An update on (small) quantum cohomology

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Introduction

Quantum cohomology originated from two sources, Gromov's technique of pseudo-holomorphic curves in symplectic geometry [Gv], and Witten's topological sigma models [Wi1][Wi2]. Mathematically it was given birth to late in 1993, when Ruan and Tian established a large part of it for semipositive symplectic manifolds, including the import classes of Calabi-Yau and Fano manifolds [RuTi]. More recently, several people succeeded in establishing quantum cohomology in the generality suggested in the system of axioms of Kontsevich and Manin [KoMa], both in the algebraic [Be1][LiTi1] and in the symplectic categories [FkOn][LiTi2][Ru2][Si2].

The purpose of this paper is to revisit some of the problems the author was able to study during the last three years in the light of the new methods. The result is a rather incomplete and personal view on quantum cohomology with an emphasis on how to compute small quantum cohomology rings from a good knowledge of the classical cohomology ring together with a minimal geometric input. This point of view is almost complementary to the problem in mirror symmmetry where such arguments do not help much. In view of the already vast literature I have only included such references that are directly related to the topics discussed here. I apologize to the many whose beautiful and deep contributions could not even be mentioned without overly enlarging the perspective of this note.

The first chapter gives a (rather formal) presentation of the new formulation of GW-theory, on which quantum cohomology is build. We do not comment on the actual construction of the invariants, for which we refer the interested reader to the forthcoming surveys [Be2] (algebraic) and [Si3] (symplectic). Rather we focus on how one can actually compute GW-invariants. The point I want to make there (if any) is that on one hand there is a closed formula for the invariants in the algebraic setting, which can sometimes be applied directly, while on the other hand it is occasionally easier and more instructive to argue symplectically. The latter point is illustrated by certain projective bundles over Fano manifolds (Proposition 1.1). The mentioned algebraic formula (Theorem 1.4) involves only the Chern class of an index bundle and the scheme theoretic structure of the relevant moduli space via Fulton's canonical class c_F. For future computations of GW-invariants it will be crucial to control the behaviour of c_F under basic geometric operations like decomposition into irreducible components of a reduced space. A systematic study of Fulton's canonical class would thus be highly desirable.

In the second chapter we present various versions of quantum cohomology rings and discuss them from the point of view of commutative algebra, i.e. as flat (analytic) deformations of Gorenstein Artinian $\mathbb{C}$-algebras. Applications include Gröbner basis computations of GW-invariants from a presentation of quantum cohomology rings (section 2.7) and residue formulas (Corollary 2.5) that in nice cases specialize to formulas of Vafa-Intriligator type (Proposition 2.6). For experts: The finer structure on the total space of the deformation that comes from the WDVV-equation will not be commented on here.

Chapter 3 outlines joint work with G. Tian on the (small) quantum cohomology of $\mathcal{N}_g$, the moduli space of stable 2-bundles of fixed determinant of odd degree over a genus g Riemann surface. It is shown that the recursion for the quantum relations proposed by physicists' [BeJoSaVa] is equivalant to a quantum version of Thaddeus' intersection-theoretic recursion involving the γ-class (Proposition 3.2 and Lemma 3.3). We then present a method to prove this "quantum intersection recursion" using an algebraic degeneration of $\mathcal{N}_g$ due to Gieseker. The proof is complete up to a conjectural vanishing of contributions of curves with irreducible components inside some bad locus.

Even before a rigorous definition of quantum cohomology was available, Batyrev has studied the case of arbitrary toric manifolds. It seemed to be worthwile to reconsider his arguments with the general theory at hand. The result that at least in the non-Fano case the investigation of quantum cohomology of toric varieties is still a rewarding and presumably treatable problem is presented in Chapter 4.

1 GW-invariants

1.1 Virtual fundamental classes

By the effort of several people we have now very satisfactory definitions of GW-invariants at our disposal, both in the algebraic [BeFa][Be1][LiTi1] and in the symplectic category [FkOn][LiTi2][Ru2][Si1]. Algebraically, the object under study is a projective scheme M, smooth over a field K, not necessarily algebraically closed or of characteristic zero. Symplectically, one looks at a compact symplectic manifold (M, ω) with *any* fixed tame almost complex structure J (tame: $\omega(X, JX) > 0\, \forall X \in TM \setminus \{0\}$). A common feature of all approaches is the use of a compactification of the space of pseudo-holomorphic maps from closed Riemann surfaces to M (respectively, morphisms from connected algebraic curves, proper and smooth over the base field K, to M) that has been proposed by Kontsevich under the name of "stable maps" [KoMa]: A *stable k-pointed (complex) curve in M* is a tuple $(C, \mathbf{x}, \varphi)$ with

- C is a complete, connected, reduced algebraic curve over $\mathbb{C}$ (respectively, K) with at most ordinary double points
- $\mathbf{x} = (x_1, \ldots, x_k)$ with pairwise distinct $x_i \in C_{\text{reg}}$ (respectively, $C_{\text{reg}}(K)$)
- $\varphi : C \to M$ is pseudo-holomorphic with respect to J (respectively, a morphism of K-schemes)
- $\mathrm{Aut}(C, \mathbf{x}, \varphi) = \{\sigma : C \to C \text{ biregular} \mid \sigma(\mathbf{x}) = \mathbf{x}, \varphi \circ \sigma = \varphi\}$ is finite

The last condition is referred to as *stability condition.* Here φ being pseudo-holomorphic means that φ is continuous and for any irreducible component $C_i \subset C$, $\varphi|_{C_i}$ is a morphism of almost complex manifolds. The arithmetic genus $g(C) = h^1(C, \mathcal{O}_C)$ of C is called *genus* of $(C, \mathbf{x}, \varphi)$.

The concept of stable curves in a manifold or K-variety should be viewed as natural generalization of the notion of (Deligne-Mumford) stable curves [DeMu]. The only difference is that the base $\mathrm{Spec}\, K$ (or a parameter space T) is now replaced by M (respectively $T \times M$).

