every coroot in $R^\vee$ is conjugate to a coroot in $\Pi(A)$. Since W maps the lattice $\mathbb{Z}\Pi(A)$ to $\mathbb{Z}\Pi(A)$, we also have $R^\vee \subset \mathbb{Z}\Pi(A)$, so the coefficients $n_{\alpha,\beta}$ are indeed integral. This finishes the proof of the claim that (3) maps a Weyl chamber A to a basis $\Pi(A)$ of $R^\vee$.

Next we consider the map (4) in the opposite direction. Given a basis Π of $R^\vee$, the set $A(\Pi) := \{\lambda \in V_\mathbb{Q} \mid \langle \lambda, \alpha^\vee\rangle > 0 \text{ for all } \alpha^\vee \in \Pi\}$ is an indeed a Weyl chamber: The condition $\langle A(\Pi), \alpha^\vee\rangle > 0$ for all $\alpha^\vee \in \Pi$ implies $\langle (A(\Pi), \beta^\vee\rangle > 0$ for all $\beta^\vee \in R^+(\Pi)$, and this in turn means that $A(\Pi)$ is disjoint from all reflecting hyperplanes. Further, it is an intersection of half spaces cut out by the reflecting hyperplanes, hence convex.

Next we check that (3) and (4) are mutually inverse. Every Weyl chamber B is the intersection of the half spaces cut out by its walls that contain B by Theorem 4.5.6.(5), so $A(\Pi(B)) = B$. Conversely, if $\Psi \subset R^\vee$ is a basis with $A = A(\Psi)$, then $R^+(\Psi) = \{\alpha^\vee \in R^\vee \mid \langle A, \alpha^\vee\rangle > 0\}$ and hence the basis $\Pi(A(\Psi)) \subset R^+(\Psi)$. But Ψ is the only subset of $R^+(\Psi)$ that is a basis, so $\Pi(A(\Psi)) = \Psi$.

Now we consider the maps (1) and (2). We first argue that (1), which sends $\Pi \mapsto R^+(\Pi)$ is injective since Π contains exactly those elements of $R^+(\Pi)$ that cannot be written as a sum of two elements of $R^+(\Pi)$. To see this, on the one hand, no $\alpha \in \Pi$ can be written as a sum in $R^+(\Pi)$. Conversely, if $\beta \in R^+(\Pi) \setminus \Pi$, then by Lemma 4.6.4 we have for all $\alpha \in \Pi$

$$s_\alpha(\beta) = \beta - \langle \beta, \alpha^\vee\rangle\alpha \in R^+(\Pi)$$

Since $\langle \beta, \alpha^\vee\rangle \geq 0$ is nonzero for at least one $\alpha \in \Pi$, we get the expression $\beta = \langle \beta, \alpha^\vee\rangle\alpha + s_\alpha(\beta)$ of β as sum of two elements in $R^+(\Pi)$.

The map (1) is also surjective: given a system of positive roots R^+ we pick a basis Π of R (which exists by (3)), for which $R^+ \cap R^+(\Pi)$ has a the maximal possible number of elements. If $R^+ \neq R^+(\Pi)$, then there exists $\alpha \in \Pi$ with $\alpha \notin R^+$. But then Lemma 4.6.4 implies that $R^+(s_\alpha(\Pi)) \cap R^+$ would have more elements than $R^+ \cap R^+(\Pi)$, a contradiction. Thus (1) is surjective, hence a bijection, and its inverse is the map (2).

Now we pick a W-invariant inner product and consider the corresponding isomorphism $i\colon V_\mathbb{Q}^* \xrightarrow{\cong} V_\mathbb{Q}$. If a basis Π corresponds to the alcove $A \subset V_\mathbb{Q}^*$, then $\Pi^\vee$ corresponds to $i(A) \subset V_\mathbb{Q}$. The commutativity of the diagram is left to the reader. □

Corollary 4.6.6 (1) Every root of a root system R is contained in at least one basis of R, because every reflecting hyperplane is the wall of at least one Weyl chamber. In particular, every root system R has a basis.

(2) Given two bases Π, Π' of a root system R, there exists a unique element $w \in W$ with $w\Pi = \Pi'$ because W acts free and transitively on its Weyl chambers by Theorem 4.5.6.(3).

Lemma 4.6.7 Let R be a root system in V, $\Pi \subset R$ a basis, and $R^+ = R^+(\Pi)$ the set of positive roots. Then the vector

$$\rho = \frac{1}{2} \sum_{\alpha \in R^+} \alpha \in V$$

lies in the dominant Weyl chamber and is called the **Weyl vector**. For all $\alpha \in \Pi$ one has $s_\alpha(\rho) = \rho - \alpha$ and thus $\langle \rho, \alpha^\vee \rangle = 1$ for all $\alpha \in \Pi$.

Proof. From Lemma 4.6.4 we immediately obtain $s_\alpha(\rho) = \rho - \alpha$ for $\alpha \in \Pi$. Now we compare this with $s_\alpha(\rho) = \rho - \langle \rho, \alpha^\vee \rangle \alpha$ to deduce $\langle \rho, \alpha^\vee \rangle = 1$ for $\alpha \in \Pi$. □

Exercise 44 Let $R \supset R^+ \supset \Pi$ be a root system with a system of positive roots and the corresponding basis. Show that for any positive root $\beta \in R^+$ there exists a sequence of simple roots $\alpha_1, \dots, \alpha_n$ with $\beta = \alpha_1 + \cdots + \alpha_n$ such that every partial sum $\alpha_1 + \cdots + \alpha_i$ is also a root. Hint: aim to use Lemma 4.5.8.

End W.11 Exerc.

4.7 Classification of root systems

Definition 4.7.1 For a root system R with basis Π, the **Cartan matrix** is the $\Pi \times \Pi$-matrix

$$C(R) := (\langle \alpha, \beta^\vee \rangle)_{\alpha, \beta \in \Pi}$$

Because of Corollary 4.6.6.(2) this is independent of the chosen basis Π.

Remark 4.7.2 Cartan matrices have the following properties:

- The diagonal elements are equal to $2 = \langle \alpha, \alpha^\vee \rangle$.
- The off-diagonal entries are non-positive (see Remark 4.6.3) and integral (see Remark 4.4.15).
- By (7) we have $\langle \alpha, \beta^\vee \rangle = 0$ if and only if $\langle \beta, \alpha^\vee \rangle = 0$.
- Since we may assume V to be a Euclidean vector space, $\det(C(R)) > 0$.

Cartan matrices can be encoded graphically as **Dynkin diagrams**: the elements of Π are nodes of a graph, and between the α- and β-nodes one draws $\langle \alpha, \beta^\vee \rangle \langle \beta, \alpha^\vee \rangle$-many edges (i.e. between 0 and 3 for a reduced root system). If two roots of different length are connected, the edge is oriented to the shorter root.

Examples 4.7.3 (1) The single entry Cartan matrix (2) has the corresponding Dynkin diagram • and it describes the root system of $\mathfrak{sl}(2, \mathbb{C})$.

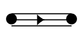
(2) For the root system for $\mathfrak{sl}(n+1, \mathbb{C})$ from Example 4.4.8 we can choose as simple roots:

$$\Pi = \{\epsilon_i - \epsilon_{i+1} \mid 1 \le i \le n\}$$

The corresponding Cartan matrix and Dynkin diagram are:

$$\begin{pmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & 2 & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 2 & -1 \\ 0 & \dots & 0 & -1 & 2 \end{pmatrix},$$

(3) The Cartan matrices and Dynkin diagrams of the root systems $A_1 \times A_1$, A_2,B_2, and G_2 from Example 4.5.11 are as follows:

$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, • •, $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$, , $\begin{pmatrix} 2 & -2 \\ -1 & 2 \end{pmatrix}$, , $\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$, 

Definition 4.7.4 If $R_1 \subset V_1$ and $R_2 \subset V_2$ are root systems over the same field, then we defined their **sum**:

$$R_1 \oplus R_2 = (R_1 \times \{0\}) \cup (\{0\} \times R_2) \subset V_1 \oplus V_2$$

This is again a root system.

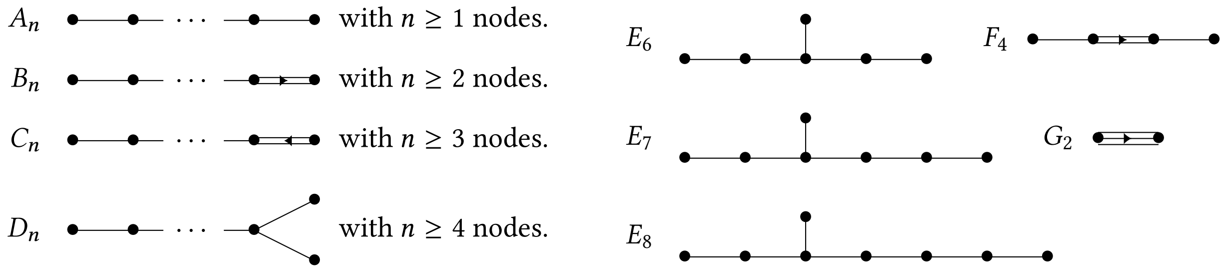
A root system is **indecomposable** if it is nonempty and not isomorphic to a sum of nonempty root systems. Otherwise it is said to be **decomposable**.

Proposition 4.7.5 Every root system $R \subset V$ has a unique decomposition $R = R_1 \sqcup \cdots \sqcup R_n$ such that R_i is an indecomposable root system in $\mathrm{span}(R_i)$ and $R \cong R_1 \oplus \cdots \oplus R_n$. This decomposition corresponds to the decomposition of the Dynkin diagram into its connected components.

Proof. Define on R the smallest equivalence relation for which $\alpha \sim \beta$ if $\langle \alpha, \beta^\vee \rangle \neq 0$. The corresponding equivalence classes provide the desired partition. □

Theorem 4.7.6 [Cartan, Killing] Let $\mathbb{F}$ be a field of characteristic $\mathrm{char}(\mathbb{F}) = 0$. The Dynkin diagrams of indecomposable root systems over $\mathbb{F}$ (up to isomorphism) are classified by the following list.

The four classical series: The five exceptional root systems:



Proof. Omitted. □

Remark 4.7.7 Together with Theorem 4.4.21 we will obtain a classification of complex simple Lie algebras via Dynkin diagrams. The classical series correspond to the classical Lie algebras from Section 1.2. In particular, Dynkin diagrams are a great way to remember the exceptional isomorphisms from Remarks 1.2.10, namely $A_1 \cong B_1 \cong C_1$, $B_2 \cong C_2$, $D_1 = \emptyset$, $D_2 \cong A_1 \times A_1$ and $D_3 \cong A_3$.

L20 End

4.8 Cartan subalgebras are conjugate

Definition 4.8.1 An endomorphism δ of an abelian group V is **locally nilpotent** if for every $v \in V$ there exists $N \in \mathbb{N}$ such that $\delta^N(v) = 0$. If V is a vector spaces over $\mathbb{F}$ of $\mathrm{char}(\mathbb{F}) = 0$, then we define for each locally nilpotent linear map $\delta\colon V \to V$ another linear endomorphism $\exp(\delta)\colon V \to V$ by:

$$\exp(\delta)(v) := \sum_{n \geq 0} \frac{\delta^n(v)}{n!} = v + \delta(v) + \frac{\delta^2(v)}{2} + \frac{\delta^3(v)}{3!} + \cdots$$

Lemma 4.8.2 Let V, W be vector spaces over $\mathbb{F}$ of $\mathrm{char}(\mathbb{F}) = 0$.

(1) For $0 \in \mathrm{End}(V)$ we have $\exp(0) = \mathrm{id}_V$. If δ, δ' are commuting locally nilpotent endomorphisms of V, then $\delta + \delta'$ is also locally nilpotent and $\exp(\delta + \delta') = \exp(\delta) \circ \exp(\delta')$. In particular, $\exp(-\delta) = \exp(\delta)^{-1}$.

(2) If the first of the following squares of linear maps commutes and δ, δ' are locally nilpotent, then the second square also commutes:

$$\begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\delta"'] & W \arrow[d, "\delta'"] \\ V \arrow[r, "f"] & W \end{tikzcd} \qquad \begin{tikzcd} V \arrow[r, "f"] \arrow[d, "\exp(\delta)"'] & W \arrow[d, "\exp(\delta')"] \\ V \arrow[r, "f"] & W \end{tikzcd}$$

If f is invertible, this implies $\exp(f \circ \delta \circ f^{-1}) = f \circ \exp(\delta) \circ f^{-1}$.

(3) If $\delta\colon V \to V$ is nilpotent (!), then $\delta^t\colon V^* \to V^*$ is nilpotent and $\exp(\delta^t) = \exp(\delta)^t$.

Proof. Exercise. □

Lemma 4.8.3 [Exponential of locally nilpotent derivation] Let A be an algebra over $\mathbb{F}$ of $\mathrm{char}(\mathbb{F}) = 0$ and $\delta\colon A \to A$ a locally nilpotent derivation. Then $\exp(\delta)\colon A \to A$ is an algebra automorphism.

Proof. Let $a, b \in A$. By the defining property of a derivation we have $\delta(ab) = \delta(a)b + a\delta(b)$ and by induction $\delta^n(ab) = \sum_i \binom{n}{i}\delta^i(a)\delta^{n-i}(b)$. This implies:

$$\exp(\delta)(ab) = \sum_{n\geq 0} \frac{\delta^n(ab)}{n!} = \sum_{i,j\geq 0} \frac{\delta^i(a)}{i!}\frac{\delta^j(b)}{j!} = (\exp(\delta)(a))(\exp(\delta)(b))$$ □

If $A = \mathfrak{g}$ is a Lie algebra, then we denote by G the subgroup of $\mathrm{Aut}(\mathfrak{g})$ that is generated by the $\exp(\mathrm{ad}_x)$ for all ad-nilpotent $x \in \mathfrak{g}$.

