Trees of tangles in infinite separation systems

Part I.

with Christian and Jakob

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Reviewing the Splinter Theorem
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**Theorem (Splinter Theorem; JMC'19)**
Let $U$ be a universe of separations and $(A_i)_{i \leq n}$ a family of subsets of $U$. If $(A_i)_{i \leq n}$ splinters then we can pick an element $a_i$ from each $A_i$ so that $\{a_1, \ldots, a_n\}$ is nested.

Typically $A_i$ is the set of separations which efficiently distinguish a given pair of profiles. $(A_i)_{i \in I}$ splinters if for all $a_i \in A_i$, $a_j \in A_j$, or $a_i$ and $a_j$ have a corner separation in $A_i$ or $A_j$. How can we extend this to the infinite?
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How can we extend this to the infinite?
The profinite approach
Apply the splinter theorem to finite restrictions.
A **profinite universe** is an inverse limit \( \bar{U} = \lim \left( \bar{U}_p \mid p \in P \right) \) of finite \( \bar{U}_p \), it consists of those separations \( \bar{s} = (\bar{s}_p \mid p \in P) \) which are compatible wrt. bonding maps \( f_{pq} : U_p \to U_q \).
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**First Observation:** If $(A_i \mid i \in I)$ in $\hat{U}$ splinters, then so does every projection to a $\hat{U}_p$. 

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$\leq$, $\lor$, $\land$ work coordinate-wise.

**First Observation:** If $(A_i | i \in I)$ in $\bar{U}$ splinters, then so does every projection to a $\bar{U}_p$. So, apply the finite Splinter Theorem!
Second Observation: If we apply the Splinter Theorem to $\tilde{U}_p$ and map the nested set to $\tilde{U}_q$ we get a splinter solution for $\tilde{U}_q$.

For every $\tilde{U}_p$ let $\mathcal{N}_p$ be the set of all nested sets which meet every $A_i \uparrow p$. 
Second Observation: If we apply the Splinter Theorem to $\hat{U}_p$ and map the nested set to $\hat{U}_q$ we get a splinter solution for $\hat{U}_q$.

For every $\hat{U}_p$ let $\mathcal{N}_p$ be the set of all nested sets which meet every $\mathcal{A}_i \uparrow p$. Splinter Theorem says that these are non-empty.
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For every $\hat{U}_p$ let $\mathcal{N}_p$ be the set of all nested sets which meet every $A_i \upharpoonright p$. Splinter Theorem says that these are non-empty.

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Consider $N \in \varprojlim (\mathcal{N}_p \mid p \in P)$. We can turn $N$ into a nested set in $\hat{U}$.

If the $A_i$ are closed, $N$ meets all of them.
Theorem (Profinite Splinter Theorem)
Let $\overline{U} = \varprojlim (\overline{U}_p \mid p \in P)$ be a profinite universe and $(A_i \mid i \in I)$ a family of non-empty closed subsets of $\overline{U}$. If $(A_i \mid i \in I)$ splinters then there is a closed nested set $N \subseteq \overline{U}$ containing at least one element from each $A_i$. 
Application to graphs

Let $A_{P,P'}$ be the set of all efficient $P-P'$-distinguishers.

How do we ensure that the $A_{P,P'}$ are closed?
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**Observation**

$\tilde{S}_k$ is closed in $\tilde{U}$. 
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**Lemma**

$A_{P,P'}$ is closed for bounded $P, P'$. 
Lemma

\( (A_{P,P'} \mid P, P' \text{ bounded, robust and distinguishable profiles in } G) \text{ splinters.} \)

Luckily, the splinter condition was designed for this. We only need robust and distinguishable.
Done!

Can we build a tree-decomposition from this?
Can we do something without inverse limits?
Relations, Crossing Numbers and Canonicity
In graph separations: Crossing number is *strongly submodular*.

Encode this in our splinter-condition.
The nestedness relation

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Let **nested** be a reflexive and symmetric relation.

