Recall that a complex of $A$-modules is a sequence of modules and $A$-linear maps $d^i : M^i \to M^{i+1}$ for $i \in \mathbb{Z}$ such that $d^{i+1} \circ d^i = 0$ for all $i$. Write a complex as $M^\bullet$. The $i$-cycles of a complex is by definition $Z^i(M^\bullet) := \ker(d^i)$ and the $i$-boundaries are $B^i(M^\bullet) = \operatorname{im}(d^{i-1})$. Clearly, $B^i(M^\bullet) \subseteq Z^i(M^\bullet)$ and we define the $i$-th cohomology of $M^\bullet$ to be $H^i(M^\bullet) = Z^i(M^\bullet)/B^i(M^\bullet)$. A morphism of complexes $f^\bullet : M^\bullet \to N^\bullet$ is given by maps $f^i : M^i \to N^i$ for all $i \in \mathbb{Z}$ such that $d_N^i \circ f^i = f^{i+1} \circ d_M^i$ for all $i$.

Show that any morphism of complexes $f^\bullet$ induces a map $H^i(f^\bullet) : H^i(M^\bullet) \to H^i(N^\bullet)$ for all $i \in \mathbb{Z}$.

A morphism $f^\bullet$ is called a quasi-isomorphism if $H^i(f^\bullet)$ is an isomorphism for all $i$. Show that the following conditions are equivalent: (1) $M^\bullet$ is exact at every $M^i$, (2) $H^i(M^\bullet) = 0$ for all $i$, (3) the map $0 \to M^\bullet$ is a quasi-isomorphism.

Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor between abelian categories (for instance, the categories of modules over some rings). If $A$ has enough injectives, the $i$-th right derived functor $R^i F$ of $F$ is constructed as follows. For any $A \in \mathcal{A}$, take an injective resolution $A \to E^\bullet$ and define $R^i F(A) = H^i(F(E^\bullet))$. This definition does not depend on the choice of injective resolution and if $0 \to A' \to A \to A'' \to 0$ is exact, then there is a long exact sequence

$$0 \to A' \to A \to A'' \to R^1 F(A') \to R^1 F(A) \to R^1 F(A'') \to \ldots$$

$$\ldots \to R^i F(A') \to R^i F(A) \to R^i F(A'') \to R^{i+1} F(A') \to \ldots$$

If $0 \to A \to E \to M \to 0$ is exact and $E$ is injective, show that $R^i F(A) \simeq R^{i-1} F(M)$ for $i \geq 2$ and that $R^1 F(A) = \ker(F(E) \to F(M))$. More generally, show that if

$$0 \to A \to E^0 \to \ldots \to E^m \to M \to 0$$

is exact and all $E^i$ are injective, then $R^i F(A) \simeq R^{i-m-1} F(M)$ for $i \geq m + 2$ and $R^{m+1} F(A) = \ker(F(E^m) \to F(M))$.

Write down the corresponding “dimension shifting” statement for left derived functors of a right exact functor $F$ which are constructed using projective resolutions and convince yourself that a similar proof works in this case as well.

Let $M$ be an $A$-module. Consider the endofunctor $\operatorname{Mod}A \to \operatorname{Mod}A$ defined by $N \to N \otimes M$ and $f \mapsto f \otimes \operatorname{id}_M$. This functor is right exact and $\operatorname{Mod}A$ has enough projectives, so there exist left derived functors defined by $\operatorname{Tor}_i(M, N) = H^i(P_\bullet \otimes M)$, where $P_\bullet$ is any projective resolution of $N$. It is a fact
that Tor\(_i(M, N) = Tor\(_i(N, M) = H^i(P'_\bullet \otimes N), \) where P'_\bullet is any projective resolution of M. Furthermore, if 0 → N' → N → N'' → 0 is exact, we get a long exact sequence
\[ \cdots \rightarrow Tor_1(N', M) \rightarrow Tor_1(N, M) \rightarrow Tor_1(N'', M) \rightarrow N' \otimes M \rightarrow N \otimes M \rightarrow N'' \otimes M \rightarrow 0. \]

Suppose that a ∈ A is not a zero-divisor. Show that Tor\(_0(A/a, M) ≃ M/aM, Tor\(_1(A/a, M) ≃ \{ m ∈ M \mid am = 0 \} \) and Tor\(_n(A/a, M) = 0 \) for all n ≥ 2.

Show that the following conditions are equivalent: (1) N is flat, (2) Tor\(_n(M, N) = 0 \) for all n ≥ 1 and all modules M, (3) Tor\(_1(M, N) = 0 \) for all modules M.

**Exercise 4.** [2, Ex. 2.25 & 2.26]

(1) Let A be any ring and let 0 → N' → N → N'' → 0 be an exact sequence of A-modules with N'' flat. Show that N is flat if and only if N' is flat.

(2) Show that an A-module N is flat if and only Tor\(_1(A/I, N) = 0 \) for all finitely generated ideals I ⊆ A.

**References**