Theorem 4.8.4 [Cartan subalgebras are conjugate] Let $\mathfrak{g}$ be a complex semisimple Lie algebra and $\mathfrak{h}, \mathfrak{h}' \subset \mathfrak{g}$ two Cartan subalgebras. Then there exists a Lie algebra automorphism $\sigma \in \mathrm{Aut}(\mathfrak{g})$ that sends $\sigma(\mathfrak{h}) = \mathfrak{h}'$.

Proof sketch. Consider $\mathfrak{h}_{\mathrm{reg}} := \mathfrak{h} \setminus \bigcup_{\alpha\in R(\mathfrak{g},\mathfrak{h})} \ker(\alpha)$ which is Zariski-open in $\mathfrak{h}$. We have $\mathfrak{h} = \ker(\mathrm{ad}_h\colon \mathfrak{g} \to \mathfrak{g})$ for all $h \in \mathfrak{h}_{\mathrm{reg}}$. Now consider the map:

$$\begin{aligned} \mathfrak{g}_\alpha \times \cdots \mathfrak{g}_\beta \times \mathfrak{h}_{\mathrm{reg}} &\to \mathfrak{g} \\ (x, \ldots, y, h) &\mapsto (\exp(\mathrm{ad}_x) \circ \cdots \circ \exp(\mathrm{ad}_y))(h) \end{aligned}$$

where the product runs over all roots in some fixed order. This map has surjective differential at every tuple $(0, \ldots, 0, h)$ and its image thus (here we use a special case of the differential dominance criterion from algebraic geometry) contains a Zariski-open set in $\mathfrak{g}$. The same holds for $\mathfrak{h}'$ and so their images intersect. This means we can find $h \in \mathfrak{h}_{\mathrm{reg}}$ and $h' \in \mathfrak{h}'_{\mathrm{reg}}$ as well as $\tau_1, \tau_2 \in G$ (as defined before the theorem) with $\tau_1(h) = \tau_2(h')$ and hence $\sigma := \tau_2^{-1}\tau_1\colon h \mapsto h'$. But this implies $\sigma(\mathfrak{h}) = \mathfrak{h}'$ (Exercise). □

Exercise 45 Prove Lemma 4.8.2.

Exercise 46 Consider $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$ with standard basis e, f, h. Define the automorphism $\sigma := \exp(\mathrm{ad}_e) \circ \exp(\mathrm{ad}_{-f}) \circ \exp(\mathrm{ad}_e)$ of $\mathfrak{g}$ and compute its action on the basis e, f, h. Next compute $\exp(e)$ and $\exp(-f)$ and show they are elements of $\mathrm{SL}(2, \mathbb{C})$ (the special linear group of complex 2×2 matrices of determinant 1). Compute $s = \exp(e)\cdot\exp(-f)\cdot\exp(e)$ and show that conjugation $z \mapsto szs^{-1}$ defines an automorphism of $\mathfrak{g}$. How does it compare to σ?

End W.12 Exerc.

4.9 Constructing semisimple Lie algebras

Definition 4.9.1 Let I be a set and $\mathbb{F}$ a field. Then a **free Lie algebra on I** over $\mathbb{F}$ is a Lie algebra L over $\mathbb{F}$, together with a map $\mathrm{can}\colon I \to L$, such that for any Lie algebra $\mathfrak{g}$ over $\mathbb{F}$, precomposition with can provides a bijection

$$\mathrm{LieAlg}_{\mathbb{F}}(L, \mathfrak{g}) \xrightarrow{-\circ\mathrm{can}} \mathrm{Set}(I, \mathfrak{g})$$

between Lie algebra morphisms from L to $\mathfrak{g}$ and maps of sets from I to $\mathfrak{g}$. This this called the **universal property** of the free Lie algebra on I over $\mathbb{F}$.

Remark 4.9.2 (Uniqueness of free Lie algebras) Let $\mathrm{can}\colon I \to L$ and $\mathrm{can}'\colon I \to L'$ be two free Lie algebras over the same set I and field $\mathbb{F}$, then there exist unique Lie algebra morphisms $\phi\colon L \to L'$ and $\psi\colon L' \to L$ such that $\phi \circ \mathrm{can} = \mathrm{can}'$ and $\psi \circ \mathrm{can}' = \mathrm{can}$. Moreover, ϕ and ψ are mutual inverses. To see this, note that there is a unique Lie algebra morphism $\zeta\colon L \to L$ such that $\zeta \circ \mathrm{can} = \mathrm{can}$. Since $\zeta = \mathrm{id}_L$ and $\zeta = \psi \circ \phi$ satisfy this condition, they are equal. Similarly one shows $\phi \circ \psi = \mathrm{id}_{L'}$.

Remark 4.9.3 (Existence of free Lie algebras) It is not hard to explicitly construct a free Lie algebra on a given set I. Here we only give a sketch. One first constructs a *free algebra A* on I over $\mathbb{F}$ with multiplication denoted by $\cdot$ (Idea: a basis is given by bracketed words of length ≥ 1 with letters drawn from I). This satisfies a universal property similar as in Definition 4.9.1, but for all $\mathbb{F}$-algebras. To obtain a Lie algebra L from A, one takes the quotient by the 2-sided ideal R generated by elements of the form $a \cdot a$ and $(a \cdot (b \cdot c)) + (b \cdot (c \cdot a)) + (c \cdot (a \cdot b))$ for $a, b, c \in A$. This inherits the desired universal property from A: any map $\phi\colon I \to \mathfrak{g}$ extends uniquely to an algebra morphism $\hat{\phi}\colon A \to \mathfrak{g}$, but then $\hat{\phi}(R) = 0$ because $\mathfrak{g}$ is a Lie algebra, so this map descends uniquely to the quotient, thus providing the desired Lie algebra morphism $L = A/R \to \mathfrak{g}$.

By the preceding remarks, we have a free Lie algebra for every set I and $\mathbb{F}$, and it is uniquely determined up to unique isomorphism. We thus call the result *the* free Lie algebra on I over $\mathbb{F}$ and denote it by $\mathrm{LA}_{\mathbb{F}}(I)$

Definition 4.9.4 Let $\mathbb{F}$ be a field, I a set, and $T \subset \mathrm{LA}_{\mathbb{F}}(I)$ a subset of the free Lie algebra on I over $\mathbb{F}$

(1) Let $\langle T\rangle_L$ denote the Lie-ideal generated by T in $\mathrm{LA}_{\mathbb{F}}(I)$, i.e. the intersection of all ideals of $\mathrm{LA}_{\mathbb{F}}(I)$ that contain T. Then the **Lie algebra with generators I and relations T** is defined as the quotient $\mathrm{LA}_{\mathbb{F}}(I)/\langle T\rangle_L$.

(2) Suppose that $\mathfrak{g}$ is a Lie algebra over $\mathbb{F}$ and $I \subset \mathfrak{g}$. Then we say $\mathfrak{g}$ **is presented by the generators I with relations T** if the map $\phi\colon \mathrm{LA}_{\mathbb{F}}(I) \to \mathfrak{g}$ induced by the inclusion $I \hookrightarrow \mathfrak{g}$ descends to an isomorphism $\mathrm{LA}_{\mathbb{F}}(I)/\langle T\rangle_L \to \mathfrak{g}$.

The latter condition combines the following requirements:

- We say that I generates $\mathfrak{g}$ if $\phi\colon \mathrm{LA}_{\mathbb{F}}(I) \to \mathfrak{g}$ is surjective.
- We can compare the kernel of this morphism ϕ with $\langle T\rangle_L$. If $\ker(\phi) \supset \langle T\rangle_L$, then we get an induced morphism:

$$\mathrm{LA}_{\mathbb{F}}(I)/\langle T\rangle_L \to \mathrm{LA}_{\mathbb{F}}(I)/\ker(\phi) \to \mathfrak{g}$$

  Informally this containment means that *the relations T among the generators I are satisfied in* $\mathfrak{g}$. Moreover, if ϕ was surjective, then so will be the induced map on the quotient.
- Third, the map on the quotient will be injective if T also generates the kernel of ϕ, i.e. *if T describes all relations between the generators I in* $\mathfrak{g}$.

L21 End

Theorem 4.9.5 [Presentation of complex semisimple Lie algebras by generators and relations] (1) Let $\mathfrak{g}$ be a complex semisimple Lie algebra with Cartan subalgebra $\mathfrak{h}$ and $\Pi \subset R := R(\mathfrak{g},\mathfrak{h})$ a basis of the corresponding root system. If we choose for every $\alpha \in \Pi$ a basis element $x_\alpha \in \mathfrak{g}_\alpha$ of the corresponding root space, then there exist elements $y_\alpha \in \mathfrak{g}_{-\alpha}$ such that $[x_\alpha, y_\alpha] = \alpha^\vee$. Set $h_\alpha := \alpha^\vee$ for the $\alpha \in \Pi$. Then these elements satisfy the following relations for all $\alpha, \beta \in \Pi$:

$$\begin{aligned}
[x_\alpha, y_\alpha] &= h_\alpha \\
[x_\alpha, y_\beta] &= 0 \quad \text{if } \alpha \neq \beta \\
[h_\alpha, h_\beta] &= 0 \\
[h_\alpha, x_\beta] &= \langle \beta, \alpha^\vee\rangle x_\beta \\
[h_\alpha, y_\beta] &= -\langle \beta, \alpha^\vee\rangle y_\beta \\
\mathrm{ad}_{x_\alpha}^{1-\langle \beta,\alpha^\vee\rangle}(x_\beta) &= 0 \quad \text{if } \alpha \neq \beta \\
\mathrm{ad}_{y_\alpha}^{1-\langle \beta,\alpha^\vee\rangle}(y_\beta) &= 0 \quad \text{if } \alpha \neq \beta
\end{aligned}$$

Furthermore, $\mathfrak{g}$ is presented by the generators $x_\alpha, y_\alpha, h_\alpha$ for $\alpha \in \Pi$ with the listed relations (the last two are called **Serre relations**).

(2) Given an abstract root system R with basis Π, then the complex Lie algebra $\mathfrak{g} = \mathfrak{g}_{R,\Pi}$ generated by the symbols $x_\alpha, y_\alpha, h_\alpha$ for $\alpha \in \Pi$ with the listed relations is semisimple. The images of the generators h_α form the basis of a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ and there is an isomorphism of root systems $R \cong R(\mathfrak{g},\mathfrak{h})$ sending $\beta \mapsto (h_\alpha \mapsto \langle \beta, \alpha^\vee\rangle)$.

Partial proof. We will only show a part of (1). That we can find a y_α for every x_α such that $[x_\alpha, y_\alpha] = h_\alpha$ follows from the Definition 4.4.12 of $\alpha^\vee$ as special basis element of the 1-dimensional subspace $[\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}] \subset \mathfrak{g}$. The second relation holds since $[x_\alpha, y_\beta] \in [\mathfrak{g}_\alpha, \mathfrak{g}_{-\beta}] \subset g_{\alpha-\beta} = 0$ since $\alpha - \beta \notin R$ by definition of a basis. The third is clear since $\mathfrak{h}$ is abelian. The fourth and fifth relations follow from the definition of the root spaces. For the sixth relation we observe

$$\mathrm{ad}_{x_\alpha}^{1-\langle \beta,\alpha^\vee\rangle}(x_\beta) \in \mathfrak{g}_{\beta+\alpha-\langle \beta,\alpha^\vee\rangle\alpha} = \mathfrak{g}_{\alpha+s_\alpha(\beta)} = 0$$

since $\alpha + s_\alpha(\beta) = s_\alpha(-\alpha+\beta) \notin R$ since $\alpha - \beta \notin R$ as observed above. The last relation follows analogously. Using the notation from (2), we thus obtain a Lie algebra morphism

$$\mathfrak{g}_{R,\Pi} \to \mathfrak{g}$$

and it remains to show that this is an isomorphism. We first consider surjectivity, i.e. that $\mathfrak{g}$ is generated by the $x_\alpha, h_\alpha, y_\alpha$ for $\alpha \in \Pi$. This follows from Exercise 44 together with the fact $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] = \mathfrak{g}_{\alpha+\beta}$ when $\alpha, \beta, \alpha+\beta \in R$ from Theorem 4.4.9. Thus $\mathfrak{g}_{R,\Pi} \twoheadrightarrow \mathfrak{g}$. The injectivity of this morphism follows from (2), which we will not prove. □

As a consequence we can now assemble a proof for the classification result stated in Theorem 4.4.21, namely that complex semisimple Lie algebras are classified, up to isomorphism, by complex reduced root systems up to isomorphism. More generally, the analogous statement holds over every algebraically closed field of characteristic zero.

Proof of Theorem 4.4.21. First, recall from Theorem 4.4.18 that a choice of Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ indeed determines a complex reduced root system $R(\mathfrak{g}, \mathfrak{h})$. Then Theorem 4.8.4 shows that all Cartan subalgebras are conjugate in $\mathfrak{g}$, which means that the isomorphism class of the root system $R(\mathfrak{g}, \mathfrak{h})$ is independent of the choice of $\mathfrak{h}$. It is also clear that isomorphic Lie algebras have isomorphic root systems. Thus the map

$$\left\{\begin{matrix}\text{isomorphism classes of}\\ \text{complex semisimple Lie algebras}\end{matrix}\right\} \to \left\{\begin{matrix}\text{isomorphism classes of}\\ \text{complex reduced root systems}\end{matrix}\right\}$$

induced by $\mathfrak{g} \mapsto R(\mathfrak{g}, \mathfrak{h}) \subset \mathfrak{h}^*$ is well-defined. Theorem 4.9.5.(2) shows that this map is surjective, since it constructs a complex semisimple Lie algebra for every abstract root system with basis (which exists by Corollary 4.6.6 and is unique up to action by automorphisms given by the Weyl group Theorem 4.6.5) by giving a presentation. Theorem 4.9.5.(1) shows the map is injective, since any two complex semisimple Lie algebras with isomorphic root systems are both isomorphic to the Lie algebra with the mentioned presentation. □

Now we consider the classification of complex *simple* Lie algebras.

