Lie Algebras
Theorem 3.2.2

Jacob Stegemann

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We refine the statement and the proof of (Thm 3.2.2(1)) from `https://www.math.uni-hamburg.de/home/wedrich/Lie-Algebras.pdf` pages 22 ff.

Recall the statement:

Theorem 3.2.2. *Let V be a finite dimensional vector space and $\mathfrak{g} \subset \mathfrak{gl}(V)$ a Lie subalgebra, such that*

$$\forall x \in \mathfrak{g} : x \text{ is nilpotent as an endomorphism of } V.$$

Then the following hold

(1) If $V \neq 0$, then there exists $v \in V, v \neq 0$, such that

$$\forall x \in \mathfrak{g} : x(v) = 0$$

Notation: $\mathfrak{g}v = 0$.

(2) [...]

However the condition $\mathfrak{g} \subset \mathfrak{gl}(V)$ is very restrictive. So restrictive in fact that the proof of (2) as given in the lecture uses a more general version of (1), which can be stated as follows:

Theorem 3.2.2 (more general). *Let $\mathfrak{g}$ be a Lie algebra and V be a finite dimensional $\mathfrak{g}$ representation, such that*

$$\forall x \in \mathfrak{g} : \rho(x) \text{ is nilpotent as an endomorphism of } V.$$

Then the following hold

(1) If $V \neq 0$, then there exists $v \in V, v \neq 0$, such that

$$\forall x \in \mathfrak{g} : \rho(x)(v) = 0 \quad (i.e.\ x.v = 0)$$

(2) [...]

But it turns out that this general version actually follows from the special case $\mathfrak{g} \subset \mathfrak{gl}(V)$. To see this, we define $\mathfrak{g}' := \mathrm{im}(\rho) \subset \mathfrak{gl}(V)$ and consider the factorization

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "\rho"] \arrow[rr, bend right, "\rho"'] & \mathfrak{g}' \arrow[r, hook] & \mathfrak{gl}(V). \end{tikzcd}$$

If now the original theorem holds for $\mathfrak{g}'$, then the new theorem will hold for $\mathfrak{g}$, since the elements of $\mathfrak{g}'$ are exactly those of the form $\rho(x)$ for $x \in \mathfrak{g}$.

Thus we are left to prove the original theorem:

Proof. The proof proceeds by induction on $\dim \mathfrak{g}$. To be precise, we prove:

$$\forall d \in \mathbb{N}\ \forall V\ \forall \mathfrak{g} \subset \mathfrak{gl}(V), \dim \mathfrak{g} = d : [...],$$

in particular V is **not** fixed.

For $\dim \mathfrak{g} = 1$, we have $\mathfrak{g} = \mathrm{span}\{x\}$. Since $x : V \to V$ is nilpotent and $V \neq 0$, its kernel is nonzero and we can choose $v \in \ker(x), v \neq 0$.

Now let $\dim \mathfrak{g} > 1$ and assume the statement holds for all Lie algebras of smaller dimension.

The remaining part of the proof consists of two steps:

(1) Decompose $\mathfrak{g}$ into smaller parts to apply the induction hypothesis.

(2) Find $v \in V, v \neq 0$ such that $\mathfrak{g}v = 0$.

Step 1 (decompose $\mathfrak{g}$)*:*

The idea is to write $\mathfrak{g} = \mathfrak{h} + \mathrm{span}\{l\}$ and use the induction hypothesis on $\mathfrak{h}$. However there are two ways to go about this: Either we first choose $l \in \mathfrak{g}$ to have some nice properties and then $\mathfrak{h}$ to fill up the rest, or we choose $\mathfrak{h} \subset \mathfrak{g}$ first and then $l \notin \mathfrak{h}$ to interact nicely with $\mathfrak{h}$.
We will follow the second approach: One reason for the first approach being hard is that we need to assure that $\mathfrak{h}$ is still a Lie algebra to apply the induction hypothesis. It is not at all obvious that $\mathfrak{h}$ has dimension exactly one less than $\mathfrak{g}$ (e.g. 0, $\{1\text{-dim} \subset \mathfrak{g}\}$ and $\mathfrak{g}$ could be the only subalgebras of $\mathfrak{g}$ (e.g. $\mathbb{R}^3$ with $\times$)).
This is why we define $\mathfrak{h}$ as:

Let $\mathfrak{h} \subsetneq \mathfrak{g}$ be a maximal[1] Lie subalgebra.

