98j:17020 17B50 17B20

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The classification of the simple modular Lie algebras. VI. Solving the final case. (English. English summary)

Trans. Amer. Math. Soc. **350** (1998), *no.* 7, 2553–2628.

FEATURED REVIEW.

The problem of classifying the finite-dimensional simple Lie algebras of characteristic $p > 0$ is a long-standing one. Work on this question during the last thirty years has been directed by the Kostrikin-Shafarevich Conjecture of 1966, which states: Over an algebraically closed field of characteristic $p > 5$ a finite-dimensional restricted simple Lie algebra is classical or of Cartan type.

The classical algebras are the analogues of the finite-dimensional simple complex Lie algebras, while the Cartan-type algebras correspond to the four infinite families W, S, H, K (Witt, special, Hamiltonian, contact) of infinite-dimensional complex Lie algebras of Cartan. If the notion of Cartan-type algebra is expanded to include the simple algebras arising from Cartan-type algebras by twisting by an automorphism, then one can formulate the Generalized Kostrikin-Shafarevich Conjecture by removing the restrictedness assumption in the statement above.

The Generalized Kostrikin-Shafarevich Conjecture is now a theorem for $p > 7$. First announced by Strade and R. L. Wilson in 1991, its proof is spread over a number of papers. As the title suggests, the paper under review, which solves the final case, is the capstone of work on the classification problem.

The main contributions to the proof include A. I. Kostrikin and I. R. Shafarevich's construction and investigation of the Cartan-type algebras [Izv. Akad. Nauk SSSR Ser. Mat. **33** (1969), 251–322; MR **40** #5680]; generalizations of their construction by V. G. Kac [Uspehi Mat. Nauk **26** (1971), no. 3(159), 199–200; MR **47** #293; Izv. Akad. Nauk SSSR Ser. Mat. **38** (1974), 800–834; MR **51** #5685], Wilson [Bull. Amer. Math. Soc. **75** (1969), 987–991; MR **42** #3135; J. Algebra **40** (1976), no. 2, 418–465; MR **54** #366] and G. Y. Shen [Chinese Ann. Math. Ser. B **4** (1983), no. 3, 329–346; MR 85k:17010]; Kac's Recognition Theorem [Izv. Akad. Nauk SSSR Ser. Mat. **34** (1970), 385–408; MR **43** #2033] and its reworking by the reviewer, T. Gregory, and A. Premet (in preparation); R. E. Block's determination [Ann. of Math. (2) **90** (1969), 433–459; MR **40** #4319] of the differentiably simple Lie algebras which allowed the finite-dimensional semisimple Lie algebras of characteristic p to be described; B. Weisfeiler's work [J. Algebra **53** (1978), no. 2, 344–361; MR 80b:17011; Bull. Amer.

Math. Soc. **84** (1978), no. 1, 127–130; MR 81f:14027] on graded Lie algebras, which is critical in the local analysis of simple algebras; the classification by Block and Wilson [J. Algebra **114** (1988), no. 1, 115–259; MR 89e:17014] of the restricted simple Lie algebras (thereby proving the original (restricted) Kostrikin-Shafarevich Conjecture); and Strade's monumental series of articles starting in 1989 [Part I, Ann. of Math. (2) **130** (1989), no. 3, 643–677; MR 91a:17023; Part II, J. Algebra **151** (1992), no. 2, 425–475; MR 93j:17042; Part III, Ann. of Math. (2) **133** (1991), no. 3, 577–604; MR 92g:17024; Part IV, Ann. of Math. (2) **138** (1993), no. 1, 1–59; MR 94k:17039; Part V, Abh. Math. Sem. Univ. Hamburg **64** (1994), 167–202; MR 95h:17023].