Theorem 4.9.6 Given a complex simple Lie algebra $\mathfrak{g}$, then the choice of a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ determines an indecomposable complex root system $R(\mathfrak{g}, \mathfrak{h}) \subset \mathfrak{h}^*$. This map

$$\begin{matrix}\left\{\begin{matrix}\text{complex simple}\\ \text{Lie algebras}\end{matrix}\right\} & \to & \left\{\begin{matrix}\text{complex indecomposable}\\ \text{root systems}\end{matrix}\right\}\\ \mathfrak{g} & \mapsto & R(\mathfrak{g}, \mathfrak{h}) \subset \mathfrak{h}^*\end{matrix}$$

induces a bijection between isomorphism classes of complex simple Lie algebras and isomorphism classes of indecomposable complex root systems.

Proof. Based on Theorem 4.4.21 it remains to show that a complex semisimple Lie algebra is simple if and only if its root system is indecomposable. Indeed, every decomposition of the root system would induce a decomposition of the Lie algebra into a direct sum of corresponding ideals. Conversely, every such decomposition into a direct sum of ideals leads to a decomposition of the associated root system. □

The Killing classification of Theorem 3.1.2 now follows from Theorem 4.9.6 and the classification of indecomposable root systems from Theorem 4.7.6.

Remark 4.9.7 The classification from Theorem 4.4.21 is part of a chain of further identifications, which will not be treated here.

$$\left\{\begin{matrix}\text{isomorphism classes of}\\ \text{compact Lie groups with trivial center}\end{matrix}\right\} \to \left\{\begin{matrix}\text{isomorphism classes of}\\ \text{compact Lie algebras over } \mathbb{R}\end{matrix}\right\} \to \left\{\begin{matrix}\text{isomorphism classes of}\\ \text{complex semisimple Lie algebras}\end{matrix}\right\}$$

The first map is induced by forming the Lie algebra associated to the Lie group (the tangent space at the identity), and the second map is induced by a complexification procedure.

Exercise 47 Write down explicit presentations (following Theorem 4.9.5) for the complex semisimple Lie algebras corresponding to the root systems from Example 4.5.11.

5 Representation theory of complex semisimple Lie algebras

Throughout this section we let $\mathfrak{g}$ denote a complex semisimple Lie algebra and $\mathfrak{h}$ a Cartan subalgebra. All vector spaces are defined over the field of complex numbers $\mathbb{C}$ unless specified otherwise.

In this section, we will often use capital letters to denote elements of Lie algebras, e.g. $X \in \mathfrak{g}$, $H \in \mathfrak{h}$. For a $\mathfrak{g}$-representation $\rho\colon \mathfrak{g} \to \mathfrak{gl}(V)$, we will further abbreviate the notation for the action on a vector $v \in V$ as: $Xv := X \cdot v := \rho(X)(v)$.

5.1 Weights

Definition 5.1.1 (1) The elements of $\mathfrak{h}^*$ are called **weights of** $\mathfrak{g}$ (relative to $\mathfrak{h}$).

(2) For every representation V, the **weight space** for a weight $\lambda \in \mathfrak{h}^*$ is defined as:

$$V_\lambda = \{v \in V \mid Hv = \lambda(H)v \text{ for all } H \in \mathfrak{h}\}$$

If $V_\lambda \neq 0$, then λ is a **weight of the representation** V. The set of weights of V is denoted $P(V)$.

(3) For every system of positive roots R^+, one defines a partial order on $\mathfrak{h}^*$ by setting

$$\lambda \geq \mu \iff \lambda - \mu \in \mathbb{Z}_{\geq 0}R^+$$

A greatest element λ in $P(V)$, i.e. if $\lambda \geq \mu$ for all $\mu \in P(V)$, is called the **highest weight** of V with respect to R^+. A nonzero vector $0 \neq v \in V_\lambda$ is then called a **highest weight vector** of V. For given V, a highest weight need not exist, but it is unique if it does. (For partial orders, there is also the notion of a maximal element. This is λ, such that $\mu \geq \lambda$ implies $\mu = \lambda$, but not every element needs to be comparable to λ. If a greatest element exists, then it is the unique maximal element.)

(4) The **lattice of integral weights**, or short **weight lattice**, is:

$$\mathcal{X} := \{\lambda \in \mathfrak{h}^* \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z} \text{ for all } \alpha \in R\}$$

All roots are integral weights, i.e. $R \subset \mathcal{X}$ and the Weyl group preserves the weight lattice $W\mathcal{X} = \mathcal{X}$.

(5) For a given basis $\Pi = \{\alpha_1, \dots, \alpha_r\}$ of the root system we consider the **fundamental dominant weights** $\Lambda_i = \Lambda_{\alpha_i}$, which form the basis of $\mathfrak{h}^*$ dual to the basis of coroots $\alpha_i^\vee$ of $\mathfrak{h}$. In formulas: $\langle \Lambda_i, \alpha_j^\vee \rangle = \delta_{i,j}$. The fundamental dominant weights also form a $\mathbb{Z}$-basis for the weight lattice.

(6) The elements of

$$\mathcal{X}^+(R^+) = \mathbb{Z}_{\geq 0}\Lambda_1 + \cdots + \mathbb{Z}_{\geq 0}\Lambda_r = \{\lambda \in \mathfrak{h}^* \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}_{\geq 0} \text{ for all } \alpha \in R^+\}$$

are called **dominant integral weights**. $\mathcal{X}^+(R^+)$ is the intersection of the weight lattice $\mathcal{X}$ with the closure of the dominant Weyl chamber with respect to R^+.

Examples 5.1.2 (1) The weights of the adjoint representation are the roots together with the zero vector in $\mathfrak{h}^*$.

(2) Let us consider the case $\mathfrak{sl}(2, \mathbb{C})$ with Cartan subalgebra $\mathfrak{h} = \mathbb{C}h$ and $R^+ = \{\alpha\}$ (we may choose $\alpha = 2h^*$ so that $\alpha^\vee = h$.). We identify weights with complex numbers along $x\alpha/2 \mapsto x$ (equivalently $\lambda \mapsto \langle \lambda, \alpha^\vee \rangle$). The simple representation $L(m)$ then has weights $\{m, m-2, \dots, 2-m, -m\}$, the partial order is the obvious one, and m is the highest weight of $L(m)$. The weight lattice is $\mathbb{Z} \subset \mathbb{C}$, the dominant integral weights are $\mathbb{Z}_{\geq 0}$, and these are exactly the highest weights of the simple representations. L22 End

We are interested in classifying finite-dimensional complex representations of the complex semisimple Lie algebra $\mathfrak{g}$. By Weyl's Theorem 4.2.5, this reduces to classifying simple representations, which we will see to be in bijection with dominant integral weights.

Lemma 5.1.3 Let V be a $\mathfrak{g}$-representation. Then $\mathfrak{g}_\alpha V_\lambda \subset V_{\lambda+\alpha}$ for all $\alpha \in R$ and $\lambda \in \mathfrak{h}^*$.

Proof. Given $H \in \mathfrak{h}$, $X \in \mathfrak{g}$ and $v \in V$, we have $HXv = [H, X]v + XHv$. If $v \in V_\lambda$ we have $Hv = \lambda(H)v$ and if $X \in \mathfrak{g}_\alpha$ we have $[H, X] = \alpha(H)X$. In this case $HXv = (\lambda(H) + \alpha(H))Xv = (\lambda + \alpha)(H)Xv$. □

Lemma 5.1.4 With respect to a system of positive roots R^+ we consider the subalgebras $\mathfrak{n} = \bigoplus_{\alpha\in R^+} \mathfrak{g}_{-\alpha}$ and $\mathfrak{n}^+ = \bigoplus_{\alpha\in R^+} \mathfrak{g}_\alpha$ of $\mathfrak{g}$. Let V be a $\mathfrak{g}$-representation and $v \in V_\lambda$ such that $\mathfrak{n}^+ \cdot v = 0$. By restriction we may view V as an $\mathfrak{n}$-representation and consider the $\mathfrak{n}$-subrepresentation $V' \subset V$ generated by v. Then V' is already a $\mathfrak{g}$-representation, $V'_\lambda = \mathbb{C}v$ and $V' = \bigoplus_{\mu\leq\lambda} V'_\mu$.

Proof. By Lemma 5.1.3 we have $V' \cap V_\lambda = \mathbb{C}v$ and $V' = \bigoplus_{\mu\leq\lambda} V'_\mu$. It thus suffices to prove $X \cdot V' \subset V'$ for all $X \in \mathfrak{b} := \mathfrak{h} \oplus \mathfrak{n}^+$. For $r \in \mathbb{Z}_{\geq 0}$ we consider the subspace of V' defined as

$$V'(r) = \operatorname{span}_{\mathbb{C}}\{Y_1 \cdots Y_i v \mid i \leq r, Y_i \in \mathfrak{n}\}.$$

Then we have $V' = \bigcup_{r\geq 0} V'(r)$ and we shall prove by induction in r that $X \cdot V'(r) \subset V'(r)$ for $X \in \mathfrak{b}$. The base of the induction at $r = 0$ follows from the assumptions $v \in V_\lambda$ and $\mathfrak{n}^+ \cdot v = 0$. For the induction step with $r \geq 1$ we write:

$$XY_1 \cdots Y_r \cdot v = Y_1 X Y_2 \cdots Y_r \cdot v + [X, Y_1] Y_2 \cdots Y_r \cdot v$$

On the right hand side in the first term we see (using induction) an element $XY_2 \cdots Y_r \cdot v \in V'(r-1)$, thus $Y_1 X Y_2 \cdots Y_r \cdot v \in V'(r)$. In the second term, we use $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{n}$ to split the Lie bracket $[X, Y_1] = \tilde{X} + \tilde{Y}$ with $\tilde{X} \in \mathfrak{b}$ and $\tilde{Y} \in \mathfrak{n}$ and again use the induction hypothesis to conclude $[X, Y_1] Y_2 \cdots Y_r \cdot v \in V'(r)$. □

Theorem 5.1.5 Let V be a simple $\mathfrak{g}$-representation.

(1) If the set of weights $P(V)$ contains a maximal element, then it is a greatest element, i.e. a highest weight.

(2) If V has a highest weight λ, then $\dim_{\mathbb{C}}(V_\lambda) = 1$.

(3) If V is finite-dimensional, then it is the direct sum of its weight spaces and has a highest weight.

There are infinitely dimensional (even simple) $\mathfrak{g}$-representations which have no weights at all, or which have a nonempty set of weights without a maximal element. We will see examples later.

Proof. If $v \in V_\lambda$ with $\lambda \in P(V)$ maximal, then $\mathfrak{g}_\alpha v = 0$ for all positive roots $\alpha \in R^+$, so $\mathfrak{n}^+ \cdot v = 0$. Lemma 5.1.4 provides a subrepresentation V' with $V'_\lambda = \mathbb{C}v \neq 0$ and $V' = \bigoplus_{\mu\leq\lambda} V'_\mu$. Since V is simple, we have $V' = V$, which shows (1) and (2).

If V is of finite dimension, then V decomposes as $V = \bigoplus_{\lambda\in\mathfrak{h}^*} V_\lambda$ thanks to Theorem 4.3.1.(2) (see also Remark 4.4.5.(1)). The poset of weights of V is finite and not empty, thus has a maximal element, which is a highest weight by (1). This shows (3). □

Theorem 5.1.6 If V is a finite-dimensional $\mathfrak{g}$-representation, then all of its weights are integral, i.e. $P(V) \subset \mathcal{X}$, and the set of weights $P(V)$ is stable under the Weyl group. If a highest weight exists, then it is dominant and integral.

Proof. Fix a root $\alpha \in R$ and consider the subalgebra $\mathfrak{sl}(2,\mathbb{C})_\alpha := \mathfrak{g}^\alpha = \mathfrak{g}_\alpha \oplus \mathbb{C}\alpha^\vee \oplus \mathfrak{g}_{-\alpha}$ which is isomorphic to $\mathfrak{sl}(2,\mathbb{C})$. By restricting V to this subalgebra, we deduce from Consequences 2.1.5 that $\langle\lambda, \alpha^\vee\rangle \in \mathbb{Z}$ for any $\lambda \in P(V)$.

Let E_α denote a generator of the 1-dimensional vector space $\mathfrak{g}_\alpha$, $E_{-\alpha}$ one of $\mathfrak{g}_{-\alpha}$, fix a weight λ of V and set $m = \langle\lambda, \alpha^\vee\rangle$. For $v \in V_\lambda$ and $v \neq 0$, we have:

$$(E_{-\alpha})^m \cdot v \neq 0 \text{ if } m \geq 0; \quad (E_\alpha)^{-m} \cdot v \neq 0 \text{ if } m \leq 0$$

Either way we have $V_{\lambda - m\alpha} \neq 0$, so $s_\alpha(\lambda) = \lambda - \langle\lambda, \alpha^\vee\rangle\alpha = \lambda - m\alpha$ is another weight of V.

If $\lambda \in P(V)$ is not dominant, then there exists $\alpha \in \Pi$ with $\langle\lambda, \alpha^\vee\rangle\alpha < 0$. Then $s_\alpha(\lambda) \in P(V)$ and $s_\alpha(\lambda) \geq \lambda$, so λ is not maximal. Conversely it follows that maximal weights of the finite-dimensional representation V are dominant. □

Theorem 5.1.7 Let V and V' be simple $\mathfrak{g}$-representations with the same highest weight λ, then V and V' are isomorphic.

Proof. Let $v \in V_\lambda$ and $v' \in V'_\lambda$ be nonzero vectors and $D \subset V \oplus V'$ the $\mathfrak{g}$-subrepresentation generated by (v, v'). By Lemma 5.1.4 D is the direct sum of its weight spaces and $D_\lambda = \mathbb{C}(v, v')$.