The set $\mathcal{C}(M)$ of stable curves in M modulo isomorphism wears a natural topology, the Gromov-topology, making $\mathcal{C}(M)$ a Hausdorff topological space with compact connected components. In the algebraic setting this set can be identified with the K-rational points of an algebraic K-scheme, also denoted $\mathcal{C}(M)$, with proper connected components. In any of the general approaches to GW-invariants cited above one ends up with a homology class $[\![\mathcal{C}(M)]\!] \in H_*(\mathcal{C}(M), \mathbb{Q})$ (or rational Chow class), the *virtual fundamental class*. The notation is motivated by the fact that if for $R \in H_2(M; \mathbb{Z})$, $2g + k \geq 3$, the part

$$\mathcal{C}_{R,g,k}(M) \;=\; \Big\{(C, \mathbf{x}, \varphi) \in \mathcal{C}(M) \,\Big|\, g(C) = g, \sharp\mathbf{x} = k, \varphi_*[C] = R\Big\}$$

of $\mathcal{C}(M)$ is a manifold of the minimal dimension allowed by the Riemann-Roch theorem (the *expected dimension*)

$$d(M, R, g, k) \;:=\; 2 \dim \mathcal{M}_{g,k} + 2c_1(M, \omega) \cdot R + (1 - g) \cdot \dim M\,,$$

then $[\![\mathcal{C}(M)]\!]|_{\mathcal{C}_{R,g,k}(M)} = [\mathcal{C}_{R,g,k}(M)]$, the usual fundamental class with respect to the natural orientation given by the $\bar\partial_J$-operator. Here $\mathcal{M}_{g,k} = \mathcal{C}_{0,g,k}(\text{point})$ denotes the coarse moduli space of (Deligne-Mumford-) stable k-pointed curves of genus g. In general one should think of $[\![\mathcal{C}(M)]\!]$ as the limit of the fundamental classes (that might or might not exist in reality) of a sufficiently generic perturbation of $\mathcal{C}(M)$.

In the algebraic setting we now restrict for simplicity to the case $K = \mathbb{C}$. We can then use singular homology theory. Fixing R, g, k there are two continuous maps from $\mathcal{C}_{R,g,k}(M)$: The evaluation map ($k > 0$)

$$\mathrm{ev} : \mathcal{C}_{R,g,k}(M) \longrightarrow M^k, \quad (C, \mathbf{x}, \varphi) \longmapsto (\varphi(x_1), \ldots, \varphi(x_k))\,,$$

and the forgetful map ($2g + k \geq 3$)

$$p : \mathcal{C}_{R,g,k}(M) \longrightarrow \mathcal{M}_{g,k}, \quad (C, \mathbf{x}, \varphi) \longmapsto (C, \mathbf{x})^{\mathrm{st}}\,,$$

where $(C, \mathbf{x})^{\mathrm{st}}$ is the unique stable curve won by successive contraction of unstable components of $(C, \mathbf{x})$. Given the virtual fundamental class one may define, for any $R \in H_2(M; \mathbb{Z})$, $g \geq 0$, $k > 0$, $2g + k \geq 3$, a *GW-correspondence*

$$\begin{aligned} \mathrm{GW}^{M,J}_{R,g,k} : H^*(M; \mathbb{Q})^{\otimes k} \;&\longrightarrow\; H_*(\mathcal{M}_{g,k}; \mathbb{Q}) \\ \alpha_1 \otimes \ldots \otimes \alpha_k \;&\longmapsto\; p_*\Big([\![\mathcal{C}_{R,g,k}(M, J)]\!] \cap \mathrm{ev}^*(\alpha_1 \times \ldots \times \alpha_k)\Big)\,. \end{aligned}$$

It should be remarked here that the GW-correspondences can in principle be constructed from any compactification $\bar{\mathcal{C}}$ of a moduli space of maps from pointed stable curves to M dominated by $\mathcal{C}_{R,g,k}(M,J)$, i.e. with a map $\sigma : \mathcal{C}_{R,g,k}(M,J) \to \bar{\mathcal{C}}$ such that ev and p factor over σ and $\bar{\mathrm{ev}} : \bar{\mathcal{C}} \to M^k$, $\bar{p} : \bar{\mathcal{C}} \to \mathcal{M}_{g,k}$. In fact, setting $[\![\bar{\mathcal{C}}]\!] := \sigma_*[\![\mathcal{C}_{R,g,k}(M,J)]\!]$ as a virtual fundamental class for $\bar{\mathcal{C}}$, we get

$$\mathrm{GW}^{M,J}_{R,g,k}(\alpha_1 \otimes \ldots \otimes \alpha_k) \;=\; \bar{p}_*\Big([\![\bar{\mathcal{C}}]\!] \cap \bar{\mathrm{ev}}^*(\alpha_1 \times \ldots \times \alpha_k)\Big) .$$

For instance, in the algebraic setting natural candidates for $\bar{\mathcal{C}}$ are Hilbert or Chow scheme compactifications of graphs of maps (or rather k-fold fibered products of the universal objects to account for the marked points), viewed as subschemes or algebraic cycles on $\mathcal{C}_{g,k} \times M$ (in an orbifold sense, cf. 1.3 below). The reason why stable curves are often preferable is their much easier deformation theory.

The whole point of the theory is invariance under deformations of J inside the space of almost complex structures taming some symplectic form (or smooth projective deformations of M in the algebraic setting). The proof is by producing a relative virtual fundamental class for the family of spaces of stable curves in M over the parameter space ($S = [0,1]$ say) of the defomation, that restricts to the absolute virtual fundamental class for any fixed $J = J_s$ (cap product with the pull-back of the point class $\delta_s \in H^0_{\{s\}}(S)$). The (deformation class) of the symplectic structure on M being understood we will thus write $\mathrm{GW}^M_{R,g,k}$.