The Block-Wilson classification marked a major breakthrough in the theory in that not only did it determine the restricted simple Lie algebras, but it also provided a framework for the classification of the nonrestricted simple Lie algebras as well. It was Strade's insight that p-envelopes could be used to replace the restrictedness assumption. A p-envelope for a Lie algebra L is a restricted Lie algebra $(L_p, [p])$ and an embedding $\psi\colon L \to L_p$ such that the p-subalgebra generated by $\psi(L)$ is L_p. Since Cartan subalgebras and tori in characteristic-p Lie algebras need not be conjugate, and the root space decompositions may look quite different depending on which Cartan subalgebra or torus is used, a judicious choice of torus needs to be made. What seems to work best for classification purposes is a torus T of maximal dimension in the p-envelope L_p such that for each root α there exists a value $1 \le i \le p-1$ such that $\alpha([L_{i\alpha}, L_{-i\alpha}]) = 0$. Such a torus is termed optimal, and Strade in 1989 showed that optimal tori always exist for L simple of characteristic $p > 7$. In that same paper Strade analyzed the 1-sections $L(\alpha) = \sum_{i=0}^{p-1} L_{i\alpha}$ and 2-sections $L(\alpha, \beta) = \sum_{i,j=0}^{p-1} L_{i\alpha + j\beta}$ of L determined by roots α and β with respect to an optimal torus.

The characterization of the 1-sections enabled the reviewer, J. M. Osborn and the author [Trans. Amer. Math. Soc. **341** (1994), no. 1, 227–252; MR 94c:17035] to show that in each 1-section $L(\alpha)$ relative to an optimal torus T there is a unique subalgebra $Q(\alpha)$ satisfying the following properties: (1) The solvable radical $\operatorname{rad} L(\alpha)$ of $L(\alpha)$ is contained in $Q(\alpha)$; (2) $Q(\alpha)$ is solvable or $Q(\alpha)/\operatorname{rad} Q(\alpha)$ is a classical simple algebra; and (3) $\dim L(\alpha)/Q(\alpha) \le 2$. The sum $Q = \sum_\alpha Q(\alpha)$ is either L or a maximal T-invariant subalgebra. When L is restricted, then the Mills-Seligman axiomatic characterization of the classical algebras can be used to show that if $Q = L$, then L is classical. Otherwise, in the restricted case, it can be argued that Q is a maximal subalgebra of the type required to apply Kac's Recognition Theorem. The nonrestricted case does not behave so nicely. What one can say in the nonrestricted case is that the subalgebra Q equals L or it can be embedded in a maximal subalgebra. In most situations a maximal subalgebra containing Q is the appropriate choice for applying the recognition theorem. Further analysis to show that result requires knowledge of the 1-sections and 2-sections. The socle of a semisimple Lie algebra is the direct sum of ideals which are of the form $S_i \otimes A(m_i;\underline{1})$, where S_i is a simple Lie algebra and $A(m_i;\underline{1})$ is a truncated polynomial algebra in m_i indeterminates for some $m_i \in \{0,1,2,\cdots\}$. The sum $\sum_i S_i$ of the simple algebras is called the core. The core of a nonsolvable 1-section is by definition just the core of $L(\alpha)/\operatorname{rad} L(\alpha)$, which in this case must be a simple algebra. At this juncture in the classification it is convenient to divide considerations into four cases: (a) Every 1-section is solvable. (b) Every 1-section is solvable or has core sl(2), and a nonsolvable 1-section exists. (c) There exists a 1-section with a core which is a nonclassical simple Lie algebra, and no 2-section has the simple Hamiltonian Lie algebra $\mathrm{H}(2;\underline{1};\Phi(\tau))^{(1)}$ as its core. (d) There exist a 1-section with a core which is a nonclassical simple Lie algebra and a 2-section with $\mathrm{H}(2;\underline{1};\Phi(\tau))^{(1)}$ as its core.

