Complex Geometry
WS 16/17

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# Exercises 7

1. Consider the action of the multiplicative group $\mathbb{R}^\times = \mathbb{R} \setminus \{0\}$ on $X = \mathbb{R}^2 \setminus \{(0,0)\}$ defined by
$$\mathbb{R}^\times \times X \longrightarrow X, \quad \lambda.(x,y) = (\lambda \cdot x, \lambda^{-1} \cdot y).$$

(a) Show that the action is free.
(b) Show that the preimage of $\big([0,1] \times \{1\}\big) \times \big(\{1\} \times [0,1]\big) \subset X \times X$ under the map
$$\mathbb{R}^\times \times X \longrightarrow X \times X, \quad (\Lambda, p) \longmapsto (p, \Lambda.p)$$
is not compact. Conclude that the action is not proper.
(c) Show that the qotient $X/\mathbb{R}^\times$ is homeomorphic to the real line with origin doubled, a non-Hausdorff topoological space.
(d) Similarly, show that the holomorphic action of $\mathbb{C}^*$ on $Y = \mathbb{C}^2 \setminus \{(0,0)\}$ defined by $\lambda.(z,w) = (\lambda \cdot z, \lambda^{-1} w)$ is not proper. Describe the quotient $Y/\mathbb{Z}$ in analogy with (c).

2. Let $E$ be an elliptic curve, viewed as a one-dimensional complex Lie group, and let $\tau \in E$ be an $n$-torsion point (that is, $n \cdot \tau = \tau + \ldots + \tau = 0$ in $E$, but $k \cdot \tau \neq 0$ for $0 < k < n$). Let $\Sigma$ be a closed Riemann surface $\Sigma$ and $\sigma \in \mathrm{Aut}(\Sigma)$ a biholomorphism of order $n$, that is, $\sigma^n = \mathrm{id}$. Consider the following action of $G = \mathbb{Z}/n$ on $E \times \Sigma$:
$$k \cdot (x,y) = \big(x + k\tau, \sigma^k(y)\big).$$
Show:

(a) The action is proper and free.
(b) There is a holomorphic map of complex manifolds
$$\pi : X = (E \times \Sigma)/G \longrightarrow \Sigma/G$$
with fibres elliptic curves. Classsify the possible elliptic curves arising as such fibres.