How does this tie up with the original definition of *GW-invariants* in [Ru1], [RuTi]? These were defined for semipositive symplectic manifolds or in dimensions up to 6 by fixing a Riemann surface Σ of genus g, k pairwise distinct points $x_i \in \Sigma$, submanifolds $A_1, \ldots, A_k$, $B_1, \ldots, B_l \subset M$ Poincaré-dual to $\alpha_1, \ldots, \alpha_k$, $\beta_1, \ldots, \beta_l \in H^*(M)$ and a sufficiently generic J through

$$\Phi^M_{R,g}(\alpha_1, \ldots, \alpha_k \mid \beta_1, \ldots, \beta_l) \;:=\; \sharp \left\{ \varphi : \Sigma \to M \;\middle|\; \begin{array}{l} \bar{\partial}_J \varphi = 0,\ \varphi_*[\mathbb{P}^1] = R \\ \varphi(x_i) \in A_i,\ \mathrm{im}\varphi \cap B_j \neq \emptyset \end{array} \right\} ,$$

if the dimensions match, and 0 otherwise. Here "$\sharp$" means an algebraic sum taking into account signs and multiplicities. In case $g = 0$ we will often drop the index g. Let $q : \mathcal{M}_{g,k+l} \to \mathcal{M}_{g,k}$ be the morphism forgetting the last l points (and stabilizing) and choose $x_1, \ldots, x_k \in \Sigma$ in such a way that $\mathrm{Aut}(\Sigma, (x_1, \ldots, x_k))$ is trivial. Then $\mathcal{M}_{g,k}$ is smooth in $P = (\Sigma, (x_1, \ldots, x_k))$. Let $\delta \in H^{\dim \mathcal{M}_{g,k}}_{\{P\}}(\mathcal{M}_{g,k}; \mathbb{Z})$ be the positive generator. It turns out that

$$\Phi^M_{R,g}(\alpha_1, \ldots, \alpha_k \mid \beta_1, \ldots, \beta_l) \;=\; \int \mathrm{GW}^M_{R,g,k+l}(\alpha_1 \otimes \ldots \otimes \alpha_k \otimes \beta_1 \otimes \ldots \otimes \beta_l) \cap q^*\delta .$$

Note that for the purpose of the computation of these invariants it suffices to know the push-forward class $\mathrm{ev}_*[\![\mathcal{C}_{R,g,k+l}(M)]\!] \in H_*(M^{k+l})$, that sometimes, especially if $l = 0$ and the cohomology of M is small in dimension $d(M, R, g, k+l)$, might be easier to handle.

1.2 Computation of GW-invariants I: The symplectic approach

To actually compute GW-invariants, according to the definition, one may take up a symplectic or algebraic point of view. In fact, it can be shown that in cases where both theories are applicable, i.e. for complex projective manifolds, the two definitions yield the same result [Si4][1]. Either approach has its virtues. In this and the next section I want to illustrate this starting with a computation that is more easily done by symplectic methods, even in the projective case.

Let (N, ω) be a positive sympletic manifold, i.e. $c_1(N, J) \cdot \varphi_*[\mathbb{P}^1] > 0$ for any J-holomorphic rational curve $\varphi : \mathbb{P}^1 \to N$, J some ω-tame almost complex structure on N. A large class of such manifolds are provided by Fano-manifolds, which are projective algebraic manifolds with ample anticanonical bundle. Examples will be given below. Let E be a complex vector bundle on N of rank r. The associated projective bundle (of lines) $\pi : P = \mathbb{P}(E) \to N$ has a distinguished deformation class of symplectic structures. A symplectic form $\tilde{\omega}$ on P can be constructed from any hermitian metric on $\mathcal{O}_P(1)$ by adding a sufficiently large multiple of $\pi^*\omega$. Under the assumptions

- $c_1(E) = 0$
- there exists a section $s : N \to P$ with $s^* c_1(\mathcal{O}_P(1)) = 0$

we want to relate certain GW-invariants on P to GW-invariants on N.

Proposition 1.1 *Let $R \in H_2(N; \mathbb{Z})$, $\alpha_1, \ldots, \alpha_k \in H^*(N)$ and let $\sigma \in H^*(P)$ be Poincaré-dual to $s_*[N]$. Then*

$$\Phi^N_R(\alpha_1, \ldots, \alpha_k) \;=\; \Phi^P_{s_*R}(\pi^*\alpha_1, \ldots, \pi^*\alpha_k, \sigma)\,.$$

This relation is the key to the computation of quantum cohomology rings of certain moduli spaces of stable bundles, cf. Chapter 3.
Sketch of proof. Using $c_1(E) = 0$ and $s^*c_1(\mathcal{O}_P(1)) = 0$ one checks that the dimensions match on the left-hand side iff they do on the right-hand side. By Ruan's original definition [Ru1] we have the following recipe for the computation of $\Phi^N_R(\alpha_1, \ldots, \alpha_k)$: Choose oriented submanifolds $A_1, \ldots, A_k \subset N$ Poincaré-dual to $\alpha_1, \ldots, \alpha_k$ (replace α_i by some multiple if necesary) and k generic points $x_1, \ldots, x_k \in \mathbb{P}^1$. Then for a generic almost complex structure J on N

$$\Phi^N_R(\alpha_1, \ldots, \alpha_k) \;=\; \sharp \left\{ \varphi : \mathbb{P}^1 \to M \;\middle|\; \bar{\partial}_J \varphi = 0,\ \varphi_*[\mathbb{P}^1] = R,\ \varphi(x_i) \in A_i \right\}\,,$$

where again "$\sharp$" means counting with signs according to the natural orientations involved (there is no multiplicity to be observed in case when no B_j's are present). For simplicity let us assume $\dim_{\mathbb{C}} N > 2$. Then J can be chosen in such a way that the finitely many relevant $\varphi : \mathbb{P}^1 \to N$ are embeddings and pairwise disjoint.

