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Mathematical Systems and Control Theory – 3rd Exercise Sheet.

Discussion of the solutions in the exercise on November 27, 2019.

Problem 1 (controllability): Check the following systems for controllability:

a) $\dot{x}(t) = x(t) + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} u(t);$

b) $\dot{x}(t) = \begin{bmatrix} 0 & 1 & & & \\ & \ddots & \ddots & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ -\alpha_0 & -\alpha_1 & \dots & \dots & -\alpha_{n-1} \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix} u(t).$

Problem 2 (Hautus-Popov test for stabilizable systems): Let $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{n \times m}$ be given matrices. Show that the following statements are equivalent:

a) The pair (A, B) is stabilizable.

b) In the Kalman decomposition

$$\left(V^\mathsf{T} A V, V^\mathsf{T} B\right) = \left(\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \right) \qquad \text{with orthogonal } V \in \mathbb{R}^{n \times n},$$

it holds that $\Lambda(A_{22}) \subset \mathbb{C}^-$.

c) If $v \neq 0$ is a left eigenvector of A associated with the eigenvalue λ with $\mathrm{Re}(\lambda) \geq 0$, then $v^\mathsf{H} B \neq 0$.

d) It holds $\operatorname{rank} \begin{bmatrix} A - \lambda I & B \end{bmatrix} = n$ for all $\lambda \in \mathbb{C}$ with $\mathrm{Re}(\lambda) \geq 0$.

Problem 3 (state and feedback transformations): Let (A_1, B_1), $(A_2, B_2) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m}$. We say that

- (A_1, B_1) and (A_2, B_2) are *system equivalent*, if there exists a nonsingular matrix $T \in \mathbb{R}^{n \times n}$ such that

  $$(A_2, B_2) = \left(T^{-1} A_1 T, T^{-1} B_1\right),$$

  and we write $(A_1, B_1) \sim_\mathrm{s} (A_2, B_2)$;

- (A_1, B_1) and (A_2, B_2) are *feedback equivalent*, if there exists a nonsingular matrix $T \in \mathbb{R}^{n \times n}$ and a matrix $F \in \mathbb{R}^{m \times n}$ such that

  $$(A_2, B_2) = \left(T^{-1}(A_1 + B_1 F) T, T^{-1} B_1\right),$$

  and we write $(A_1, B_1) \sim_\mathrm{f} (A_2, B_2)$.

a) Show that system and feedback equivalence are indeed equivalence relations.

b) Show that if $(A_1, B_1) \sim_{\mathrm{f}} (A_2, B_2)$, then

$$\begin{aligned} (A_1, B_1) \text{ is controllable} \quad &\Leftrightarrow \quad (A_2, B_2) \text{ is controllable}, \\ (A_1, B_1) \text{ is stabilizable} \quad &\Leftrightarrow \quad (A_2, B_2) \text{ is stabilizable}. \end{aligned}$$

Problem 4 (stabilization): Consider the control problem

$$\begin{aligned} \dot{\varphi}(t) &= \omega(t), \\ j\dot{\omega}(t) &= -r\omega(t) + ku(t), \end{aligned}$$

with k, j, $r > 0$. Compute all stabilizing feedback matrices $F \in \mathbb{R}^{1\times 2}$ of this system.