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Optimization of Complex Systems – 8th Exercise Sheet.

Discussion of the solutions in the exercise on January 6, 2020.

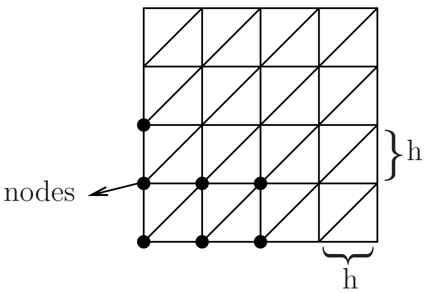
Problem 1 (2D Poisson FEM): Consider the two-dimensional boundary value problem

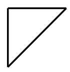
$$\begin{aligned} -\Delta y &= f \quad \text{in} \quad \Omega = (0,1)^2 \subset \mathbb{R}^2, \\ y &= 0 \quad \text{on} \quad \partial\Omega. \end{aligned}$$

In this exercise the goal is to implement a 2D FEM code for this boundary value problem. To this end, the following triangulation

$$\overline{\Omega} = \bigcup_{l=1}^{n} \Omega_l$$

should be used:



Each element is a triangle of the form  up to rotation.

As basis functions we choose the standard piecewise linear pyramid functions.

a) Write a MATLAB function `[An, fn] = getPDE(f, n)` that sets up the sparse Galerkin matrix A_n and the right hand side f_n of the linear system $A_n y_n = f_n$. The input `f` is a function handle for the right-hand side of the PDE and the input `n` corresponds to the number of inner nodes per dimension. For the solution of this problem, the following steps should be carried out:

- Write a function which gets `n` and outputs the four matrices `B`, `C`, `D`, and `E`. The matrices `B` and `D` have as many rows as there are nodes, i. e., $(n+2)^2$. The `B` matrix has two columns and contains the x_1 and the x_2 coordinate of each node. The `D` matrix has one column and indicates for each node if it is on the boundary (then the entry is 0) or an inner node (then the entry is 1). The matrices `C` and `E` have as many rows as there are triangular elements in Ω. The matrix `C` has three columns and contains the three node indices (corresponding to the row index in `B`) for each triangle. The matrix `E` has two columns and contains for each triangle the x_1 and the x_2 coordinate of its centroid. Hint: Use a lexicographical ordering of the nodes for `B` and `D`, i. e., the ordering should be

$$((0,0),\ (0,1/h),\ \ldots,\ (0,1),\ \ldots,\ (1,0),\ (1,1/h),\ \ldots,\ (1,1)).$$

  The MATLAB functions `meshgrid` and `reshape` may be useful here.

- Write a function which gets the matrices `B` and `C` and outputs vectors `b11`, `b12`, `b22`, and `d`. Each of these vectors has as many entries as there are triangular elements in Ω and their single entries b_{11}^l, b_{12}^l, b_{22}^l, and d_l are explained in the additional material provided on the website.
- Write a function which gets `b11`, `b12`, `b22`, and `d` and outputs the local stiffness matrices, i. e., the matrices $A_{n,l} = [a_{ij}^l]_{i,j=1}^3$ with

$$a_{ij}^l = \int_{\Omega_l} \nabla\varphi_i^l(x) \cdot \nabla\varphi_j^l(x)\mathrm{d}x,$$

where φ_i^l, $i = 1,\, 2,\, 3$ are the linear form functions on the element Ω_l. These are as many 3×3 matrices as there are triangular elements in Ω.

- Write a function which gets `f`, `E`, and `d` and outputs the local right hand sides, i. e., a vector with as many entries as there are triangular elements in Ω. The local right hand sides are given by $F_{n,l} = [f_1^l,\, f_2^l,\, f_3^l]_{l=1}^n$ with

$$f_i^l = \int_{\Omega_l} f(x)\varphi_i^l(x)\mathrm{d}x.$$

Hint: For the approximation of an integral over the reference triangle $\widehat{\Omega}$, you may use the quadrature rule

$$\iint_{\widehat{\Omega}} g(x_1, x_2)\mathrm{d}x_1\mathrm{d}x_2 \approx \frac{1}{2} g\left(\frac{1}{3}, \frac{1}{3}\right).$$

Then $f_1^l = f_2^l = f_3^l$ for $l = 1,\, \dots,\, n$ and so f_i^l has to be computed only once.

- Write a function which gets `D` and a node index (corresponding to a row index in `B`) and outputs the corresponding row index in `Bnew`, where `Bnew` is the reduced version of `B` after removing all the boundary nodes from `B`. Hint: The possible outputs of this function should range from 1 to n^2.
- Write a function which gets the local stiffness matrices, `C`, `D`, and `n` and outputs the global stiffness matrix `An`.
- Write a function which gets the local right hand sides, `C`, `D`, and `n` and outputs the global right hand side `fn`.

b) Write a script `PDEconv` for exploring the convergence of your code for the equation with $f(x_1,\, x_2) = 10\pi^2 \sin(3\pi x_1)\sin(\pi x_2)$ by making an error plot. The analytic solution is given by $y(x_1, x_2) = \sin(3\pi x_1)\sin(\pi x_2)$.