[1] J. Li and G. Tian also seem to prepare a paper in that direction using their definitions

By choosing a not necessarily integrable partial connection $\bar{\partial}_E$ on E (i.e. $(0,1)$-form valued with respect to J) we get an $\tilde{\omega}$-tamed almost complex structure $\tilde{J}$ on P compatible with $\pi : P \to (N, J)$ and inducing the standard complex structure on the fibres of π. The essential observation is that since $c_1(E) = 0$ and since the integrability condition is void over $\mathbb{P}^1$, $\varphi^* E$ is a *holomorphic* $\mathbb{P}^{r-1}$-bundle over $\mathbb{P}^1$ of degree zero. By slightly perturbing $\bar{\partial}_E$ along the images of the finitely many φ we may assume $\varphi^*(P, \tilde{J}) \simeq \mathbb{P}^1 \times \mathbb{P}^{r-1}$ holomorphically, because any holomorphic vector bundle over $\mathbb{P}^1$ of degree zero can be infinitesimally deformed to the trivial bundle. Now for any such $\varphi : \mathbb{P}^1 \to N$, $\varphi(x_i) \in A_i$, $i = 1, \ldots, k$, choose another point $x_{k+1} \in \Sigma$ sufficiently generic, put $q := \pi^{-1}(\varphi(x_{k+1})) \cap \operatorname{im} s$, and define

$$\tilde{\varphi} : \mathbb{P}^1 \to P$$

as the composition of the constant section $\mathbb{P}^1 \to \mathbb{P}^1 \times \{0\}$ with the map of total spaces $\varphi^* P \to P$ induced by φ. Observing $c_1(s^* E) = 0$ we get

$$\tilde{\varphi}_*[\mathbb{P}^1] = s_* R, \quad \tilde{\varphi}(x_i) \in \pi^{-1}(A_i) \ (i = 1, \ldots, k), \quad \tilde{\varphi}(x_{k+1}) \in \operatorname{im} s\,,$$

and since π is $(\tilde{J}, J)$-holomorphic any such $\tilde{\varphi}$ is of this form. It is not hard to check that the deformation theory is unobstructed at the $\tilde{\varphi}$ if it was for the φ. Unobstructedness means surjectivity of the linearization of the relevant Fredholm operator $\bar{\partial}_J$. P being positive or not we thus get an enumerative description of $\Phi^P_{s_* R}(\pi^* \alpha_1, \ldots, \pi^* \alpha_k, \sigma)$ as

$$\sharp \left\{ \tilde{\varphi} : \mathbb{P}^1 \to P \,\middle|\, \tilde{\varphi}_*[\mathbb{P}^1] = s_* R,\ \varphi(x_i) \in \tilde{\pi}^{-1}(A_i) \text{ for } i \le k,\ \tilde{\varphi}(x_{k+1}) \in s(N) \right\} .$$

A final check of orientations (thus signs) finishes the proof. $\diamond$

In case of projective algebraic N one can probably modify the approach to give a purely algebraic proof. But since one has to work with trees of rational curves and higher dimensional families this will be much harder. And the symplectic point of view shows clearly the geometric reason behind the formula, which is the possibility of lifting rational curves on N to P by differentiable triviality of P along such curves.

1.3 Computation of GW-invariants II: Relation with Fulton's canonical class

In the algebraic setting one often has good computational control over a particular variety, e.g. by additional symmetries or special geometry, but the integrable complex structure is not generic with regard to certain moduli spaces of rational (or higher genus) curves. We begin this section with some remarks concerning genericity in deformation theory, which might not be so well-known to non-experts.

It is a trivial but very remarkable and useful fact that in the algebraic category one can check for genericity just by counting dimensions. We first give a somewhat more general statement:

Proposition 1.2 *Let M be a projective algebraic manifold, $R \in H_2(M, \mathbb{Z})$, $2g+k \geq 3$. The following statements are equivalent:*

1. $[\![\mathcal{C}_{R,g,k}(M)]\!] = [\mathcal{C}_{R,g,k}(M)]$
2. *Any irreducible component of $\mathcal{C}_{R,g,k}(M)$ has dimension $d(M, R, g, k)$*
3. *$\mathcal{C}_{R,g,k}(M)$ is a locally complete intersection of dimension $d(M, R, g, k)$.*

Here $[\mathcal{C}_{R,g,k}(M)]$ is the ordinary fundamental class defined for any algebraic variety [Fu, 1.5]. In the situation of the proposition the Gromov-Witten invariants based on (R, g, k) can thus be considered to be *enumerative* in the usual algebraic-geometric manner.

Before turning to the simple proof of the proposition we want to recall some facts on obstruction theory. An obstruction space for the deformation theory of the triple $(C, \mathbf{x}, \varphi)$ is the complex vector space $\mathrm{Ext}^1([\varphi^*\Omega_M \to \Omega_C(|x|)], \mathcal{O}_C)$ classifying extensions of the complex $[\varphi^*\Omega_M \to \Omega_C(|x|)]$ by the complex $[0 \to \mathcal{O}_C]$ [Fl],[Ra],[LiTi2]. In the notation of [Fl] this space is written $T^2(\iota/M, \mathcal{O}_C)$, i.e. one thinks of deforming the inclusion $\iota : |x| \hookrightarrow C$ as a morphism *over* M. Since $T^2(|x|/C) = 0$ we have $T^2(\iota/M, \mathcal{O}_C) = T^2(C/M)$ which fits into the exact sequence [Fl, I,2.25]

$$\begin{array}{ccccccc}
T^0(C)/T^0(C/M) & \longrightarrow & \mathrm{Ext}^0(\varphi^*\Omega_M, \mathcal{O}_C) & \longrightarrow & T^1(C/M) & \longrightarrow & T^1(C) \\
\| & & \| & & \| & & \| \\
0 & \longrightarrow & \Gamma(C, \varphi^*T_M) & \longrightarrow & T^1(C/M) & \longrightarrow & \mathrm{Ext}^1(\Omega_C, \mathcal{O}_C) \\
 & & & & & & \\
 & \longrightarrow & \mathrm{Ext}^1(\varphi^*\Omega_M, \mathcal{O}_C) & \longrightarrow & T^2(C/M) & \longrightarrow & T^2(C) \\
 & & \| & & \| & & \| \\
 & \longrightarrow & H^1(C, \varphi^*T_M) & \longrightarrow & T^2(C/M) & \longrightarrow & 0
\end{array}$$

In particular, the obstructions vanish if $H^1(C, \varphi^*T_M) = 0$. $T^1(C)$ is the Zariski tangent space to the deformations of C which are unobstructed ($T^2(C) = 0$). I should also remark that since we already have nice moduli spaces $\mathcal{M}_{g,k}$ for pointed curves one can also consider an easier deformation problem, namely relative $\mathcal{M}_{g,k}$, i.e. with $(C, \mathbf{x})$ fixed. The Zariski tangent and obstruction spaces to this deformation problem are just $\Gamma(C, \varphi^*T_M)$ and $H^1(C, \varphi^*T_M)$. This has been used by Behrend in his treatment of GW-invariants [Be1].