Every proper subrepresentation $D' \subset D$ is also a sum of its weight spaces and so

$$D' \subset \bigoplus_{\mu\neq\lambda} D_\lambda. \tag{8}$$

Let p_1, p_2 denote the projection maps from $V \oplus V'$ to V and V' respectively. These can be restricted to subspaces and (8) implies $p_1(D') \neq V$ and $p_2(D') \neq V'$. Since V and V' are simple, and since images of representations under projections are again representations, we must have $p_1(D') = 0$ and $p_2(D') = 0$, hence $D' = 0$. Thus D is simple.

An argument analogous to Schur's Lemma 4.2.1 now implies that the nonzero maps p_1 and p_2 are isomorphisms. Thus $V \cong V'$. □

One detail is still missing before we can conclude with the following summary.

Theorem 5.1.8 Let $\mathfrak{g}$ be a complex semisimple Lie algebra. Then the finite-dimensional simple $\mathfrak{g}$-representations are in bijection with the integral dominant weights of $\mathfrak{g}$.

It remains to show that every dominant integral weight actually appears as highest weight of a simple $\mathfrak{g}$-representation. This will be proved using the notion of a universal enveloping algebra. L23 End

---

Exercise 48 Let $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$ and $V = \mathbb{C}^n$ the vector representation. Show that $\wedge^i(V)$ has highest weight $\varpi_i = \epsilon_1 + \cdots + \epsilon_i$ with highest weight vector $w_1 \wedge \cdots \wedge w_i$ for a suitable basis $\{w_j\}$ of $\mathbb{C}^n$. For every dominant weight $\lambda \in \mathcal{X}^+$ show that a simple representation of highest weight λ arises as a direct summand of a suitable tensor product of representations $\wedge^i(V)$.

Exercise 49 Verify the statements made in Examples 5.1.2.(2). End W.13 Exerc.

5.2 Universal enveloping algebras

Definition 5.2.1 Let $\mathfrak{g}$ be a Lie algebra (not necessarily semisimple) over a field $\mathbb{F}$. A **universal enveloping algebra** of $\mathfrak{g}$ is a pair (U, can), where U is an associative, unital $\mathbb{F}$-algebra and

$$\mathrm{can}\colon \mathfrak{g} \longrightarrow U$$

is a $\mathbb{F}$-linear map and, more specifically, a Lie algebra morphism with respect to the commutator on U (see Construction 1.2.4), such that the following **universal property** is satisfied: for every associative, unital $\mathbb{F}$-algebra A and a Lie algebra morphism

$$\phi\colon \mathfrak{g} \longrightarrow A$$

(with respect to the commutator on A), there exists a unique morphism of unital $\mathbb{F}$-algebras $\tilde{\phi}\colon U \longrightarrow A$, such that $\phi = \tilde{\phi} \circ \mathrm{can}$. This is expressed by the diagram:

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\mathrm{can}"] \arrow[dr, "\phi"'] & U \arrow[d, "\exists! \tilde{\phi}"] \\ & A \end{tikzcd}$$

Remarks 5.2.2 (1) A priori it is not clear that every Lie algebra has a universal enveloping algebra, but if one exists, then it is unique up to canonical isomorphism. This is a standard consequence of the universal property and sometimes we will refer to **the** universal enveloping algebra of a given Lie algebra.

(2) In fact, universal enveloping algebras do exist for all Lie algebras, as we will see in Theorem 5.2.6. Moreover, $\mathfrak{g} \mapsto U(\mathfrak{g})$ is a functor that is a left adjoint to the functor that interprets associative, unital algebras as Lie algebras with respect to the commutator, see Construction 1.2.4.

(3) A universal enveloping algebra U of $\mathfrak{g}$ is generated as an associative, unital algebra by the image of $\mathfrak{g}$ under can. This is because the subalgebra of U generated by this image already satisfies the universal property of a universal enveloping algebra. Furthermore, we will see in Theorem 5.2.6 that can is injective. Therefore, we will sometimes simply interpret can as an inclusion $\mathfrak{g} \subset U$ and write X for the image $\mathrm{can}(X) \in U$ of an element $X \in \mathfrak{g}$.

Examples 5.2.3 (1) For the 0-dimensional Lie algebra $\mathfrak{g} = 0$ over $\mathbb{F}$, the field $U = \mathbb{F}$ with the unique map $\mathrm{can}\colon 0 \longrightarrow \mathbb{F}$ is a universal enveloping algebra.

(2) For a 1-dimensional Lie algebra spanned by an element X, i.e. $\mathfrak{g} = \mathbb{F}X$, the polynomial ring $\mathbb{F}[X]$ serves as universal enveloping algebra with respect to the map:

$$\begin{aligned} \mathrm{can}\colon \mathfrak{g} &\to \mathbb{F}[X] \\ aX &\mapsto aX \end{aligned}$$

Note that $\mathfrak{g}$ is abelian and the given map is a Lie algebra morphism since $\mathbb{F}[X]$ is commutative.

(3) An n-dimensional abelian Lie algebra spanned by $X_1, \dots, X_n$ has $\mathbb{F}[X_1, \dots, X_n]$ as universal enveloping algebra. In general, universal enveloping algebras can be considered as generalizations of polynomial rings to *non-commuting* variables.

Recall that a (left-)**module** for an associative, unital $\mathbb{F}$-algebra A is an $\mathbb{F}$-vector space V together with a unital algebra morphism $\phi\colon A \to \mathrm{End}_{\mathbb{F}}(V)$. We will use the notation $a \cdot v := \phi(a)(v)$ for $a \in A$ and $v \in V$.

Lemma 5.2.4 Let $\mathrm{can}\colon \mathfrak{g} \to U$ be a universal enveloping algebra of $\mathfrak{g}$. Then the restriction along can induces an equivalence of categories:

$$\{\text{modules for the } \mathbb{F}\text{-algebra } U\} \xrightarrow{\simeq} \{\text{representations of } \mathfrak{g}\}$$

Proof. Given a U-module (V, ϕ) with $\phi\colon U \to \mathrm{End}_{\mathbb{F}}(V)$ we obtain a $\mathfrak{g}$-representation $(V, \phi \circ \mathrm{can})$ with $\phi \circ \mathrm{can}\colon \mathfrak{g} \to \mathrm{End}_{\mathbb{F}}(V)$. Conversely, given a $\mathfrak{g}$-representation (V, ρ), the universal property guarantees the existence of a unique $\tilde{\rho}\colon U \to \mathrm{End}_{\mathbb{F}}(V)$, i.e. a U-module structure on V, such that $\tilde{\rho} \circ \mathrm{can} = \rho$. These assignments are mutually inverse and functorial, as one can easily check. □

Definition 5.2.5 Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{F}$ and $U(\mathfrak{g})$ a universal enveloping algebra of $\mathfrak{g}$. Consider the trivial representation $\rho\colon \mathfrak{g} \to \mathbb{F}$. By Lemma 5.2.4 this corresponds to a unital algebra morphism $\epsilon\colon U(\mathfrak{g}) \to \mathbb{F}$ with $\epsilon(X) = 0$ for all $X \in \mathrm{can}(\mathfrak{g}) \subset U(\mathfrak{g})$, the so called **augmentation**. The kernel of ϵ is a 2-sided ideal in $U(\mathfrak{g})$, called the **augmentation ideal**:

$$U^+ := \ker(\epsilon)$$

The goal of this section is the following result.

Theorem 5.2.6 [Poincaré–Birkhoff–Witt] (1) Every Lie algebra $\mathfrak{g}$ has a universal enveloping algebra $U(\mathfrak{g})$.

(2) If $\{X_i\}_{i \in I}$ is a basis of $\mathfrak{g}$ and $\leq$ a total order on the index set I, then the monomials $X_{i(1)} \cdots X_{i(r)}$ for $i(1) \leq i(2) \leq \cdots \leq i(r)$ for $r \geq 0$ form an $\mathbb{F}$-basis of $U(\mathfrak{g})$. For $r = 0$ this includes the empty monomial, which is identified with the unit $1 \in U(\mathfrak{g})$.

Note that $X_{i(1)} \cdots X_{i(r)}$ should be interpreted as $\mathrm{can}(X_{i(1)}) \cdots \mathrm{can}(X_{i(r)})$, see Remarks 5.2.2.(3). The proof is spread out over the rest of the section.

Definition 5.2.7 Let V be a vector space over $\mathbb{F}$. The **tensor algebra** is the associative unital $\mathbb{F}$-algebra

$$T(V) = \bigoplus_{r \geq 0} V^{\otimes r} = \mathbb{F} \oplus V \oplus (V \otimes V) \oplus \cdots$$

with multiplication given by the tensor product, i.e. defined on elementary tensors by:

$$(v_1 \otimes v_2 \otimes \cdots \otimes v_r)(w_1 \otimes \cdots \otimes w_t) := v_1 \otimes \cdots \otimes v_r \otimes w_1 \otimes \cdots \otimes w_t.$$

The **exterior algebra** on V is the associative unital $\mathbb{F}$-algebra

$$\wedge(V) = T(V)/\langle v \otimes v \mid v \in V \rangle$$

and the **symmetric algebra** on V is the associative unital $\mathbb{F}$-algebra

$$S(V) = T(V)/\langle v \otimes w - w \otimes v \mid v, w \in V \rangle.$$

These algebras, $T(V)$, $\wedge(V)$, and $S(V)$ are examples of **graded algebras**, i.e. associative algebras A with a decomposition as vector spaces

$$A = \bigoplus_{r \geq 0} A_r = A_0 \oplus A_1 \oplus A_2 \oplus \cdots$$

such that the multiplication maps $A_r \times A_s \to A_{r+s}$ for all $r, s \geq 0$.

The tensor, exterior, and symmetric algebras also satisfy universal properties. We only need the first kind.

Lemma 5.2.8 Let V be a vector space over $\mathbb{F}$ and denote by $c\colon V \longrightarrow T(V)$ the evident $\mathbb{F}$-linear inclusion. If A is a associative unital $\mathbb{F}$-algebra and $\phi\colon V \longrightarrow A$ an $\mathbb{F}$-linear map, then there exists a unique morphism of associative unital algebras $\tilde{\phi}\colon T(V) \longrightarrow A$ such that $\tilde{\phi} \circ c = \phi$. This is expressed by the diagram:

$$\begin{tikzcd} V \arrow[r, "c"] \arrow[dr, "\phi"'] & T(V) \arrow[d, "\exists!\tilde{\phi}"] \\ & A \end{tikzcd}$$

Proof. As V generates $T(V)$ as an algebra, there exists at most one algebra morphism $\tilde{\phi}\colon T(V) \longrightarrow A$ with prescribed values on V. We define it on elementary tensors in $T(V)$ as:

$$\tilde{\phi}(v_1 \otimes \cdots \otimes v_r) := \phi(v_1) \cdots \phi(v_r)$$

It is straightforward to check that this defines an unital algebra morphism. □

Theorem 5.2.9 Let $I(\mathfrak{g})$ denote the 2-sided ideal of $T(\mathfrak{g})$ that is generated by all elements of the form $x \otimes y - y \otimes x - [x, y]$ where $x, y \in \mathfrak{g}$. Then $U(\mathfrak{g}) := T(\mathfrak{g})/I(\mathfrak{g})$ together with the map

$$\mathrm{can}\colon \mathfrak{g} \overset{c}{\hookrightarrow} T(\mathfrak{g}) \twoheadrightarrow U(\mathfrak{g})$$

is a universal enveloping algebra of $\mathfrak{g}$. L24 End

This theorem establishes Theorem 5.2.6.(1).

Proof. Denote the canonical projection $T(\mathfrak{g}) \longrightarrow U(\mathfrak{g})$ by π. We first check that $\mathrm{can} = \pi \circ c$ is a Lie algebra morphism. Let $x, y \in \mathfrak{g}$, then we have:

$$\begin{aligned} \mathrm{can}([x, y]) = \pi([x, y]) &= \pi(x \otimes y - y \otimes x) \\ &= \mathrm{can}(x)\mathrm{can}(y) - \mathrm{can}(y)\mathrm{can}(x) = [\mathrm{can}(x), \mathrm{can}(y)] \end{aligned}$$

where we have used that $x \otimes y - y \otimes x - [x, y]$ is in the kernel of π.

As unital algebra $U(\mathfrak{g})$ is generated by $\mathrm{can}(\mathfrak{g})$, so for any associative unital algebra A, a Lie algebra morphism $\mathfrak{g} \to A$ extends in at most one way to a unital algebra morphism $U(\mathfrak{g}) \longrightarrow A$. It thus remains that such an extension exists. To this end, consider the diagram

$$\begin{tikzcd} \mathfrak{g} \arrow[r, hook, "c"] \arrow[dr, "\phi"'] & T(\mathfrak{g}) \arrow[r, two heads] \arrow[d] & U(\mathfrak{g}) \arrow[dl] \\ & A & \end{tikzcd}$$

where ϕ is a Lie algebra morphism. By Lemma 5.2.8, ϕ extends to a unital algebra morphism $\hat{\phi}\colon T(\mathfrak{g}) \longrightarrow A$. Since ϕ is a morphism of Lie algebras, we obtain $\hat{\phi}(I(\mathfrak{g})) = 0$. Thus $\hat{\phi}$ factors uniquely through a unital algebra morphism $\tilde{\phi}\colon U(\mathfrak{g}) \longrightarrow A$. □

Proof of Theorem 5.2.6.(2). Let $\mathfrak{g}$ be a Lie algebra and $U(\mathfrak{g})$ its universal enveloping algebra as constructed in Theorem 5.2.9, both over the field $\mathbb{F}$. Let $\{X_i\}_{i \in I}$ be an $\mathbb{F}$-basis with ordered index set I.