[1]It suffices to choose $\mathfrak{h}$ maximal with respect to $\subset$ instead of dim.

We still need to find $l \in \mathfrak{g}$ such that $\mathfrak{h} + \operatorname{span}\{l\} = \mathfrak{g}$. Obviously $l \notin \mathfrak{h}$ is necessary. But how do we make sure that $\mathfrak{h}+\operatorname{span}\{l\}$ is not just any subset of $\mathfrak{g}$ but in fact all of it?
This is where the maximality of $\mathfrak{h}$ comes into play. If we choose $l \in \mathfrak{g}, l \notin \mathfrak{h}$ such that $\mathfrak{h} + \operatorname{span}\{l\}$ is again a Lie algebra, then

$$\mathfrak{h} \overset{l \notin \mathfrak{h}}{\subsetneq} \mathfrak{h} + \operatorname{span}\{l\} \subset \mathfrak{g}$$

implies $\mathfrak{h} + \operatorname{span}\{l\} = \mathfrak{g}$.

But when is $\mathfrak{h} + \operatorname{span}\{l\}$ again a Lie algebra?
The only non trivial condition is "closed under $[-,-]$". I.e. we need to require for l to satisfy $[\mathfrak{h}, l] \subset \mathfrak{h} + \operatorname{span}\{l\}$. It's hard to motivate at this point, but to find v in the end, we actually want $[\mathfrak{h}, l] \subset \mathfrak{h}$. The intuition is that this ties the action of l more closely to that of $\mathfrak{h}$. For more details please refer to the $(*)$-Problem at the end of this document.
To summarize, we want to find $l \in \mathfrak{g}, l \notin \mathfrak{h}$ such that $[\mathfrak{h}, l] \subset \mathfrak{h}$.[a]

[a] To clarify: This will not finish the proof, it just decomposes $\mathfrak{g}$ into more handy pieces, namely $\mathfrak{h}$ and l.

The **ingenious idea** of the proof is to find l (more precisely $\bar{l}$) as the vector given by the induction hypothesis applied to $\mathfrak{h}$ (more precisely $\overline{\operatorname{ad}}(\mathfrak{h})$). For this we need a clever representation of $\mathfrak{h}$. But what representation should we choose?

We try to reverse engineer this clever representation U of $\mathfrak{h}$. The induction hypothesis would give $u \in U, u \neq 0$ such that $\mathfrak{h}u = 0$.
Somehow we have to find the connections between:

$$\begin{aligned} u \in U \quad &\longleftrightarrow \quad l \in \mathfrak{g}, \\ u \neq 0 \quad &\longleftrightarrow \quad l \notin \mathfrak{h}, \\ \mathfrak{h}u = 0 \quad &\longleftrightarrow \quad [\mathfrak{h}, l] \subset \mathfrak{h}. \end{aligned}$$

Introducing a 0 on the right hand side of $l \notin \mathfrak{h}$ is easy, if we just quotient by $\mathfrak{h}$ (thus $u = \bar{l}$). Since we need $l \in \mathfrak{g}$ we can already guess that $U = \mathfrak{g}/\mathfrak{h}$. After taking this quotient, the third correspondence can be refined to:

$$\mathfrak{h}\bar{l} = 0 \quad \longleftrightarrow \quad \overline{[\mathfrak{h}, l]} = 0.$$

The left hand side says $\forall y \in \mathfrak{h} : y.\bar{l} = 0$ and the right hand says $\forall y \in \mathfrak{h} : \overline{[y, l]} = 0$. Thus a reasonable guess would be to define the action of $y \in \mathfrak{h}$ on $\overline{x} \in U = \mathfrak{g}/\mathfrak{h}$ as $y.\overline{x} := \overline{[y, x]} = \overline{\mathrm{ad}_y(x)}$.