Proof (of proposition). Deformation theory exhibits a formal or analytic germ of a semiuniversal deformation space by $t_2 = \dim T^2(\iota/M, \mathcal{O}_C)$ equations in a t_1-dimensional complex vector space, $t_1 = \dim T^1(\iota/M, \mathcal{O}_C)$. But by a Riemann-Roch computation

$$t_2 - t_1 \;=\; d(M, R, g, k)\,.$$

This shows the equivalence of (2) and (3) and also with the fact that the virtual fundamental class is top-dimensional on each irreducible component. The multiplicities can then be checked at closed points lying in only one irreducible component. $\diamond$

To actually conclude genericity of the integrable complex structure on M in the sense of the symplectic approach we need generic smoothness of $\mathcal{C}_{R,g,k}(M)$ or, equivalently, generic vanishing of the obstruction spaces.

Proposition 1.3 *Assume one of the equivalent conditions in Proposition 1.2 be fulfilled. Then there are equivalent*

1. *$[\mathcal{C}_{R,g,k}(M)]$ has no multiple components, i.e. any irreducible component of $\mathcal{C}_{R,g,k}(M)$ is generically reduced.*
2. *Any irreducible component of $\mathcal{C}_{R,g,k}(M)$ contains some $(C,\mathbf{x},\varphi)$ with $T^2(\iota : |x| \hookrightarrow C/M, \mathcal{O}_C) = 0$ (sufficient: $H^1(C,\varphi^*T_M) = 0$)*
3. *Any irreducible component of $\mathcal{C}_{R,g,k}(M)$ contains some $(C,\mathbf{x},\varphi)$ with $\dim T^1(\iota : |x| \hookrightarrow C/M, \mathcal{O}_C) = d(M,R,g,k)$ (sufficient: $\dim H^0(C,\varphi^*T_M) = c_1(M)\cdot R + (1-g)\dim M = d(M,R,g,k) - \dim \mathcal{M}_{g,k}$)* $\diamond$

Case 2 is applicable for instance if $g = 0$ and M is what is called *convex*, which by definition means $H^1(\mathbb{P}^1,\varphi^*T_M) = 0$ for any $\varphi : \mathbb{P}^1 \to M$. Examples include manifolds with globally generated tangent bundles, like generalized flag varieties G/P, P a parabolic subgroup of the semisimple Lie group G.

Unfortunately, it happens quite often that $\mathcal{C}_{R,g,k}(M)$ has larger than expected dimension, in the case of which one really has to deal with the somewhat non-explicit virtual fundamental classes. There is however a closed formula for this class in terms of Fulton's canonical class and the associated index bundle [Si2]. Recall that if X is an algebraic variety embeddable in a smooth scheme N (e.g. X projective) with ideal sheaf $\mathcal{I}$ the *normal cone of X in N* is $C_{X/N} = \mathrm{Spec}_X \oplus_d \mathcal{I}^d/\mathcal{I}^{d+1}$. $C_{X/N}$ is a closed subscheme of the normal bundle $N_{X/N} = \mathrm{Spec}_X S^\bullet(\mathcal{I}/\mathcal{I}^2)$, which is not a vector bundle but the linear space over X associated to the conormal sheaf (the fiber dimensions may jump). Such cones have a Segre class $s(C_{X/N}) \in A_*(X)$, a Chow class on X. Fulton shows that the class

$$c_F(X) \ := \ c(T_N|_X) \cap s(C_{X/N}) \in A_*(X)$$

does not depend on the choice of embedding $X \hookrightarrow N$ [Fu, Expl.4.2.6]. We will call this class *Fulton's canonical class*. For smooth X, $c_F(X)$ coincides with the total Chern class $c(T_X) \cap [X]$.

We apply this to $X = \mathcal{C}_{R,g,k}(M)$ with $g = 0$. Fixing an embedding $i : M \hookrightarrow \mathbb{P}^n$ there is a canonical choice for the smooth space N into which X embeds, namely $\mathcal{C}_{i_*R,0,k}(\mathbb{P}^n)$. I purposely wrote "space" because as a scheme or projective variety $\mathcal{C}_{i_*R,0,k}(\mathbb{P}^n)$ is *not* in general smooth, no matter that the deformation theory is always unobstructed.

This situation is familiar in the case of $\mathcal{M}_{g,k} = \mathcal{C}_{0,g,k}(\mathbb{P}^0)$, where the presence of finite automorphism groups spoil both smoothness and the existence of a universal family. A by-pass of this problem is to avoid taking quotients by finite groups and work in a category of algebraic or analytic orbifolds (smooth case) or orbispaces (non-smooth case). An orbispace consists of a scheme (respectively complex space) X together with a covering by affine schemes exhibited as quotients of affine schemes by finite groups acting algebraically (respectively an open covering by finite group quotients of complex spaces). The transition functions are required to admit liftings to (open sets in) the uniformizing spaces.

This works in fact well in the analytic category: For a complex projective variety M, $\mathcal{C}_{R,g,k}(M)$ has the structure of an analytic orbispace. It is a fine moduli space in the category of analytic orbispaces, the universal family and universal morphism being the natural morphism of analytic orbispaces $\pi : \mathcal{C}_{R,g,k+1}(M) \to \mathcal{C}_{R,g,k}(M)$ and $\mathrm{ev}_{k+1} : \mathcal{C}_{R,g,k+1}(M) \to M$ given by forgetting the last marked point and stabilizing, and by evaluating the map at the $(k+1)$-st point.