We first check that the monomials $X_{i(1)} \cdots X_{i(r)}$ with $i(1) \leq i(2) \leq \cdots \leq i(r)$ for $r \geq 0$ span $U(\mathfrak{g})$. Let $U_r \subset U(\mathfrak{g})$ denote the subset spanned by all monomials (not necessarily ordered indices) of length at most r. We claim that U_r is actually spanned by monomials *with ordered indices* and prove this by induction. The case $r = 1$ is trivial. For $r \geq 2$ we consider an arbitrary monomial $X_{i(1)} \cdots X_{i(r)}$. Then $X_{i(j)}X_{i(j+1)} = X_{i(j+1)}X_{i(j)} + [X_{i(j)}, X_{i(j+1)}]$ and we can expand the commutator $[X_{i(j)}, X_{i(j+1)}] = \sum_k a_k X_k$. This implies that the class in U_r/U_{r-1} represented by $X_{i(1)} \cdots X_{i(r)}$ does not depend on the order of the factors; the indices may be assumed to be ordered. Then $X_{i(1)} \cdots X_{i(r)}$ can be expanded as a sum of an ordered monomial of length r and ordered monomials of shorter length. In summary, ordered monomials span.

Next we claim that the monomials $X_{i(1)} \cdots X_{i(r)}$ with $i(1) \leq i(2) \leq \cdots \leq i(r)$ for $r \geq 0$ are linearly independent in $U(\mathfrak{g})$. A usual strategy for showing linear independence of such elements is to show that they act in a linearly independent way on a representation. To this end, consider the vector space S over $\mathbb{F}$ with basis $Y_{i(1)}\cdots Y_{i(r)}$ with $i(1)\le i(2)\le\cdots\le i(r)$ for $r\ge 0$.

Claim: There exists an action of $\mathfrak{g}$ on S, such that

$$X_i Y_{i(1)}\cdots Y_{i(r)} = Y_i Y_{i(1)}\cdots Y_{i(r)} \quad \text{whenever } i(1)\le i(2)\le\cdots\le i(r)$$

We will show this claim as part of Lemma 5.2.10. Assuming it holds, we can now consider S as a $U(\mathfrak{g})$-module. For $i(1)\le i(2)\le\cdots\le i(r)$ we now find:

$$X_{i(1)}\cdots X_{i(r)}\cdot 1_S = Y_{i(1)}\cdots Y_{i(r)}$$

where 1_S denotes the basis element of S for the length zero monomial. The $Y_{i(1)}\cdots Y_{i(r)}$ were assumed to be linearly independent in S, which now implies the linear independence of the ordered monomials $X_{i(1)}\cdots X_{i(r)}$ in $U(\mathfrak{g})$. □

It remains to state and prove the announced auxiliary lemma. To this end, retain notation from the preceding proof and denote by S_r the subspace of S spanned by polynomials of length r. We have $S_0=\mathbb{F}1_S$, $S=\bigoplus_{r\in\mathbb{Z}_{\ge 0}} S_r$. Consider S as a polynomial ring, we also have an associative and commutative multiplication such that $S_rS_s\subset S_{r+s}$ for all $r,s\in\mathbb{Z}_{\ge 0}$. We also write $S_{\le r}=\bigoplus_{s=0}^{r} S_s$ and set $S_{\le r}=0$ when $r<0$. We will also use multiindices $\underline{i}=(i(1),\ldots,i(r))\in I^r$ and write $Y_{\underline{i}}:=Y_{i(1)}\cdots Y_{i(r)}$. A multiindex $\underline{i}$ is **monotone** if $Y_{\underline{i}}$ is ordered. We also write $i\le\underline{i}$ if $i\le i(j)$ for all $1\le j\le r$.

Lemma 5.2.10 There exists a unique family of bilinear maps $\phi_r\colon \mathfrak{g}\times S_{\le r}\to S_{\le r+1}$ for $r\in\mathbb{Z}$, denoted $(x,p)\mapsto xp$ such that

(1) ϕ_r extends ϕ_{r-1};

(2) $X_iY_{\underline{i}}=Y_iY_{\underline{i}}$ if $\underline{i}\in I^r$ and $i\le\underline{i}$;

(3) $X_iY_{\underline{i}}\in Y_iY_{\underline{i}}+S_{\le r}$ if $\underline{i}\in I^r$ and $i\in I$;

(4) $X_i(X_jp)-X_j(X_ip)=[X_i,X_j]p$ for all $i,j\in I$ and $p\in S_{\le r-1}$.

Proof. For $r<0$ these are just the zero maps. Now we proceed by induction. Suppose ϕ_r has already been constructed. Then we need to show that it extends in a unique way to ϕ_{r+1}, such that the properties (2)-(4) are still satisfied.

For $i\in I$ and a monotone $\underline{i}\in I^{r+1}$ we have to define $\phi_{r+1}(X_i,Y_{\underline{i}})=X_iY_{\underline{i}}\in S$. If $i\le\underline{i}$, then we set $X_iY_{\underline{i}}:=Y_iY_{\underline{i}}$ and (2) is satisfied. Otherwise we write $\underline{i}=(j,\underline{j})$ with $j\in I$ and $\underline{j}\in I^r$, and we know $i>j$. To satisfy (1)-(4) we must have:

$$\begin{aligned}X_iY_{\underline{i}} &= X_i(X_jY_{\underline{j}})\\ &= X_j(X_iY_{\underline{j}})+[X_i,X_j]Y_{\underline{j}}\\ &= X_j(Y_iY_{\underline{j}})+X_jq+[X_i,X_j]Y_{\underline{j}}\\ &= Y_j(Y_iY_{\underline{j}})+X_jq+[X_i,X_j]Y_{\underline{j}}\end{aligned}$$

where we have abbreviated $q=X_iY_{\underline{j}}-Y_iY_{\underline{j}}$. As $q\in S^{\le r}$, the terms in the last row are already defined, and so the computation serves as definition of $X_iY_{\underline{i}}$ in case $i\not\le\underline{i}$.

The required properties of ϕ_{r+1} are immediate, except maybe (4). Here we have to show:

$$X_i(X_jY_{\underline{j}})-X_j(X_iY_{\underline{j}})=[X_i,X_j]Y_{\underline{j}}$$

for all $i,j\in I$ and $\underline{j}\in I^r$. We call this assertion $A(i,j,\underline{j})$. Clearly we have $A(i,i,\underline{j})$ and by the definition of ϕ_{r+1} we also have $A(i,j,\underline{j})$ if $i>j\le\underline{j}$. Since the Lie bracket is anti-symmetric, we also have $A(i,j,\underline{j})$ if $j>i\le\underline{j}$. The remaining case is for $j\not\le\underline{j}$ and $i\not\le\underline{j}$. Here we write $\underline{j}=(k,\underline{k})$ with $k\in I$ and $\underline{k}\in I^{r-1}$, so we have $k<i$, $k<j$ and $k<\underline{k}$. Now we expand

$$\begin{aligned}X_i(X_jY_{\underline{j}}) &= X_i(X_j(X_kY_{\underline{k}}))\\ &= X_i([X_j,X_k]Y_{\underline{k}})+X_i(X_k(X_jY_{\underline{k}}))\\ &= X_i([X_j,X_k]Y_{\underline{k}})+[X_iX_k](X_jY_{\underline{k}})+X_k(X_i(X_jY_{\underline{k}}))\end{aligned}$$

where the second equation follows by induction and the third by the cases already verified. (To see this, write $X_j Y_{\underline{k}} = Y_j Y_{\underline{k}} + q$ with $q \in S_{\leq r-2}$ and note $k < j$ and $k \leq \underline{k}$.) The same equation holds after exchanging i and j. Now we get three equations:

$$\begin{aligned} X_i(X_j Y_{\underline{j}}) &= X_i([X_j, X_k] Y_{\underline{k}}) + [X_i X_k](X_j Y_{\underline{k}}) + X_k(X_i(X_j Y_{\underline{k}})) \\ X_j(X_i Y_{\underline{j}}) &= X_j([X_i, X_k] Y_{\underline{k}}) + [X_j X_k](X_i Y_{\underline{k}}) + X_k(X_j(X_i Y_{\underline{k}})) \\ [X_i, X_j] Y_{\underline{j}} &= [X_i, X_j](Y_k Y_{\underline{k}}) \end{aligned}$$

We have to show $X_i(X_j Y_{\underline{j}}) - X_j(X_i Y_{\underline{j}}) - [X_i, X_j] Y_{\underline{j}} = 0$, i.e. after subtracting the second and third equation from the first, the left hand side should vanish. We compute the right hand side of this difference as:

$$\begin{aligned} &[X_i, [X_j, X_k]] Y_{\underline{k}} + [[X_i, X_k], X_j] Y_{\underline{k}} + X_k([X_i, X_j] Y_{\underline{k}}) - [X_i, X_j](Y_k Y_{\underline{k}}) \\ &= ([X_i, [X_j, X_k]] + [X_j, [X_k, X_i]] + [X_k, [X_i, X_j]]) Y_{\underline{k}} = 0 \end{aligned}$$

□

Remarks 5.2.11 (1) A(n ascending) filtration of a vector space V is a sequence of subspaces $V_{\leq r}$ for $r \in \mathbb{Z}$ with $V_{\leq r} \subset V_{\leq r+1}$ for all $r \in \mathbb{Z}$. Any subspace and any quotient of a vector space with a filtration inherits a filtration. A linear map $\phi\colon V \longrightarrow W$ respects two given filtrations on V and W if $\phi(V_{\leq r}) \subset W_{\leq r}$. If V is graded $V = \bigoplus_{r \in \mathbb{Z}} V_r$, then V can be equipped with the filtration $V_{\leq s} := \bigoplus_{s \leq r} V_r$. For any filtration on V, one can define the associated graded vector space $\operatorname{gr} V := \bigoplus_{r \in \mathbb{Z}} V_{\leq r}/V_{\leq r-1}$. If the filtration comes from a grading, we have a natural isomorphism $V \cong \operatorname{gr} V$.

(2) A (unital) associative algebra A is filtered if it is equipped with a filtration, such that the multiplication sends $A_{\leq r} \times A_{\leq s} \longrightarrow A_{\leq r+s}$ (and $1 \in A_{\leq 0}$). The associated graded $\operatorname{gr} A$ naturally inherits the structure of a graded algebra. If the filtration comes from a grading, we have a natural isomorphism of graded algebras $A \cong \operatorname{gr} A$. Every quotient of a filtered associative algebra is again filtered.

(3) **Alternative form of the PBW theorem:** Let $\mathfrak{g}$ be a Lie algebra. The two surjections $T(\mathfrak{g}) \twoheadrightarrow S(\mathfrak{g})$ and $T(\mathfrak{g}) \xrightarrow{\cong} \operatorname{gr} T(\mathfrak{g}) \longrightarrow \operatorname{gr} U(\mathfrak{g})$ have the same kernel and thus define an isomorphism of graded, associative, unital algebras

$$\operatorname{gr} U(\mathfrak{g}) \cong S(\mathfrak{g}).$$

Exercise 50 Let e, f, h be the usual basis of $\mathfrak{sl}(2, \mathbb{C})$ with $[h, e] = 2e$, $[h, f] = -2f$, and $[e, f] = h$. Express $f^2 he \in U(\mathfrak{sl}(2, \mathbb{C}))$ as a linear combination of ordered monomials with respect to the order e, h, f.

Exercise 51 Let $\mathfrak{g}$ be a vector space and $b\colon \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ a bilinear map. Consider the ideal $I \subset T(\mathfrak{g})$ generated by $x \otimes y - y \otimes x - b(x, y)$ for $x, y \in \mathfrak{g}$. Show that the map $\mathfrak{g} \hookrightarrow T(\mathfrak{g}) \twoheadrightarrow T(\mathfrak{g})/I$ is injective if and only if b is a Lie bracket.

Exercise 52 Show that every morphism between Lie algebras extends uniquely to a unital algebra morphism between their universal enveloping algebras.

Exercise 53 Let $\mathfrak{g}$ be a finite-dimensional Lie algebra and $b\colon \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathbb{F}$ a nondegenerate invariant bilinear form. Choose a basis $\{x_1, \ldots, x_n\}$ of $\mathfrak{g}$ and denote by $\{x^1, \ldots x^n\}$ the basis dual with respect to b. I.e. $b(x_i, x^j) = \delta_{i,j}$. We set

$$C = C_b := \sum_{i=1}^{n} x_i x^i \in U(\mathfrak{g})$$

Show that $C_b \in U(\mathfrak{g})$ is independent of the choice of a basis for the Lie algebra $\mathfrak{g}$ and that $C_n \in Z(U(\mathfrak{g}))$, i.e. that $uC = Cu$ for all $u \in U(\mathfrak{g})$.

End W.14 Exerc.

5.3 Constructing highest weight modules

In this subsection we fix a system of positive roots $R^+ \subset R(\mathfrak{g}, \mathfrak{h})$ of the complex semisimple Lie algebra $\mathfrak{g}$ with Cartan subalgebra $\mathfrak{h}$.

Definition 5.3.1 For a weight $\lambda \in \mathfrak{h}^*$ consider the left ideal $I_\lambda \subset U(\mathfrak{g})$ generated by $x \in \mathfrak{g}_\alpha$ for $\alpha \in R^+$ and $H - \lambda(H)1$ for $H \in \mathfrak{h}$. The quotient $\Delta(\lambda) = U(\mathfrak{g})/I_\lambda$ is naturally a left module for $U(\mathfrak{g})$ and, by Lemma 5.2.4, a $\mathfrak{g}$-representation. It is called the **Verma module** of $\mathfrak{g}$ of highest weight $\lambda \in \mathfrak{h}^*$. The coset of $1 \in U(\mathfrak{g})$ is denoted $v_\lambda \in \Delta(\lambda)$ and called the **canonical generator** of the Verma module $\Delta(\lambda)$.

Proposition 5.3.2 For every weight $\lambda \in \mathfrak{h}^*$ we have:

(1) If $\alpha_1, \dots, \alpha_p \in R^+$ are the positive roots in a fixed order and $y_\alpha \in \mathfrak{g}_{-\alpha}$ basis elements of the root spaces for the negative roots, then the vectors $y_{\alpha_1}^{m(1)} \cdots y_{\alpha_p}^{m(p)} v_\lambda$ indexed by functions $m\colon \{1, \dots, p\} \to \mathbb{Z}_{\geq 0}$ form a $\mathbb{C}$-basis of the Verma module $\Delta(\lambda)$.

