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Fachbereich Mathematik
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Model Reduction Homework Sheet 4.

The problems will be discussed in the exercise on Thursday, June 06.

Problem 1: Let $[A, B, C, D] \in \Sigma_{n,m,p}$ be asymptotically stable with controllability Gramian P and observability Gramian Q. Let $T \in \mathbb{R}^{n \times n}$ be invertible and define $[\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D}] := \left[T^{-1}AT, T^{-1}B, CT, D\right]$ with controllability Gramian $\widetilde{P}$ and observability Gramian $\widetilde{Q}$. Show the following statements:

a) It holds that $\widetilde{P} = T^{-1}PT^{-\mathsf{T}}$ and $\widetilde{Q} = T^{\mathsf{T}}QT$.

b) The Hankel singular values are invariant under state-space transformations.

Problem 2: Let $[A, B, C, D] \in \Sigma_{n,m,p}$ be asymptotically stable with the Hankel operator $\mathcal{H}$. Show that $\mathcal{H}$ is a bounded linear operator. For boundedness, show that there exists a constant $c > 0$ (independent of u) such that

$$\|\mathcal{H}u\|_{\mathcal{L}_2([0,\infty),\mathbb{R}^p)} \leq c \cdot \|u\|_{\mathcal{L}_2((-\infty,0],\mathbb{R}^m)} \,.$$

Problem 3: Let $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{m \times m}$, and $C \in \mathbb{R}^{n \times m}$ be given matrices and consider the *Sylvester equation*

$$AX + XB = C \qquad \text{for } X \in \mathbb{R}^{n \times m}. \tag{1}$$

a) Consider the *vectorization operator*

$$\operatorname{vec} : \mathbb{R}^{n \times m} \to \mathbb{R}^{nm}, \quad X = \begin{bmatrix} x_1 & \ldots & x_m \end{bmatrix} \mapsto \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix},$$

and for $X = [x_{ij}] \in \mathbb{R}^{n \times m}$ and $Y \in \mathbb{R}^{p \times q}$ define the *Kronecker product*

$$\otimes : \mathbb{R}^{n \times m} \times \mathbb{R}^{p \times q} \to \mathbb{R}^{np \times mq}, \quad X \otimes Y = \begin{bmatrix} x_{11}Y & \ldots & x_{1m}Y \\ \vdots & & \vdots \\ x_{n1}Y & \ldots & x_{nm}Y \end{bmatrix} \in \mathbb{R}^{np \times mq}.$$

Show that for $T \in \mathbb{R}^{n \times m}$, $O \in \mathbb{R}^{m \times p}$, and $R \in \mathbb{R}^{p \times r}$ it holds that

$$\operatorname{vec}(TOR) = \left(R^{\mathsf{T}} \otimes T\right) \operatorname{vec}(O),$$

and conclude that (1) can be equivalently written as a linear system of the form

$$\Big(\left(I_m \otimes A\right) + \left(B^{\mathsf{T}} \otimes I_n\right)\Big) \operatorname{vec}(X) = \operatorname{vec}(C).$$

b) Consider the *Theorem of Stephanos:*

Let $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{m\times m}$ with $\Lambda(A) = \{\lambda_1, \ldots, \lambda_n\}$, $\Lambda(B) = \{\mu_1, \ldots, \mu_m\}$ be given. For a bivariate polynomial $p(x, y) = \sum_{i,j=0}^{k} c_{ij}x^i y^j$ we define by

$$p(A, B) := \sum_{i,j=0}^{k} c_{ij}(A^i \otimes B^j)$$

a polynomial of the two matrices. Then the spectrum of $p(A, B)$ is given by

$$\Lambda(p(A, B)) = \{p(\lambda_r, \mu_s) \mid r = 1, \ldots, n,\ s = 1, \ldots, m\}.$$

Use this theorem to show that the Sylvester equation (1) is uniquely solvable for all $C \in \mathbb{R}^{n\times m}$, if and only if $\Lambda(A) \cap \Lambda(-B) = \emptyset$.

Bonus: Prove the Theorem of Stephanos.