In the algebraic category there might be a problem with this approach, because liftings of the transition maps at $(C, \mathbf{x}, \varphi)$ with $(C, \mathbf{x})$ having non-stable components would require the use of the implicit function theorem, which does not hold in the Zariski topology. Instead one uses a more general concept, called *(Deligne-Mumford) stacks.* For this one defines the notion of (algebraic) family of stable curves in M parametrized by a base scheme T and morphisms between such families. This becomes a category F which has a forgetful functor $p : F \to \mathcal{S}\mathrm{ch}$ to the category of schemes by sending a family to the base. This functor has the property that "pull-backs" exist (which amounts to changing the base of a family) and that fixing a scheme T the morphisms in F over Id_T are all isomorphisms: $p : F \to \mathcal{S}\mathrm{ch}$ is a *fibered groupoid.* Given a scheme X one can associate to X the fibered groupoid of morphisms of another scheme T into X, the functor sending $T \to X$ to T. This determines X uniquely. And by our fibered groupoid of (families of) stable curves in M we just replace the often non-existent scheme-theoretic moduli space of such curves by this functor.

To make fibered groupoids more scheme-like, one imposes various other conditions: The analogs of the sheaf axioms for p (such that one can glue local constructions) — such fibered groupoids are called *stacks*; a couple of technical assumptions on the diagonal morphism $\Delta_F : F \to F \times_{\mathrm{Spec}\, K} F$ (representability, quasi-compactness, separatedness) plus the existence of a dominating scheme U, i.e. a smooth (!) surjective morphism $U \to F$ — *algebraic stacks* (also called *Artin stacks*). If $U \to F$ can be chosen with finite fibers we finally end up with the notion of *Deligne-Mumford stacks* (DM-stacks for shortness). A nice introduction to Deligne-Mumford stacks is in the appendix of [Vi], to which we refer for details.

Many DM-stacks have proper coverings, which is a proper surjective morphism from a scheme X to F. In this case one can define Chow groups and an intersection theory on F by doing intersection theory on X and pushing forward [Vi]. Stacks of stable curves in a smooth projective variety M always have proper coverings. The upshot of this is that for the purpose of GW-invariants one may work with the DM-stack $\mathcal{C}_{R,g,k}(M)$ as if it were a scheme, cf. [BeMa].

Whatever formulation we choose — analytic orbispaces or DM-stacks — we get a closed embedding $I : \mathcal{C}_{R,0,k}(M) \hookrightarrow \mathcal{C}_{i_*R,0,k}(\mathbb{P}^n) =: N$ into a smooth space and hence a generalization of Fulton's canonical class

$$c_F(\mathcal{C}_{R,0,k}(M)) \ := \ c(I^*T_N) \cap s(C_{\mathcal{C}_{R,0,k}(M)/N})\,.$$

This is a Chow or homology class on the underlying variety or a Chow class on the DM-stack $\mathcal{C}_{R,0,k}(M)$ as defined in [Vi], always with coefficients in $\mathbb{Q}$.

To state the closed formula for the virtual fundamental class we only need one more ingredient: From the evaluation map $\mathrm{ev} = \mathrm{ev}_{k+1} : \mathcal{C}_{R,g,k+1}(M) \to M$ one gets two (orbi-) sheaves $R^i\pi_*\mathrm{ev}^*T_M$ $(i = 0,1)$ on $\mathcal{C}_{R,g,k}(M)$, $\pi : \mathcal{C}_{R,g,k+1}(M) \to \mathcal{C}_{R,g,k}(M)$ the universal curve. $\pi_*\mathrm{ev}^*T_M$ is nothing but the relative tangent sheaf of $\mathcal{C}_{R,g,k}(M)$ over $\mathcal{M}_{g,k}$, and $R^1\pi_*\mathrm{ev}^*T_M$ is the relative obstruction sheaf (but the comparison map to the actual relative obstruction spaces $H^1(C,\varphi^*T_M)$ at some $(C,\mathbf{x},\varphi) \in \mathcal{C}_{R,g,k}(M)$ is not in general an isomorphism). By twisting ev^*T_M with a sufficiently ample line bundle one can show that the virtual sheaf $[\pi_*\mathrm{ev}^*T_M] - [R^1\pi_*\mathrm{ev}^*T_M] \in K_*(\mathcal{C}_{R,g,k}(M))$, the Grothendieck group of coherent sheaves, can actually be represented by the difference of two vector bundles, i.e. $[\pi_*\mathrm{ev}^*T_M] - [R^1\pi_*\mathrm{ev}^*T_M] \in K^*(\mathcal{C}_{R,g,k}(M))$, and thus has a Chern class. Let us denote this virtual bundle by $\mathrm{ind}^M_{R,g,k}$, the *index bundle* or *virtual tangent bundle* of $\mathcal{C}_{R,g,k}(M)$. The rank of $\mathrm{ind}^M_{R,g,k}$ is constant and coincides with $d(M,R,g,k) - \dim\mathcal{M}_{g,k}$.

Theorem 1.4 [Si2] *Let M be a projective algebraic manifold, $R \in H_2(M;\mathbb{Z})$. If $g = 0$ or $\mathcal{C}_{R,g,k}(M)$ is embeddable into a smooth space then*

$$[\![\mathcal{C}_{R,g,k}(M)]\!] \ = \ \Big\{ c(\mathrm{ind}^M_{R,g,k})^{-1} \cap c_F(\mathcal{C}_{R,g,k}(M)/\mathcal{M}_{g,k}) \Big\}_{d(M,R,g,k)}\,,$$

where $c_F(\mathcal{C}_{R,g,k}(M)/\mathcal{M}_{g,k}) \ = \ p^*c(T_{\mathcal{M}_{g,k}})^{-1} \cap c_F(\mathcal{C}_{R,g,k}(M)$ *is the* relative Fulton Chern class *and $\{\,.\,\}_d$ denotes the d-dimensional part.* $\diamond$