(2) The Verma module $\Delta(\lambda)$ has the weight space decomposition

$$\Delta(\lambda) = \bigoplus_{\mu \leq \lambda} \Delta(\lambda)_\mu$$

and the highest weight space $\Delta(\lambda)_\lambda$ is 1-dimensional and spanned by v_λ.

(3) Let $\mathcal{P}\colon \mathfrak{h}^* \to \mathbb{Z}_{\geq 0}$ denote the **Kostant partition function**, which counts in how many ways (if any) a weight can be decomposed into a non-negative integer linear combination of positive roots (and $\mathcal{P}(0) = 1$). In formulas:

$$\mathcal{P}(\lambda) = |\{m\colon R^+ \to \mathbb{Z}_{\geq 0} \mid \lambda = \sum_{\alpha \in R^+} m(\alpha)\alpha\}|$$

Then the dimensions of the weight spaces of the Verma module $\Delta(\lambda)$ are:

$$\dim_{\mathbb{C}}(\Delta(\lambda)_\mu) = \mathcal{P}(\lambda - \mu)$$

L25 End

Proof. Consider the polynomial ring $\mathbb{C}[H_1, \dots, H_r]$. For fixed scalars $\lambda_1, \dots, \lambda_r \in \mathbb{C}$ this has a $\mathbb{C}$-basis with elements $(H_1 - \lambda_1)^{n(1)} \cdots (H_r - \lambda_r)^{n(r)}$ indexed by functions $n\colon \{1, \dots, r\} \to \mathbb{Z}_{\geq 0}$. We consider the case when $H_1, \dots, H_r$ form a basis of the Cartan subalgebra $\mathfrak{h}$. Furthermore, let $x_\alpha \in \mathfrak{g}_\alpha$ for $\alpha \in R^+$ denote basis elements of the root spaces for positive roots. Now Theorem 5.2.6 implies that the products

$$y_{\alpha_1}^{m(\alpha_1)} \cdots y_{\alpha_p}^{m(\alpha_p)} (H_1 - \lambda_1)^{n(1)} \cdots (H_r - \lambda_r)^{n(r)} x_{\alpha_1}^{l(\alpha_1)} \cdots x_{\alpha_p}^{l(\alpha_p)}$$

for $m, l\colon R^+ \to \mathbb{Z}_{\geq 0}$ and $n\colon \{1, \dots, r\} \to \mathbb{Z}_{\geq 0}$ form a $\mathbb{C}$-basis of $U(\mathfrak{g})$. Similarly, for $\mathfrak{b} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R^+} \mathfrak{g}_\alpha \subset \mathfrak{g}$ we get a $\mathbb{C}$-basis of $U(\mathfrak{b})$ by omitting the factors y_α. Consider the Lie algebra morphisms $\mathfrak{b} \twoheadrightarrow \mathfrak{h} \to \mathbb{C}$, the first of which splits $\mathfrak{h} \hookrightarrow \mathfrak{b}$ and the second is the linear form λ. This induces an associative algebra morphism $U(\mathfrak{b}) \to \mathbb{C}$ whose kernel consists of the nontrivial monomials of our basis, which thus form an ideal in $U(\mathfrak{b})$. Multiplying by basis elements from $U(\mathfrak{g})$ on the left, we obtain a spanning set over $\mathbb{C}$ of a left ideal in $U(\mathfrak{g})$. In particular, the ideal I_λ is obtained by constructing the basis of $U(\mathfrak{g})$ as above and with $\lambda_i = \lambda(H_i)$ and then considering the span of basis vectors with $n \neq 0$ or $l \neq 0$. (We leave it as an exercise to show that left multiplication by a PBW basis elements of $U(\mathfrak{g})$ preserves this subspace.) The cosets of the $y_{\alpha_1}^{m(\alpha_1)} \cdots y_{\alpha_p}^{m(\alpha_p)}$ for $m\colon R^+ \to \mathbb{Z}_{\geq 0}$ thus form a $\mathbb{C}$-basis of $\Delta(\lambda)$. By definition we have $Hv_\lambda = \lambda(H)v_\lambda$ for all $H \in \mathfrak{h}$, so v_λ is a weight vector for λ. The remaining statements follow immediately. □

Lemma 5.3.3 [Universal property of Verma modules] Let M be a $\mathfrak{g}$-representation and $\lambda \in \mathfrak{h}^*$. Then there is a vector space isomorphism

$$\mathrm{Hom}_{\mathfrak{g}}(\Delta(\lambda), M) \xrightarrow{\cong} \{v \in M_\lambda \mid \mathfrak{g}_\alpha v = 0 \text{ for all } \alpha \in R^+\}$$

induced by the evaluation $\phi \mapsto \phi(v_\lambda)$ on the canonical generator $v_\lambda \in \Delta(\lambda)$.

Proof. For every module M over a ring, the evaluation at the neutral element 1_R induces a bijection $\mathrm{Hom}_R(R, M) \xrightarrow{\cong} M$. Similarly, in the context of quotients we find that for every left ideal $I \subset R$ the evaluation at $1_R + I$ induces a bijection $\mathrm{Hom}_R(R/I, M) \xrightarrow{\cong} \{m \in M \mid Im = 0\}$. Now we apply this for $R = U(\mathfrak{g})$ and $I = I_\lambda$. For $v \in M$ we have $I_\lambda v = 0$ if and only if $v \in M_\lambda$ and $\mathfrak{g}_\alpha v = 0$ for all $\alpha \in R^+$. An application of Lemma 5.2.4 finishes the proof. □

Theorem 5.3.4 Let $\lambda \in \mathfrak{h}^*$.

(1) The Verma module $\Delta(\lambda)$ has a maximal proper submodule $\mathrm{rad}(\Delta(\lambda))$.

(2) We denote by $L(\lambda) = \Delta(\lambda)/\mathrm{rad}(\Delta(\lambda))$ the quotient, which is a simple $\mathfrak{g}$-representation. This defines a bijection:

$$\begin{aligned} h^* &\xrightarrow{\cong} \{\text{simple highest weight representations up to isomorphism}\} \\ \lambda &\mapsto L(\lambda) \end{aligned}$$

(3) The simple quotient $L(\lambda)$ is finite-dimensional, $\dim(L(\lambda)) < \infty$ if and only if λ is dominant integral, i.e. $\lambda \in \mathcal{X}^+$.

This is the last missing part of Theorem 5.1.8.

Proof. (1) Every $\mathfrak{h}$-subrepresentation $N \subset \Delta(\lambda)$ (and thus every $\mathfrak{g}$-subrepresentation) decomposes into weight spaces, since $\Delta(\lambda)$ does so. To see this, let $w \in N \subset \Delta(\lambda)$ and decompose it as $w = w_1 + \cdots + w_n$ with $0 \neq w_i \in \Delta(\lambda)_{\mu_i}$ for pairwise distinct μ_i. We have to show $w_i \in N$. Suppose this were not the case for some $w \in N$ with $n > 1$ minimal. Then we can find $H \in \mathfrak{h}$ with $\mu_1(H) \neq \mu_2(H)$. Then with $w \in N$ and $H \cdot w \in N$, we also have $N \ni (H - \mu_1)w = \sum_{i=2}^{n}(\mu_i - \mu_1)w_i$ in contradiction to the minimality of n.

If N is a $\mathfrak{g}$-subrepresentation of $\Delta(\lambda)$, then $N_\lambda \neq 0$ implies $N = \Delta(\lambda)$. A proper subrepresentation thus satisfies $N \subset \bigoplus_{\mu \neq \lambda} \Delta(\lambda)_\mu$. As a consequence, the sum of all proper submodules of $\Delta(\lambda)$, denoted $\mathrm{rad}(\Delta(\lambda))$, is still a proper submodule and maximal with this property.

(2) The quotient $L(\lambda) = \Delta(\lambda)/\mathrm{rad}(\Delta(\lambda))$ is simple with highest weight λ and Theorem 5.1.7 shows that any simple representation L with highest weight λ must be isomorphic to it.

(3) One implication was shown in Theorem 5.1.6. It remains to show that $L(\lambda)$ is finite-dimensional provided λ is dominant integral. This will be done after some preparation. □

Definition 5.3.5 Recall the Weyl vector ρ from Lemma 4.6.7, the half sum of positive roots. The **dot action** of the Weyl group W on $\mathfrak{h}^*$ is defined by:

$$w \cdot \lambda := w(\lambda + \rho) - \rho \quad \text{for } w \in W, \lambda \in \mathfrak{h}^*$$

Note that the dot action has fixed point $W \cdot (-\rho) = -\rho$.

Lemma 5.3.6 For every simple root α and every weight $\lambda \in \mathfrak{h}^*$ with $\langle \lambda + \rho, \alpha^\vee \rangle \in \mathbb{Z}_{\geq 0}$ there exists an injection of $\mathfrak{g}$-representations

$$\Delta(s_\alpha \cdot \lambda) \hookrightarrow \Delta(\lambda)$$

In fact, this holds for any positive root $\alpha \in R^+$ as we will see later.

Proof. By Lemma 4.6.7 we have $\langle \rho, \alpha^\vee \rangle = 1$ and so $\langle \lambda + \rho, \alpha^\vee \rangle \in \mathbb{Z}_{\geq 0}$ if and only if $n := \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}_{\geq -1}$. Then we have $s_\alpha \cdot \lambda = s_\alpha(\lambda + \rho) - \rho = \lambda - (n+1)\alpha$ since $s_\alpha(\rho) = \rho - \alpha$.

For $x_\alpha \in \mathfrak{g}_\alpha$ and $y_\alpha \in \mathfrak{g}_{-\alpha}$ we now claim

$$x_\alpha y_\alpha^{n+1} v_\lambda = 0$$

If $n = -1$, this is clear. For $n \in \mathbb{Z}_{\geq 0}$ one can check that $y_\alpha^i v_\lambda$ forms the basis of a Verma module of $\mathfrak{g}^\alpha \cong \mathfrak{sl}(2, \mathbb{C})$ with highest weight vector v_λ. If further $[x_\alpha, y_\alpha] = \alpha^\vee$ and $\alpha^\vee$ acts on v_λ with eigenvalue $n+1$, then there exists an $(n+1)$-dimensional simple representation of $\mathfrak{sl}(2, \mathbb{C})$, which is a quotient of our Verma module. The corresponding kernel is spanned by the $y_\alpha^i v_\lambda$ with $i > n$, which thus form a submodule; in particular $x_\alpha y_\alpha^{n+1} v_\lambda = 0$. (Alternatively, one can also compute inductively $x_\alpha y_\alpha^i v_\lambda = i(n - i + 1) y_\alpha^{i-1} v_\lambda$.)

Furthermore, if α is a simple root, then $x_\beta y_\alpha^i v_\lambda = 0$ for any $\beta \in R^+ \setminus \{\alpha\}$ and $i \in \mathbb{Z}_{\geq 0}$ since $i\alpha - \beta$ is never a sum of positive roots. From $s_\alpha \cdot \lambda = \lambda - (n+1)\alpha$ we deduce $0 \neq y_\alpha^{n+1} v_\lambda \in \Delta(\lambda)_{s_\alpha \cdot \lambda}$. Thus $y_\alpha^{n+1} v_\lambda$ is of weight $s_\alpha \cdot \lambda$ and now Lemma 5.3.3 provides a nonzero morphism $\Delta(s_\alpha) \to \Delta(\lambda)$ that sends the canonical generator of $\Delta(s_\alpha)$ to $y_\alpha^{n+1} v_\langle$. By comparing bases as in Proposition 5.3.2 (with α ordered last), one checks that this map is injective. □

L26
End

Continuation of the proof of Theorem 5.3.4.(3). Lemma 5.3.6 shows that given $\lambda \in \mathfrak{h}^*$ and a simple root α with $\langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}_{\geq 0}$, a highest weight vector of $L(\lambda)$ always generates a finite-dimensional $\mathfrak{g}^\alpha$-subrepresentation of $L(\lambda)$. In every $\mathfrak{g}$-representation V, the sum W of all finite-dimensional $\mathfrak{g}^\alpha$-subrepresentations for a fixed $\alpha \in R$ is in fact a $\mathfrak{g}$-subrepresentation, see Exercise 54. Since $L(\lambda)$ is simple and if $\langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}_{\geq 0}$ for every simple root α, then $L(\lambda)$ is the sum of its finite-dimensional $\mathfrak{g}^\alpha$-subrepresentations for every such α. Using the characterization of finite-dimensional $\mathfrak{sl}(2, \mathbb{C})$-representations, we deduce $s_\alpha(P(L(\lambda))) = P(L(\lambda))$ for every simple reflection $s_\alpha \in W$, so $P(L(\lambda))$ is stable under the Weyl group. Since any Weyl group orbit $P(L(\lambda))$ intersects the dominant Weyl chamber, which contains only finitely many weights $\mu \leq \lambda$, we deduce that $P(L(\lambda))$ is finite. Since the weight spaces of $L(\lambda)$ are finite-dimensional, this implies $\dim_{\mathbb{C}}(L(\lambda)) < \infty$. □

Exercise 54 Let U be a representation of a Lie algebra $\mathfrak{a}$. The vectors $u \in U$ that lie in a finite-dimensional $\mathfrak{a}$-subrepresentation of U are called the **$\mathfrak{a}$-finite vectors** of U. If V is a representation of a finite-dimensional Lie algebra $\mathfrak{g}$ and $\mathfrak{a} \subset \mathfrak{g}$ a subalgebra, then the $\mathfrak{a}$-finite vectors of V form a $\mathfrak{g}$-subrepresentation $V_{\mathfrak{a}}$ of V.