Simple Lie algebras satisfying (b) were classified under one additional assumption by the reviewer [Comm. Algebra **18** (1990), no. 11, 3633–3638; MR 91f:17015] and in full generality by Strade (1991)—they are precisely the classical simple Lie algebras. The algebras for which (c) holds are of Cartan type (Strade, 1993), and the ones satisfying (d) are Cartan-type Lie algebras belonging to the Hamiltonian series (Strade, 1994). The Lie algebras which satisfy (a) are determined in the paper under review, and they are the last piece of the classification puzzle. For many reasons this is the most difficult case because the assumptions preclude the customary investigations using sl(2)-theory, a mainstay of classification work. The 3-sections need to be analyzed and the simple Lie algebras of toral rank 3 need to be determined. This is one of the few places where the 1-sections and 2-sections are insufficient. Strade shows in this paper, which is a real tour de force, that the simple Lie algebras satisfying (a) are the Block algebras (certain Cartan-type Lie algebras in the Hamiltonian series) or the special series algebras $\mathrm{S}(m;\underline{n};\Phi(\tau))^{(1)}$. With these results the Generalized Kostrikin-Shafarevich Conjecture for $p>7$ (as announced by Strade and Wilson in 1991) is now known to hold. Work on this final case was begun in collaboration with Wilson, and Strade in the article acknowledges Wilson's substantial influence, particularly on Sections 5 and 7 of the paper.

As the classification has evolved over the last thirty years, and especially over the last ten, many arguments needed to solve particular cases have required extension to handle subsequent cases. Now that the whole story is complete and the essential ingredients are understood, it is apparent that the exposition can be streamlined considerably. Work in this direction has already begun.

It is believed that the Generalized Kostrikin-Shafarevich Conjecture holds in characteristic 7. In characteristic 5, the Melikyan algebras provide the only known counterexamples, while in characteristic 3 and 2 there are a host of algebras that are neither classical nor Cartan type. One of the main obstacles in the prime characteristic theory (and why it is orders of magnitude harder than the characteristic-zero case) is the failure of Lie's theorem. This is particularly critical when it comes to saying that the derived algebra of a Cartan subalgebra of a simple Lie algebra L acts nilpotently on L. This last result was proven by Wilson [Trans. Amer. Math. Soc. **234** (1977), no. 2, 435–446; MR **58** #806] for $p > 7$ and by Premet [J. Algebra **167** (1994), no. 3, 641-703; MR 95f:17019] for $p = 7$. Premet also showed that, with one exception, the Cartan subalgebras of the Melikyan algebras have this property. Recent papers by Skryabin (1996) and Premet and Strade (1996) have provided the local analysis of the 1-sections and 2-sections in simple Lie algebras of characteristics 5 and 7 using "sandwich element" techniques. M. I. Kuznetsov [Comm. Algebra **19** (1991), no. 4, 1281–1312; MR 92d:17004] has developed a structural characterization of the Melikyan algebras; and, as mentioned earlier, Kac's Recognition Theorem, a major tool in the classification, has been reworked and now accommodates the Melikyan algebras, which were omitted in the original statement and proof. These results give reason to be very optimistic that the classification problem for $p = 5$ and 7 some day soon will be solved. Currently there is less optimism for the other small characteristics, although some classification results are beginning to emerge.

The list of the finite-dimensional simple Lie algebras of characteristic $p > 7$ consists of algebras having very natural characteristic-zero analogues, and it bears striking resemblance to the list of finite-dimensional complex simple Lie superalgebras. Reasons for this latter coincidence would be interesting to explore. Some of the methods involved in the classification—sandwich theory, for example—play an important role in Burnside-type problems in group theory. In studying the asymptotic behavior of finite p-groups and their associated Engel Lie algebras, one is led very naturally to Lie algebras of Cartan type. These relations hint at much deeper hidden interconnections between these topics, and why they exist is intriguing and still very much open to speculation. Rather than ending the story, Strade's paper may just be the beginning of an even bigger tale. *Georgia M. Benkart* (1-WI)

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