It is remarkable that the expression on the right-hand side depends only on the scheme-theoretic structure of $\mathcal{C}_{R,g,k}(M)$ and the total Chern class of a virtual vector bundle. It should thus be more accessible to computations than the original definition by cones. Unfortunately, Fulton's canonical class does not seem to show nice functorial behaviour. Let us illustrate the situation at the following realistic scenario: Assume given an explicit family of k-pointed stable curves in M containing all the curves with given $R \in H_2(M;\mathbb{Z})$ and genus g, with proper base T and such that generically no two curves in the family coincide. In other words, the family is induced by a birational morphism $f : T \to \mathcal{C}_{R,g,k}(M)$. $f^*\mathrm{ind}^M_{R,g,k}$ is nothing but the index bundle ind_T of the family. Now for a birational morphism $f_*(c_F(T))$ differs from $c_F(\mathcal{C}_{R,g,k}(M))$ only by classes in the exceptional locus. For instance, if we are given the blow-up $\sigma : \tilde{X} \to X$ of a smooth space in a smooth subvariety $\iota : Y \hookrightarrow X$ of codimension d, it follows from [Fu, Thm. 15.4] that

$$\sigma_*(c_F(\tilde{X})) = c_F(X) + (d-1)\iota_*c_F(Z)\,.$$

So one would expect the virtual fundamental class to be close to

$$\Big\{ c(\mathrm{ind}_T)^{-1} \cap c_F(T) \Big\}_{d(M,R,g,k)}\,.$$

In special cases contributions from the exceptional locus might be controllable, or ignorable for the computation of certain GW-invariants. For example,

Corollary 1.5 *In the situation of Theorem 1.4 let $Z \subset \mathcal{C}_{R,g,k}(M)$ be a subspace of dimension less than $d(M,R,g,k)$, $\mathcal{C}_{\mathrm{gd}} := \mathcal{C}_{R,g,k}(M) \setminus Z$ (the "good subspace"). Then $[\![\mathcal{C}_{R,g,k}(M)]\!]$ is the unique Chow class extending*

$$\left\{ c(\mathrm{ind}^M_{R,g,k}|_{\mathcal{C}_{\mathrm{gd}}})^{-1} \cap c_F(\mathcal{C}_{\mathrm{gd}}/\mathcal{M}_{g,k}) \right\}_{d(M,R,g,k)} . \qquad \diamond$$

In cases where $\mathcal{C}_{R,g,k}(M)$ has sufficiently mild singularities away from Z we can get away without computing Fulton's canonical class at all:

Corollary 1.6 *In the situation of Corollary 1.5 assume also that $\mathcal{C}_{\mathrm{gd}}$ is a locally complete intersection of dimension $d(M,R,g,k)+r$. Then the class of the obstruction sheaf $\mathrm{ob}_{\mathrm{gd}} := R^1\pi_*\mathrm{ev}^*T_M|_{\mathcal{C}_{\mathrm{gd}}}$ is in $K^*(\mathcal{C}_{\mathrm{gd}})$ and $[\![\mathcal{C}_{R,g,k}(M)]\!]$ is the unique Chow class extending*

$$c_r(\mathrm{ob}_{\mathrm{gd}}) \cap [\mathcal{C}_{\mathrm{gd}}] .$$

Proof. Let N be the smooth space into which $\mathcal{C}_{\mathrm{gd}}$ embeds, $\mathcal{I}$ the corresponding ideal sheaf. By going over to $N \times \mathcal{M}_{g,k}$ we may assume that $p : \mathcal{C}_{\mathrm{gd}} \to \mathcal{M}_{g,k}$ extends to N. Since $\mathcal{C}_{\mathrm{gd}}$ is a complete intersection the cotangent sequence relative $\mathcal{M}_{g,k}$ reads

$$0 \longrightarrow \mathcal{I}/\mathcal{I}^2 \longrightarrow \Omega_{N/\mathcal{M}_{g,k}}|_{\mathcal{C}_{\mathrm{gd}}} \longrightarrow \Omega_{\mathcal{C}_{\mathrm{gd}}/\mathcal{M}_{g,k}} \longrightarrow 0$$

with $\mathcal{I}/\mathcal{I}^2 = N^\vee_{\mathcal{C}_{\mathrm{gd}}/N}$ locally free. Dualizing we see that $\mathcal{T}_{\mathcal{C}_{\mathrm{gd}}/\mathcal{M}_{g,k}} = \mathrm{Hom}(\Omega_{\mathcal{C}_{\mathrm{gd}}/\mathcal{M}_{g,k}}, \mathcal{O}_{\mathcal{C}_{\mathrm{gd}}}) = \pi_*\mathrm{ev}^*T_M$ and in turn $\mathrm{ob}_{\mathrm{gd}}$ are in $K^*(\mathcal{C}_{\mathrm{gd}})$. With $C_{\mathcal{C}_{\mathrm{gd}}/N} = N_{\mathcal{C}_{\mathrm{gd}}/N}$ we get

$$\begin{aligned} & c(\mathrm{ind}^M_{R,g,k}|_{\mathcal{C}_{\mathrm{gd}}})^{-1} \cap c_F(\mathcal{C}_{\mathrm{gd}}/\mathcal{M}_{g,k}) \\ &\quad = \Big(c(\mathcal{T}_{\mathcal{C}_{\mathrm{gd}}/\mathcal{M}_{g,k}})^{-1} \cup c(\mathrm{ob}_{\mathrm{gd}}) \cup c(T_{N/\mathcal{M}_{g,k}}) \cup c(N_{\mathcal{C}_{\mathrm{gd}}/N})^{-1} \Big) \cap [\mathcal{C}_{\mathrm{gd}}] \\ &\quad = c(\mathrm{ob}_{\mathrm{gd}}) \cap [\mathcal{C}_{\mathrm{gd}}] . \end{aligned}$$

$\diamond$

Cf. also the recent paper [Al] for a comparison of Fulton's canonical class with the functorially much better behaved MacPherson Chern class of singular varieties in the hypersurface case.