Exercise 55 Show that a Verma module $\Delta(\lambda)$ for $\mathfrak{sl}(2,\mathbb{C})$ is simple if and only if it has no finite-dimensional quotient, which holds if and only if $\langle\lambda,\alpha\rangle \notin \mathbb{Z}_{\geq 0}$, where α denotes the positive root.

Exercise 56 For dominant λ, show that the sum over the images of $\Delta(s_\alpha \cdot \lambda) \hookrightarrow \Delta(\lambda)$, where α ranges over the positive roots, is the maximal proper submodule of $\Delta(\lambda)$. Hint: similar to the proof of Theorem 5.3.4.(3); show that the vectors $y_\alpha^{\langle\lambda,\alpha\rangle+1}$ for simple roots α generate the maximal proper submodule of $\Delta(\lambda)$, even as $U(\mathfrak{n})$-submodule.

Exercise 57 Let V be a simple representation of a semisimple Lie algebra and $0 \neq v \in V$ such that $g_\alpha v = 0$ for all $\alpha \in R^+$. Show that v is a highest weight vector of V.

5.4 Weyl character formula

We retain the notation and conventions from the previous subsection. In particular, $\mathfrak{g}$ is a complex semisimple Lie algebra with Cartan subalgebra $\mathfrak{h}$, Weyl group W, root system R, system of positive roots R^+, weight lattice $\mathcal{X}$, dominant integral weights $\mathcal{X}^+$, and Weyl vector ρ.

One goal of this section is the following.

Theorem 5.4.1 [Weyl's dimension formula] For every dominant integral weight $\lambda \in \mathcal{X}^+$, the dimension of the simple representation $L(\lambda)$ of highest weight λ is given by:

$$\dim_{\mathbb{C}}(L(\lambda)) = \frac{\prod_{\alpha\in R^+}\langle\lambda+\rho,\alpha^\vee\rangle}{\prod_{\alpha\in R^+}\langle\rho,\alpha^\vee\rangle}$$

As in the case of $\mathfrak{sl}(2,\mathbb{C})$, the dimensions of simple representations can be recovered from their characters that we generalize next.

Construction 5.4.2 Consider $\mathfrak{h}^*$ as an (abelian) group and form the group ring $\mathbb{Z}\mathfrak{h}^*$ and call it the **character ring** of $\mathfrak{g}$. For an element $\lambda \in \mathfrak{h}^*$ we write $e^\lambda \in \mathbb{Z}\mathfrak{h}^*$ for the corresponding group ring element (In this way, we have a clear distinction between $e^{\lambda+\mu}$ and $e^\lambda + e^\mu$). The e^λ for $\lambda \in \mathfrak{h}^*$ thus form a $\mathbb{Z}$-basis of $\mathbb{Z}\mathfrak{h}^*$ and the multiplication is determined by $e^\lambda e^\mu = e^{\lambda+\mu}$.

The ring $\mathbb{Z}\mathfrak{h}^*$ is an integral domain. (Any two elements are always contained in a subring $\mathbb{Z}E$ for a finitely generated subgroup $E \subset \mathfrak{h}^*$. Since E is free abelian, $\mathbb{Z}E$ is isomorphic to a ring of Laurent polynomials in multiple variables.) We denote the fraction field by $\mathrm{Frac}(\mathbb{Z}\mathfrak{h}^*)$.

For a finite-dimensional representation V of $\mathfrak{g}$, the **character** $\mathrm{Ch}(V) \in \mathbb{Z}\mathfrak{h}^*$ is defined by:

$$\mathrm{Ch}(V) = \sum_{\mu\in\mathfrak{h}^*} (\dim_{\mathbb{C}}(V_\mu))e^\mu$$

The Weyl group acts on $\mathfrak{h}^*$ and thus on $\mathbb{Z}\mathfrak{h}^*$. The character $\mathrm{Ch}(V)$ is fixed by W. In fact, suitable powers of generators of $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$ induce isomorphisms between the weight spaces for λ and $s_\alpha(\lambda)$.

Theorem 5.4.3 [Weyl's character formula] For every dominant integral weight $\lambda \in \mathcal{X}^+$, the character of the simple representation $L(\lambda)$ of highest weight λ is computed in $\mathrm{Frac}(\mathbb{Z}\mathfrak{h}^*)$ by:

$$\mathrm{Ch}(L(\lambda)) = \frac{\sum_{w\in W}(-1)^{l(w)}e^{w(\lambda+\rho)}}{\sum_{w\in W}(-1)^{l(w)}e^{w(\rho)}}$$

The proof will take some work.

Examples 5.4.4 As sanity check we observe $\mathrm{Ch}(L(0)) = e^0$. For $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{C})$ we have $\rho = \alpha/2$ and $\mathcal{X}^+ = \mathbb{Z}_{\geq 0}\rho$. For $n \in \mathbb{Z}_{\geq 0}$ we get:

$$\mathrm{Ch}(L(n\rho)) = \frac{e^{(n+1)\rho} - e^{-(n+1)\rho}}{e^\rho - e^{-\rho}} = e^{n\rho} + e^{(n-2)\rho} + \cdots + e^{-n\rho}$$

as in Consequences 2.1.5.(1) with $q := e^\rho$.

The following construction extends the character ring, so that we can define characters of certain infinite-dimensional representations such as Verma modules.

Construction 5.4.5 (1) We write $\mathbb{Z}^{\mathfrak{h}^*}$ for the set of functions $f\colon \mathfrak{h}^* \to \mathbb{Z}$. We write these as formal expressions $f = \sum_{\lambda\in h^*} f(\lambda)e^\lambda$. Note that $\mathbb{Z}\mathfrak{h}^*$ injects into $\mathbb{Z}^{\mathfrak{h}^*}$ as the finitely supported functions.

(2) If V is a representation of $\mathfrak{g}$ (or even just $\mathfrak{h}$) with finite-dimensional weight spaces, then we define $\mathrm{Ch}(V) \in \mathbb{Z}^{\mathfrak{h}^*}$ by the familiar formula:
$$\mathrm{Ch}(V) = \sum_{\mu \in \mathfrak{h}^*} (\dim_{\mathbb{C}}(V_\mu)) e^\mu$$
(Note that we have not defined a multiplication on $\mathbb{Z}^{\mathfrak{h}^*}$ that extends the multiplication on $\mathbb{Z}\mathfrak{h}^*$.)

(3) Let $\mathbb{Z}^{<}\mathfrak{h}^* \subset \mathbb{Z}^{\mathfrak{h}^*}$ denote the set of functions $f : \mathfrak{h}^* \to \mathbb{Z}$ that is supported on a union of finitely many sets of the form $\lambda - \mathbb{Z}_{\geq 0} R^+$, i.e. sets of the form $\{\lambda - \sum_{\alpha \in R^+} n(\alpha)\alpha \mid n : R^+ \to \mathbb{Z}_{\geq 0}\}$. In particular we have $\mathbb{Z}\mathfrak{h}^* \subset \mathbb{Z}^{<}\mathfrak{h}^*$ and the multiplication on the former extends to the latter by setting
$$(fg)(\nu) = \sum_{\lambda + \mu = \nu} f(\lambda) g(\mu).$$
The support condition guarantees that these sums have finitely many nonzero terms. The ring $\mathbb{Z}^{<}\mathfrak{h}^*$ is called the **extended character ring** of $\mathfrak{g}$.

Remarks 5.4.6 (1) If M, N are two $\mathfrak{h}$-representations that are the sum of their finite-dimensional weight spaces and $\mathrm{Ch}(M), \mathrm{Ch}(N) \in \mathbb{Z}^{<}\mathfrak{h}^*$, then $\mathrm{Ch}(M \otimes N) = \mathrm{Ch}(M)\,\mathrm{Ch}(N)$.

(2) The character of a Verma module is $\mathrm{Ch}(\Delta(\lambda)) = e^\lambda \prod_{\alpha \in R^+}(1 + e^{-\alpha} + e^{-2\alpha} + \cdots)$. In particular, in $\mathbb{Z}^{<}\mathfrak{h}^*$ we have
$$\Big(\prod_{\alpha \in R^+}(1 - e^{-\alpha})\Big)\,\mathrm{Ch}(\Delta(\lambda)) = e^\lambda$$
(This follows from $\prod_{\alpha \in R^+}(1 + e^{-\alpha} + e^{-2\alpha} + \cdots) = \sum_\mu \mathcal{P}(\mu) e^{-\mu}$.)

For the following, let $\overline{\kappa} : \mathfrak{h} \to \mathfrak{h}^*$ denote the isomorphism induced by the Killing form. It is characterized by $\langle \overline{\kappa}(h), h' \rangle = \kappa(h, h')$ for $h, h' \in \mathfrak{h}$. We let $(-,-)$ denote the bilinear form on $\mathfrak{h}^*$ corresponding under $\overline{\kappa}$ to the Killing form on $\mathfrak{h}$. If $\overline{\kappa}$ sends $h \mapsto \lambda$, then for all $\mu \in \mathfrak{h}^*$ we have $\mu(h) = (\lambda, \mu)$. This bilinear form is positive definite on the $\mathbb{Q}$-vector space $\mathbb{Q}R$ spanned by the roots, see Proposition 4.4.19.(2). It is also invariant under the Weyl group as we shall see.

Lemma 5.4.7 The restriction of the Killing form of a complex semisimple Lie algebra to a Cartan subalgebra is invariant under the Weyl group.

Proof. For $x, y \in \mathfrak{h}$ and $w \in W$ we compute:
$$\kappa(x, y) = \mathrm{tr}(\mathrm{ad}_x \circ \mathrm{ad}_y) = \sum_{\alpha \in R} \langle \alpha, x \rangle \langle \alpha, y \rangle$$
$$\kappa(w(x), w(y)) = \sum_{\alpha \in R} \langle \alpha, w(x) \rangle \langle \alpha, w(y) \rangle = \sum_{\alpha \in R} \langle w^{-1}(\alpha), x \rangle \langle w^{-1}(\alpha), y \rangle = \sum_{\beta \in R} \langle \beta, x \rangle \langle \beta, y \rangle \qquad \square$$

For the second equation in the second line recall that w is a product of reflections.

Lemma 5.4.8 Every endomorphism of a Verma module is multiplication by a scalar.

Proof. Consider the maps $\mathbb{C} \hookrightarrow \mathrm{End}_{\mathfrak{g}}(\Delta(\lambda)) \hookrightarrow \mathrm{End}_{\mathbb{C}}(\Delta(\lambda)_\lambda)$. The second map is injective since $\Delta(\lambda)_\lambda$ generates $\Delta(\lambda)$. The composition is bijective, since the highest weight space of a Verma module is 1-dimensional. Thus the component maps are bijections too. $\square$

Lemma 5.4.9 The Casimir operator $C = C_\kappa$ from Construction 4.2.6 acts on $\Delta(\lambda)$ by the scalar $c_\lambda = (\rho + \lambda, \rho + \lambda) - (\rho, \rho)$ using the bilinear form corresponding to the Killing form.

Remark 5.4.10 Combined with Lemma 5.3.6, this lemma also shows that $(\lambda, \lambda) = (w(\lambda), w(\lambda))$ for $\lambda \in \mathcal{X}$ and $w \in W$.

Proof of Lemma 5.4.9. By Exercise 53 we may consider $C \in Z(U(\mathfrak{g}))$ and by Lemma 5.4.8 we only have to compute the scalar by which it acts on $\Delta(\lambda)_\lambda$. For $\alpha \in R^+$ choose root vectors $x_\alpha \in \mathfrak{g}_\alpha$ and $y_\alpha \in \mathfrak{g}_{-\alpha}$ with $\kappa(x_\alpha, y_\alpha) = 1$ and an orthonormal basis $\{h_1, \dots, h_n\}$ of $\mathfrak{h}$ under the Killing form. Then we compute:

$$\begin{aligned} C &= \sum_{\alpha \in R^+} y_\alpha x_\alpha + x_\alpha y_\alpha + \sum_{i=1}^n h_i^2 \\ &= \sum_{\alpha \in R^+} 2y_\alpha x_\alpha + [x_\alpha, y_\alpha] + \sum_{i=1}^n h_i^2 \end{aligned} \tag{9}$$

Since $x_\alpha \Delta(\lambda)_\lambda = 0$, this expression acts on $\Delta(\lambda)_\lambda$ by the scalar:

$$c_\lambda = \sum_{\alpha \in R^+} \lambda([x_\alpha, y_\alpha]) + \sum_{i=1}^n \lambda(h_i)^2$$

Writing $\lambda = \overline{\kappa}(h)$, this transforms into:

$$c_\lambda = \sum_{\alpha \in R^+} \kappa(h, [x_\alpha, y_\alpha]) + \sum_{i=1}^n \kappa(h, h_i)^2$$

Now $\kappa(h, [x_\alpha, y_\alpha]) = \kappa([h, x_\alpha], y_\alpha) = \alpha(h)\kappa(x_\alpha, y_\alpha) = \alpha(h)$ and if $h = \sum_{i=1}^n d_i h_i$, then

$$\sum_{i=1}^n \kappa(h, h_i)^2 = \sum_{i=1}^n d_i^2 = \sum_{i,j=1}^n \kappa(d_i h_i, d_j h_j) = \kappa(h, h),$$

so we have $c_\lambda = 2\rho(h) + \kappa(h, h) = (2\rho, \lambda) + (\lambda, \lambda) = (\rho + \lambda, \rho + \lambda) - (\rho, \rho)$. □

L27 End

Remark 5.4.11 (Freudenthal's formula) From the proof of Lemma 5.4.9 we can already extract a formula for the characters of simple representations.