Consider the finite dimensional $\mathfrak{h}$ representation

$$\begin{aligned} \overline{\mathrm{ad}} : \mathfrak{h} &\to \mathfrak{gl}(\mathfrak{g}/\mathfrak{h}) \\ y &\mapsto \overline{\mathrm{ad}}_y := \overline{\mathrm{ad}_y} \end{aligned}$$

To see that this indeed is a representation, we can define it formally by first considering the representation

$$\mathrm{ad}|_{\mathfrak{h}} : \mathfrak{h} \hookrightarrow \mathfrak{g} \xrightarrow{\mathrm{ad}} \mathfrak{gl}(\mathfrak{g})$$

and then taking the quotient representation

$$\overline{\mathrm{ad}} : \mathfrak{h} \to \mathfrak{gl}(\mathfrak{g}/\mathfrak{h}).$$

But now $\mathfrak{h} \not\subset \mathfrak{gl}(\mathfrak{g}/\mathfrak{h})$, so we cannot simply apply our induction hypothesis. We fall back to the same trick as before, by considering $\mathfrak{h}' := \overline{\mathrm{ad}}(\mathfrak{h})$, i.e. the image of the representation. Then $\dim \mathfrak{h}' \leq \dim \mathfrak{h} < \dim \mathfrak{g}$, since we chose $\mathfrak{h} \subsetneq \mathfrak{g}$.

We have to check that $\mathfrak{h}'$ acts by nilpotent endomorphisms. Since $\mathfrak{g} \subset \mathfrak{gl}(V)$ acts by nilpotent endomorphisms, we can apply (Lem 3.2.1) on every element of $\mathfrak{g}$, to conclude that for every $x \in \mathfrak{g}$ the endomorphism $\mathrm{ad}_x \in \mathfrak{gl}(\mathfrak{g})$ is nilpotent. This is inherited to $\overline{\mathrm{ad}}$ as follows: For every $y \in \mathfrak{h}$ the endomorphism $\overline{\mathrm{ad}}_y \in \mathfrak{gl}(\mathfrak{g}/\mathfrak{h})$ is nilpotent.

Thus we can apply the induction hypothesis, to find $\bar{l} \in \mathfrak{g}/\mathfrak{h}, \bar{l} \neq 0$, such that $\mathfrak{h}'\,\bar{l} = 0$.

Now all we have to do, is to follow our ideas backwards to see them unfold. We designed our representation, so that $l \in \mathfrak{g}, l \notin \mathfrak{h}$ and $[\mathfrak{h}, l] \subset \mathfrak{h}$ would be true, so this will be the next thing to show.

This implies $l \in \mathfrak{g}, l \notin \mathfrak{h}$ and

$$\begin{aligned}
\mathfrak{h}' \bar{l} = 0 &\iff \forall y' \in \overline{\mathrm{ad}}(\mathfrak{h}) : \; y'(\bar{l}) = 0 \\
&\iff \forall y \in \mathfrak{h} : \; \underbrace{\overline{\mathrm{ad}}_y(\bar{l})}_{=\overline{[y,l]}} = 0 \\
&\iff \forall y \in \mathfrak{h} : \; [y, l] \in \mathfrak{h} \\
&\iff [\mathfrak{h}, l] \subset \mathfrak{h}.
\end{aligned}$$

By maximality of $\mathfrak{h}$ and $[\mathfrak{h}, l] \subset \mathfrak{h}$ together with $l \notin \mathfrak{h}$ we now have

$$\mathfrak{h} + \operatorname{span}\{l\} = \mathfrak{g}.$$

Step 2 (Find v)*:*

To proof the theorem we also need $v \in V, v \neq 0$ such that $\mathfrak{g}v = 0$, i.e. $\mathfrak{h}v = 0$ and $l(v) = 0$. Let $W := \{v \in V \mid \mathfrak{h}v = 0\}$ be the vector space of potential candidates. Our wanted vector v lives in W, so we need to have that $\ker l \cap W \neq 0$, for $v \neq 0$ to exist.
But how would we know that l has a nonzero kernel element in W?
Since $l \in \mathfrak{g}$ we do know for certain that l has a nonzero kernel in V, since it is a nilpotent map $V \to V$ (compare this to the induction start). But it could lie anywhere.
The **next trick** is to require that l restricts to a well defined map $l|_W : W \to W$, i.e. $lW \subset W$. Then l being nilpotent implies that $l|_W$ is nilpotent as well, thus it has nonzero kernel, which is exactly what we need.

Define $W := \{v \in V \mid \mathfrak{h}v = 0\}$.

