University of Hamburg
Department of Mathematics
Dr. Matthias Voigt

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Mathematical Systems and Control Theory – 5th Exercise Sheet.

Discussion of the solutions in the exercise on January 08, 2020.

Problem 1 (Ackermann formula): Let $(A, B) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times 1}$ be controllable with controller normal form

$$T^{-1}AT = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ -\alpha_0 & -\alpha_1 & -\alpha_2 & \dots & -\alpha_{n-1} \end{bmatrix}, \quad T^{-1}B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

and let $\mathcal{L} := \{\mu_1, \dots, \mu_n\}$ be closed under complex conjugation. In the following, $e_i \in \mathbb{R}^n$ denotes the i-th unit vector for $i = 1, \dots, n$, and

$$\Psi(x) := (x - \mu_1) \cdots (x - \mu_n).$$

Show the following statements:

a) With the controllability matrix $\mathcal{K}(A, B)$ it holds that

$$e_1^\mathsf{T} T^{-1} \mathcal{K}(A, B) = e_n^\mathsf{T}.$$

b) It holds that

$$e_1^\mathsf{T} \left(T^{-1}AT\right)^k = \begin{cases} e_{k+1}^\mathsf{T} & \text{for } k = 0, \dots, n-1, \\ [-\alpha_0, \dots, -\alpha_n] & \text{for } k = n. \end{cases}$$

c) Let $\beta_0, \dots, \beta_{n-1} \in \mathbb{C}$ be such that $\Psi(x) = x^n + \beta_{n-1}x^{n-1} + \dots + \beta_1 x + \beta_0$. Then it holds that

$$e_1^\mathsf{T} \Psi\big(T^{-1}AT\big) = [\beta_0 - \alpha_0, \dots, \beta_{n-1} - \alpha_{n-1}].$$

d) Let $F := -e_n^\mathsf{T} \mathcal{K}(A, B)^{-1} \Psi(A)$. Then F solves the pole placement problem for (A, B) and $\mathcal{L}$, i. e., $\Lambda(A + BF) = \mathcal{L}$.

Problem 2 (eigenvalues, eigenvectors, and pole placement): Let $(A, B) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times 1}$ be controllable and given in controller normal form. Furthermore, let $\mathcal{L} = \{\mu_1, \dots, \mu_n\}$ be closed under complex conjugation and $F = [f_1, \dots, f_n] \in \mathbb{R}^{1\times n}$ such that $\Lambda(A + BF) = \mathcal{L}$. Show the following statements:

a) If $\lambda \in \mathcal{L} \cap \Lambda(A)$, then every eigenvector of A for the eigenvalue λ is also an eigenvector of $A + BF$ for the eigenvalue λ.

b) If holds that $v = [v_1, \dots, v_n]^\mathsf{T} \in \mathbb{C}^n$ is an eigenvector of A for the eigenvalue λ, if and only if

$$v_k = \lambda^{k-1} v_1 \text{ for } k = 1, \dots, n \text{ and } \sum_{k=1}^{n} (-\alpha_{k-1}) v_k = \lambda v_n.$$

Find an analogous formula for the eigenvectors of $A + BF$.

c) Let $\lambda \in \mathcal{L} \setminus \Lambda(A)$ and $x := (\lambda I_n - A)^{-1}B$. Then $Fx = 1$.
Hint: show that xF has the eigenvalue 1, if $A + BF$ has the eigenvalue λ.

d) If $\lambda \in \mathcal{L} \setminus \Lambda(A)$, then $(\lambda I_n - A)^{-1}B$ is an eigenvector of $A + BF$ for the eigenvalue λ.

Problem 3 (partial stabilization): Let $(A, B) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m}$ be stabilizable. Often only a few eigenvalues of A are unstable, i. e., with non-negative real part. Therefore, it is not necessary to stabilize the entire system, but only to move the few unstable eigenvalues into the open left half-plane by state feedback. Assume that

$$\Lambda(A) = \Lambda_- \cup \Lambda_+$$

with $\Lambda_- \subset \mathbb{C}^-$ and $\Lambda_+ \subset \overline{\mathbb{C}^+}$.
Develop an algorithm for the partial stabilization based on the Schur form of A. The algorithm should compute $F \in \mathbb{R}^{m \times n}$ such that $\Lambda(A + BF) = \Lambda_- \cup \{\mu_{k+1}, \ldots, \mu_n\}$ for given $\mu_{k+1}, \ldots, \mu_n \in \mathbb{C}^-$.