First, for $\mathfrak{sl}(2, \mathbb{C})$ with standard basis e, h, f and its $(m+1)$-dimensional representation $L(m\rho)$, the element $fe \in U(\mathfrak{sl}(2, \mathbb{C}))$ acts on every nonzero weight space $L(m\rho)_{m\rho - i\alpha}$ by the scalar $i(m - i + 1)$. Setting $\mu = m\rho - i\alpha$, this can be expressed as:

$$i(m - i + 1) = \sum_{j=1}^i 1(m - 2(i - j)) = \sum_{j \geq 1} (\dim_\mathbb{C}(L(m\rho)_{\mu + j\alpha})) \langle \mu + j\alpha, \alpha^\vee \rangle$$

The right-hand side is zero for all μ with $L(m\rho)_\mu = 0$. Thus for all finite-dimensional representations of $\mathfrak{sl}(2, C)$ and all weights μ we have

$$\operatorname{tr}(fe|_{V_\mu}) = \sum_{j \geq 1} (\dim_\mathbb{C}(V_{\mu + j\alpha})) \langle \mu + j\alpha, \alpha^\vee \rangle$$

In the context of a general complex semisimple $\mathfrak{g}$, given $x \in \mathfrak{g}_\alpha$ and $y \in \mathfrak{g}_{-\alpha}$ with $[x, y] = \alpha^\vee$, then $\kappa(h, [x, y]) = \alpha(h)\kappa(x, y)$ and $\kappa(\alpha^\vee, \alpha^\vee) = 2\kappa(x, y)$, see (4). Conversely, any x_α and y_α with $\kappa(x_\alpha, y_\alpha) = 1$ give rise to an $\mathfrak{sl}(2)$-triple x_α, $\alpha^\vee$, $\kappa(\alpha^\vee, \alpha^\vee)/2y_\alpha$. For $\lambda \in \mathbb{Q}R$ we write $|\lambda| := \sqrt{(\lambda, \lambda)}$. For the trace of the Casimir operator on $L(\lambda)_\mu$ we get from the proof of Lemma 5.4.9:

$$\begin{aligned} \operatorname{tr}(C|_{L(\lambda)_\mu}) &= (\dim_\mathbb{C}(L(\lambda)_\mu))(|\lambda + \rho|^2 - |\rho|^2) \\ \operatorname{tr}(C|_{L(\lambda)_\mu}) &= \sum_{\alpha \in R^+} (2/\kappa(\alpha^\vee, \alpha^\vee)) \sum_{j \geq 1} (\dim_\mathbb{C}(L(\lambda)_{\mu + j\alpha})) \langle \mu + j\alpha, \alpha^\vee \rangle + \dim_\mathbb{C}(L(\lambda)_\mu)(|\mu + \rho|^2 - |\rho|^2) \end{aligned}$$

The first equation follows since C acts by the same scalar on $L(\lambda)$ as on $\Delta(\lambda)_\lambda$; the trace on the weight space $L(\lambda)_\mu$ is the shown multiple of this scalar. The second equation follows from the expression (9): the first summand is rewritten using our expression for $\operatorname{tr}(fe|_{V_\mu})$ and the remaining two as in the proof of Lemma 5.4.9, but with μ in place of λ and with a factor $\dim_\mathbb{C}(L(\lambda)_\mu)$ appearing in the trace of the relevant scalar endomorphisms.

Comparing the two formulas using $\overline{\kappa}(\alpha^\vee) = 2\alpha/(\alpha, \alpha)$ gives **Freudenthal's formula**

$$(\dim_\mathbb{C}(L(\lambda)_\mu))(|\lambda + \rho|^2 - |\mu + \rho|^2) = 2 \sum_{\alpha \in R^+} \sum_{j \geq 1} (\dim_\mathbb{C}(L(\lambda)_{\mu + j\alpha}))(\mu + j\alpha, \alpha)$$

which enables an inductive computation of the weight spaces of simple representations based on the dimensions of the weight spaces of higher weights.

Lemma 5.4.12 [Composition series of Verma modules] Every Verma module $\Delta(\lambda)$ has finite length (the length of a chain of proper submodules with simple subquotients is finite) and every simple subquotient is a simple module $L(\mu)$ with $\mu \leq \lambda$ and $(\mu+\rho, \mu+\rho) = (\lambda+\rho, \lambda+\rho)$.

Proof. The second statement follows from Theorem 5.1.5.(1) and Lemma 5.4.9 since the Casimir operator must act on every subquotient of $\Delta(\lambda)$ by the scalar c_λ. This implies that there are only finitely many μ that are candidates for highest weights of subquotients of $\Delta(\lambda)$. To see this, note first that $\mu \leq \lambda$ implies $\mu = \lambda + \nu$ with $\nu \in \mathbb{Z}R$. We argue that there are only finitely many $\nu \in \mathbb{Z}R$ such that $(\lambda+\rho, \lambda+\rho) = (\lambda+\nu+\rho, \lambda+\nu+\rho)$. For $\lambda \in \mathbb{R}R$ this is straightforward because the bilinear form $(-,-)$ is positive definite by Proposition 4.4.19 and every discrete, compact subset is finite. For general λ the equation is equivalent to $(\nu,\nu) + 2(\lambda+\rho, \nu) = 0$ and so all its solutions are contained in the subspace $A := \{\nu \in \mathbb{Q}R \mid (\lambda, \nu) \in \mathbb{Q}\}$. Now one can find $\lambda' \in \mathbb{Q}R$ with $(\lambda', \nu) = (\lambda, \nu)$ for all $\nu \in A$ and by replacing λ by λ' we return to the known case.

Every nonzero subquotient S of $\Delta(\lambda)$ has itself a simple subquotient (a general fact reminiscent of Exercise 38), thus there exists some μ with $(\mu+\rho, \mu+\rho) = (\lambda+\rho, \lambda+\rho)$ and $S_\mu \neq 0$. The length of a properly descending filtration of $\Delta(\lambda)$ can thus be estimated by:

$$l(\Delta(\lambda)) \leq \sum \dim_{\mathbb{C}}(\Delta(\lambda)_\mu)$$

where the sum runs over all $\mu \leq \lambda$ with $(\mu+\rho, \mu+\rho) = (\lambda+\rho, \lambda+\rho)$. □

Theorem 5.4.13 [Kostant's character formula] For every $\lambda \in \mathcal{X}^+$, the character of the simple module $L(\lambda)$ is the alternating sum over the characters of Verma modules with highest weights in the orbit of λ under the dot action of W. In formulas:

$$\mathrm{Ch}(L(\lambda)) = \sum_{w \in W} (-1)^{l(w)} \mathrm{Ch}(\Delta(w \cdot \lambda))$$

Example 5.4.14 For $\mathfrak{sl}(2, \mathbb{C})$ and $m \in \mathbb{Z}_{\geq 0}$, the embedding of Verma modules from Lemma 5.3.6 induces a short exact sequence

$$\Delta((-m-2)\rho) \hookrightarrow \Delta(m\rho) \twoheadrightarrow L(m\rho)$$

and we obtain $\mathrm{Ch}(L(m\rho)) = \mathrm{Ch}(\Delta(m\rho)) - \mathrm{Ch}(\Delta((-m-2)\rho)) = \mathrm{Ch}(\Delta(m\rho)) - \mathrm{Ch}(\Delta(s_\alpha \cdot m\rho))$ as claimed.

Proof. For $\lambda \in \mathbb{Q}R$ we write $|\lambda| = \sqrt{(\lambda, \lambda)}$. Lemma 5.4.12 says that we can express the character of $\Delta(\lambda)$ as

$$\mathrm{Ch}(\Delta(\lambda)) = \sum_{\substack{\mu \leq \lambda \\ |\mu+\rho| = |\lambda+\rho|}} a_l^\mu \, \mathrm{Ch}(L(\mu))$$

for some $a_\lambda^\mu \in \mathbb{Z}_{\geq 0}$ with $a_\lambda^\lambda = 1$. These numbers form an upper triangular matrix with ones on the diagonal, which can be inverted over $\mathbb{Z}$. Thus we can write:

$$\mathrm{Ch}(L(\lambda)) = \sum_{\substack{\mu \leq \lambda \\ |\mu+\rho| = |\lambda+\rho|}} b_\lambda^\mu \, \mathrm{Ch}(\Delta(\mu))$$

for suitable $b_\lambda^\mu \in \mathbb{Z}$ with $b_\lambda^\lambda = 1$. This holds for arbitrary $\lambda \in \mathfrak{h}^*$ (if we avoid the notation $|.|$). Now we assume that λ is dominant, then $L(\lambda)$ is finite-dimensional by Theorem 5.3.4.(3) and hence $\mathrm{Ch}(L(\lambda))$ is invariant under the Weyl group by Theorem 5.1.6. Now we multiply both sides of our equation by $\prod_{\alpha \in R^+}(e^{\alpha/2} - e^{-\alpha/2}) = e^\rho \prod_{\alpha \in R^+}(1 - e^{-\alpha})$ to get

$$\left(\prod_{\alpha \in R^+} (e^{\alpha/2} - e^{-\alpha/2})\right) \mathrm{Ch}(L(\lambda)) = \sum_\mu b_\lambda^\mu e^{\mu+\rho} = \sum_\nu d_\nu e^\nu$$

where we write $d_\nu = b_\lambda^{\nu-\rho}$ and note $d_{\lambda+\rho} = 1$ and $d_\nu = 0$ if $|\nu| \neq |\lambda+\rho|$ or $\nu \not\leq \lambda+\rho$. The left-hand side is negated by every simple reflection (see Lemma 4.6.4), so we also must have $d_\nu = (-1)^{l(w)} d_{w(\nu)}$ for all $w \in W$. In particular, we must have $d_\nu = 0$ unless $|\nu| = |\lambda+\rho|$ and $w(\nu) \leq \lambda+\rho$ for every $w \in W$. In the following lemma we will also see that $d_\nu = 0$ unless $\nu \in W(\lambda+\rho)$. Upon passing from ν back to $\mu+\rho$ we obtain the desired character formula. □

Lemma 5.4.15 Let $\mu \in \mathcal{X}^+$ and $\nu \in \mathcal{X}$. If $|\nu| = |\mu|$ and $w(\nu) \leq \mu$ for all $w \in W$, then $\nu \in W(\mu)$.

Proof. Every integral weight has a W-conjugate in $\mathcal{X}^+$ and their "absolute values" $|-|$ agree by Remark 5.4.10. Without loss of generality, we assume $\nu \in \mathcal{X}^+$. It remains to prove for $\mu, \nu \in \mathcal{X}^+$ that $\nu \leq \mu$ and $|\nu| = |\mu|$ imply $\nu = \mu$.

Since the scalar product of any vector from the dominant Weyl chamber with a positive root is positive, we must have $(\mu - \nu, \nu) \geq 0$. Then $0 = (\mu, \mu) - (\nu, \nu) = (\mu - \nu, \mu - \nu) + (\nu, \mu - \nu) + (\mu - \nu, \nu) \geq 0$ and the final equality forces $\mu = \nu$. □

We are now ready to prove Weyl's character formula.

Proof of Theorem 5.4.3. In the proof of Kostant's character formula we obtained the formula

$$\left(\prod_{\alpha \in R^+} (e^{\alpha/2} - e^{-\alpha/2})\right) \mathrm{Ch}(L(\lambda)) = \sum_{w \in W} (-1)^{l(w)} e^{w(\lambda+\rho)}$$

Dividing this by its specialization at $\lambda = 0$ produces Weyl's character formula. □

Remark 5.4.16 The specialization of the above formula at $\lambda = 0$ is called **Weyl's denominator formula**:

$$e^\rho \prod_{\alpha \in R^+} (1 - e^{-\alpha}) = \prod_{\alpha \in R^+} (e^{\alpha/2} - e^{-\alpha/2}) = \sum_{w \in W} (-1)^{l(w)} e^{w(\rho)}$$

Finally we prove Weyl's dimension formula.

Proof of Theorem 5.4.1. It is tempting to use the ring homomorphism $a\colon \mathbb{Z}\mathfrak{h}^* \to \mathbb{Z}$ with $a(e^\lambda) = 1$ for all $\lambda \in \mathfrak{h}^*$. When applied to Weyl's character formula, this unfortunately only returns the useless relation $0 \dim(L(\lambda)) = 0$. An abstract version of de l'Hospital's rule comes to the rescue. Consider the subring $\mathbb{Z}\mathcal{X} \subset \mathbb{Z}\mathfrak{h}^*$ and for $\alpha \in R^+$ the group homomorphism $\partial_\alpha\colon \mathbb{Z}\mathcal{X} \to \mathbb{Z}\mathcal{X}$ define by $\partial_\alpha(e^\mu) = \langle \mu, \alpha^\vee \rangle e^\mu$. One can check easily that ∂_α is a derivation and ∂_α and ∂_β commute for $\alpha, \beta \in R^+$. Define $D = \prod_{\alpha \in R^+} \partial_\alpha \in \mathrm{End}(\mathbb{Z}\mathcal{X})$. Combined with the ring homomorphism a we find $a(D(e^\mu)) = \prod_{\alpha \in R^+} \langle \mu, \alpha^\vee \rangle$. Lemma 4.6.4 then implies $a(D(e^{w(\mu)})) = (-1)^{l(w)} a(D(e^\mu))$ for all $w \in W$. Now we apply $a \circ D$ to

$$\left(e^\rho \prod_{\alpha \in R^+} (1 - e^{-\alpha})\right) \mathrm{Ch}(L(\lambda)) = \sum_{w \in W} (-1)^{l(w)} e^{w(\lambda+\rho)}$$

and obtain

$$(a \circ D)\left(e^\rho \prod_{\alpha \in R^+} (1 - e^{-\alpha})\right) a(\mathrm{Ch}(L(\lambda))) = |W| \prod_{\alpha \in R^+} \langle \lambda + \rho, \alpha^\vee \rangle$$

because if one of the factors $(1 - e^{-\alpha})$ is not hit by a derivation, it vanishes under a. Now we note $a(\mathrm{Ch}(L(\lambda))) = \dim(L(\lambda))$ and divide the equation by its specialization at $\lambda = 0$ to obtain Weyl's dimension formula. Note that the latter is non-zero since ρ lies in the dominant Weyl chamber and thus not on any reflecting hyperplane. □

L28
End