Claim: $lW \subset W$.

Proof: Let $w \in W$, then we have to show $l(w) \in W$, i.e. $\forall y \in \mathfrak{h} : y(l(w)) = 0$. Although looking weird, this is just the action of $\mathfrak{g}$ on V (recall $\mathfrak{g} \subset \mathfrak{gl}(V)$). Let $y \in \mathfrak{h}$, then

$$y(l(w)) = y.(l.w) \overset{\text{action}}{=} l.(y.w) + [y, l].w \overset{\substack{y \in \mathfrak{h} \\ w \in W}}{=} 0 + [y, l].w = 0,$$

where the last equality holds, since $[\mathfrak{h}, l] \subset \mathfrak{h}$.

Now since $l \in \mathfrak{g}$ is a nilpotent map $V \to V$, it restricts to a nilpotent map $l|_W : W \to W$. Since $\dim \mathfrak{h} < \dim \mathfrak{g}$ and $\mathfrak{h} \subset \mathfrak{gl}(V)$, we have that $W \neq 0$ by the induction hypothesis. Together with $l|_W$ being nilpotent, this implies that there exists a $v \in \ker l|_W, v \neq 0$.

Then by definition of W we finally have $v \in V, v \neq 0$ with

$$\mathfrak{h}v = 0 \quad \text{and} \quad l(v) = 0 \qquad \Rightarrow \qquad \mathfrak{g}v = 0. \qquad \square$$

Remark. A few questions to think about:

(a) Where did we use $V \neq 0$? It's obvious that for $V = 0$ there is no $v \neq 0$, but where exactly did we use this fact?

(b) Did we really need that $\mathfrak{g}$ acts nilpotently? More precisely, would it suffice to require

$$\forall x \in \mathfrak{g}\ \forall W \subset V, W \neq 0 : (xW \subset W \;\Rightarrow\; \ker x|_W \neq 0)$$

instead? (This property is how we used l being nilpotent in the end).

(c) Why did we, in the beginning of the proof, mention, that "V is **not** fixed"?

(d) What happens if we define $\mathfrak{h} \subsetneq \mathfrak{g}$ to be maximal with respect to dim instead of $\subset$?

(e) Why exactly is $W \neq 0$?

(∗) What is the intuition behind the implication $[\mathfrak{h}, l] \subset \mathfrak{h} \;\Rightarrow\; lW \subset W$? Use this to explain that $[\mathfrak{h}, l] \subset \mathfrak{h} + \operatorname{span}\{l\}$ does not imply $lW \subset W$.

Hints:

(1) The proof of $lW \subset W$ used that $\mathfrak{h} + \operatorname{span}\{l\} \subset \mathfrak{g} \subset \mathfrak{gl}(V)$ is a representation.

(2) What does this implication say for $\mathfrak{h} = \operatorname{span}\{x\}$?

(3) Intuitively W can be thought of as some kind of "kernel" of $\mathfrak{h}$.

(4) What happens when generalizing to $\mathfrak{a}, \mathfrak{b} \subset \mathfrak{g}$?

(5) Prove that for every subspace $U \subset V$ the following relations hold

$$\begin{aligned} [\mathfrak{b}, \mathfrak{a}]U &\subset \mathfrak{a}\mathfrak{b}U + \mathfrak{b}\mathfrak{a}U, \\ \mathfrak{a}\mathfrak{b}U &\subset [\mathfrak{a}, \mathfrak{b}]U + \mathfrak{b}\mathfrak{a}U, \\ \mathfrak{b}\mathfrak{a}U &\subset [\mathfrak{b}, \mathfrak{a}]U + \mathfrak{a}\mathfrak{b}U. \end{aligned}$$

(6) Draw a picture like this

$$\begin{tikzcd}
 & lW \arrow[r, "\mathfrak{h}"] & \mathfrak{h}lW \\
W \arrow[ur, "l"] \arrow[rr, "{[\mathfrak{h},l]\subset\mathfrak{h}}"] \arrow[dr, "\mathfrak{h}"'] & & 0 \in W \\
 & 0 \in W \arrow[r, "l"'] & l(0) \in W
\end{tikzcd}$$

and apply (5), to conclude and "visually see" $\mathfrak{h}lW = 0$.

(7) Ask Jacob ;)