**Crossing** means ‘not nested’.

A **corner of $a$ and $b$** is an element $c$ of $\mathcal{A}$, such that anything that crosses $c$ also crosses $a$ or $b$. 
Split \((\mathcal{A}_i \mid i \in I)\) into ‘levels’:

\[ |i| : I \rightarrow \mathbb{N}_0 \]

Think of \(|i|\) as ‘the order of the elements of \(\mathcal{A}_i\).’
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The \textbf{k-crossing number} of \(a\) is the number of elements of \(\mathcal{A}\) that cross \(a\) and lie in some \(\mathcal{A}_i\) with \(|i| = k\).
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\textit{We take care here not to count multiplicities.}
The thin splinter theorem

\((A_i \mid i \in I)\) thinly splinters if:

1. For every \(i \in I\) all elements of \(A_i\) have finite \(k\)-crossing number for all \(k \leq |i|\).
(A_i | i ∈ I) thinly splinters if:

1. For every i ∈ I all elements of A_i have finite k-crossing number for all k ≤ |i|.
2. If a_i ∈ A_i and a_j ∈ A_j cross with |j| < |i|, then A_i contains some corner of a_i and a_j that is nested with a_j.
(\mathcal{A}_i \mid i \in I) \textbf{ thinly splinters} if:

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3. If \( a_i \in \mathcal{A}_i \) and \( a_j \in \mathcal{A}_j \) cross with \( |i| = |j| = k \), then
   - either \( \mathcal{A}_i \) contains a corner of \( a_i \) and \( a_j \) with strictly lower \( k \)-crossing number than \( a_i \),
   - or else \( \mathcal{A}_j \) contains a corner of \( a_i \) and \( a_j \) with strictly lower \( k \)-crossing number than \( a_j \).
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2. If \(a_i \in \mathcal{A}_i\) and \(a_j \in \mathcal{A}_j\) cross with \(|j| < |i|\), then \(\mathcal{A}_i\) contains some corner of \(a_i\) and \(a_j\) that is nested with \(a_j\).
3. If \(a_i \in \mathcal{A}_i\) and \(a_j \in \mathcal{A}_j\) cross with \(|i| = |j| = k\), then
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   - or else \(\mathcal{A}_j\) contains a corner of \(a_i\) and \(a_j\) with strictly lower \(k\)-crossing number than \(a_j\).

**Theorem (Thin Splinter Theorem)**

If \((\mathcal{A}_i \mid i \in I)\) thinly splinters, then there is a canonical set \(N \subseteq \mathcal{A}\) which meets every \(\mathcal{A}_i\) and is pairwise nested.
Proof

Construct $N_0 \subseteq N_1 \subseteq N_2 \ldots$, s.t. $N_k$ takes care of all $A_i$ with $|i| \leq k$.

Set $N_{-1} := \emptyset$. In step $k$:

Let $N_k^+$ consist of
from each $A_i$ with $|i| = k$
among those elements nested with $N_{k-1}$
all of minimum $k$-crossing number.

Set $N_k := N_{k-1} \cup N_k^+$. Need to show:

- For each $A_i$ we had elements to choose from.
- $N_k$ is nested.
Proof

- For each $A_i$, $|i| = k$, we had elements to choose from.

That is, $A_i$ has an element that is nested with $N_{k-1}$.

By (1) every element of $A_i$ crosses only finitely many elements of $N_{k-1}$.

Let $a_i$ be one that crosses as few as possible.

Suppose it crosses some $a_j \in N_{k-1}$, then $a_j \in A_j$ with $|j| < k$.

By (2), $a_i$ and $a_j$ have a corner in $A_i$ that is nested with $a_j$.

This corner was a better choice for $a_i$. 
• $N_k$ is nested.

Every element of $N_k^+$ is nested with $N_{k-1}$ by construction. Only need to show that $N_k^+$ is nested.

Suppose $a_i$ and $a_j$ in $N_k^+$ cross. By (3) there is a corner of $a_i$ and $a_j$ in $A_i$ or $A_j$, with a strictly lower $k$-crossing number than the corresponding $a_i$ or $a_j$. 

\[\square\]