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Optimization of Complex Systems – 1st Exercise Sheet.

Discussion of the solutions in the exercise on October 28, 2019.

Problem 1 (variational inequality): Let U_{ad} be convex, $f : U_{\mathrm{ad}} \to \mathbb{R}$ be continuously differentiable, and let $\bar{u}$ be a local minimizer of the problem

$$\min_{u \in U_{\mathrm{ad}}} f(u).$$

Show that then the variational inequality

$$f'(\bar{u})(u - \bar{u}) \geq 0 \quad \forall\, u \in U_{\mathrm{ad}}$$

is satisfied.
Hint: The set U_{ad} is convex if and only if $u_1,\, u_2 \in U_{\mathrm{ad}} \;\Rightarrow\; tu_1 + (1-t)u_2 \in U_{\mathrm{ad}}\ \forall\, t \in [0,1]$.

Problem 2 (minimizers of convex functionals): Let $U_{\mathrm{ad}} \subseteq \mathbb{R}^n$ be convex and $f : U_{\mathrm{ad}} \to \mathbb{R}$ be strictly convex, that is,

$$f(tu_1 + (1-t)u_2) < tf(u_1) + (1-t)f(u_2)$$

for all $t \in (0,1)$ and $u_1 \neq u_2$. Let $\bar{u} \in U_{\mathrm{ad}}$ be a local minimizer of the problem

$$\min_{u \in U_{\mathrm{ad}}} f(u). \tag{1}$$

a) Show that $\bar{u}$ is a global minimizer of (1) and that this global minimizer is unique.

b) Let $y_{\mathrm{d}} \in \mathbb{R}^n$, $S \in \mathbb{R}^{n \times n}$ be invertible, and $\lambda > 0$. Show that the functional

$$f(u) = \frac{1}{2}\left\|Su - y_{\mathrm{d}}\right\|^2 + \frac{\lambda}{2}\left\|u\right\|^2$$

is strictly convex.

Problem 3 (linear-quadratic optimal control): Consider the optimal control problem

$$\min f(z,u) := \frac{1}{2} z_N^{\mathsf{T}} Q z_N + \sum_{i=0}^{N-1} f_i(z_i, u_i),$$

$$\text{subject to } z_{i+1} = A_i z_i + B_i u_i + c_i, \quad i = 0,\, \ldots,\, N-1,$$

where $z_0 \in \mathbb{R}^n$ is given and fixed, $Q \in \mathbb{R}^{n \times n}$ is symmetric, $A_i \in \mathbb{R}^{n \times n}$, $B_i \in \mathbb{R}^{n \times m}$, and $c_i,\, z_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^m$ for $i = 0,\, \ldots,\, N-1$, and $f_i : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ are given differentiable functions.
Formulate the Lagrangian of this problem and derive the KKT optimality system.

Problem 4 (classification of PDEs): Let $\Omega \in \mathbb{R}^n$ be a domain and consider a *linear, second-order PDE* with the unknown $y \colon \Omega \to \mathbb{R}$ of the form

$$-\sum_{i,k=1}^{n} a_{ik}(x) \frac{\partial^2 y(x)}{\partial x_i \partial x_k} + \sum_{i=1}^{n} b_i(x) \frac{\partial y(x)}{\partial x_i} + c(x) y(x) = f(x) \quad \forall\, x \in \Omega,$$

where $a_{ik},\, b_i,\, c,\, f : \Omega \to \mathbb{R}$ are given mappings for all $i,\, k = 1,\, \ldots,\, n$. Assume that $\frac{\partial^2 y(\cdot)}{\partial x_i \partial x_k} = \frac{\partial^2 y(\cdot)}{\partial x_k \partial x_i}$ for all $i,\, k = 1,\, \ldots,\, n$. Then we can choose w. l. o. g. $a_{ki}(\cdot) = a_{ik}(\cdot)$ for all $i,\, k = 1,\, \ldots,\, n$. (Why?)
Define the matrix $A(x) := [a_{ik}(x)]_{i,k=1}^{n}$. Then a linear, second-order PDE of the form above is called

i) *elliptic* in $x \in \Omega$ if $A(x)$ is definite, i. e., all eigenvalues of $A(x)$ are either strictly positive or strictly negative;

ii) *hyperbolic* in $x \in \Omega$ if $A(x)$ has one strictly negative eigenvalue and $n-1$ strictly positive eigenvalues (or vice versa);

iii) *parabolic* in $x \in \Omega$ if $A(x)$ has one eigenvalue equal to zero and $n-1$ strictly positive (or strictly negative) eigenvalues and $\operatorname{rank}\left(\begin{bmatrix} A(x) & b(x) \end{bmatrix}\right) = n$, where $b(x) = \begin{bmatrix} b_1(x), \ldots, b_n(x) \end{bmatrix}^\mathsf{T}$.

We say that the PDE is elliptic/hyperbolic/parabolic if it is elliptic/hyperbolic/parabolic in all $x \in \Omega$. Decide of which type the following PDEs are and explain your decision:

a) the *Poisson equation*

$$-\Delta y(x) = f(x) \quad \text{for } x \in \Omega,$$

b) the *heat equation*

$$\frac{\partial y(x,t)}{\partial t} - \Delta y(x,t) = f(x,t) \quad \text{for } (x,t) \in \Omega := \Omega_{\text{space}} \times (0,T),$$

c) the *wave equation*

$$\frac{\partial^2 y(x,t)}{\partial t^2} - \Delta y(x,t) = f(x,t) \quad \text{for } (x,t) \in \Omega := \Omega_{\text{space}} \times (0,T).$$