2 On the algebraic structure of quantum cohomology rings

2.1 Quantum cohomology rings

Quantum cohomology is based on the observation of Witten that by degenerating the domain from a Riemann surface to a nodal curve one expects to observe many relations between GW-invariants. These will be responsible for the fact that certain deformations of the multiplicative structure of the cohomology ring involving GW-invariants will be associative.

Now that GW-invariants are based on virtual fundamental classes the relations reduce to the simple statement that virtual fundamental classes are compatible with restriction to the divisors $D_i \subset \mathcal{M}_{g,k}$ of nodal curves. With the general technique of virtual fundamental classes at hand the proof does not present much difficulty, cf. the papers quoted at the beginning of Chapter 1.

Especially important is the case $g = 0$ where the relations can be cast in a family of deformations of the ring structure on $H^*(M)$, the quantum cohomology rings. Let $\{\gamma_i\}$ be a basis of $H^*(M)$ and $\gamma_i^\vee \in H^*(M)$ be the Poincaré-dual basis. The quantum product depends on the choice of another cohomology class η, which we fix for the time being. For α, $\beta \in H^*(M)$ one defines

$$\alpha *_\eta \beta \ := \ \sum_i \sum_{R\in H_2(M;\mathbb{Z})} \Big(\sum_{r\geq 0} \frac{1}{r!} \Phi^M_R(\alpha, \beta, \gamma_i \mid \underbrace{\eta, \dots, \eta}_{r \text{ times}}) \gamma_i^\vee \Big) [R]\,,$$

which a priori takes values in $H^*(M) \otimes_{\mathbb{Z}} \mathbb{Z}[[H_2(M;\mathbb{Z})]]$. Unless $\eta = 0$, referred to as *small* case (small quantum cohomology ring etc.), there is already one convergence assumption to be made for the inner bracket. Namely,

For any $R \in H_2(M;\mathbb{Z})$ *and* α, β, $\gamma \in H^*(M)$

$$\textstyle\sum_{r\geq 0} \frac{1}{r!} \Phi^M_R(\alpha, \beta, \gamma \mid \underbrace{\eta, \dots, \eta}_{r \textit{ times}}) \qquad \text{(conv 1)}$$

converges.

(One could remedy this by introducing formal variables for η too, but this does not help in producing a ring structure.) Conjecturally, this should alwas be true for η suffciently small, but is already non-trivial to check for $\mathbb{P}^2$, say.

Note that $\mathbb{Z}[[H_2(M;\mathbb{Z})]]$ contains copies of $\mathbb{Z}[[t, t^{-1}]]$ and thus the multiplicative structure on $\mathbb{Z}[H_2(M;\mathbb{Z})]$ does not extend. On the other hand, contributions to $\alpha *_\eta \beta$ are non-zero only if R can be represented by a rational curve (respectively, pseudo-holomorphic rational curve). An even smaller but more intrinsically defined set of classes is the monoid $\mathcal{R}\mathrm{C}\,(M) \subset H_2(M;\mathbb{Z})$ generated by classes R such that some GW-invariant Φ_R does not vanish. Any symplectic form ω' that tames J (or ample $\mathbb{Q}$-class in the algebraic setting) evaluates positively on $\mathcal{R}\mathrm{C}\,(M) \setminus \{0\}$ by the analog of the Wirtinger theorem. And tameness is an open condition. Letting $|\,.\,|$ be any norm on $H_2(M;\mathbb{Q})$ we thus conclude the existence of a $\lambda > 0$ with

$$\omega(R) \ \geq \ \lambda \cdot |R| \quad \forall\, R \in \mathcal{R}\mathrm{C}\,(M)\,.$$

In other words, $\mathbb{P}(\mathcal{R}\mathrm{C}\,(M))$ is bounded away from $\mathbb{P}(\omega^\perp) \subset \mathbb{P}(H_2(M;\mathbb{Q}))$. So with the proviso of (conv 1), $*_\eta$ defines a product on $H^*(M) \otimes_{\mathbb{Z}} \mathbb{Z}[[\mathcal{R}\mathrm{C}\,(M)]]$.[2] The

[2] A similar way to get a sensible domain of definition for quantum multiplication is the use of the so-called Novikov-ring, the partial completion of $H^*(M) \otimes \mathbb{Z}[H_2(M;\mathbb{Z})]$ with respect to an order on $H_2(M;\mathbb{Z})$ defined by the symplectic from ω, cf. [McSa, 9.2]. And in the algebraic setting it is often natural to admit coefficients in the nef cone $\mathrm{NE}\,(M) \supset \mathcal{R}\mathrm{C}\,(M)$, which by the Hodge index theorem is also strongly convex

degeneration relations show that $*_\eta$ is associative. The contribution to $R = 0$ is nothing but the ordinary cup product.

To get an actual analytic deformation of the cup product ring structure on $H^*(M)$ one can try to replace $[R]$ by $e^{-\omega(R)}$ (or $e^{i(B+i\omega)(R)}$, $B \in H^*(M)$, as one should do in mirror symmetry). Stronger convergence assumptions are to be made:

For any $\alpha, \beta, \gamma \in H^*(M)$, $\Phi^M_R(\alpha, \beta, \gamma \mid \eta, \dots, \eta)$ *is exponentially bounded in* $\omega(R)$ *and*

$$\sum_{r\geq 0} \tfrac{1}{r!}\Big(\sum_{R\in\mathcal{R}\mathrm{C}\,(M)} \Phi^M_R(\alpha,\beta,\gamma \mid \underbrace{\eta,\dots,\eta}_{r\ \text{times}})e^{-t\omega(R)}\Big) \tag{conv 2}$$

converges for $t \gg 0$.

Then for ω sufficiently positive

$$\alpha *_{\eta,\omega} \beta \ := \ \sum_i \sum_{r\geq 0} \frac{1}{r!} \sum_{R\in\mathcal{R}\mathrm{C}\,(M)} \Phi^M_R(\alpha,\beta,\gamma_i \mid \underbrace{\eta,\dots,\eta}_{r\ \text{times}})e^{-\omega(R)}$$

defines a ring structure on $H^*(M;\mathbb{C})$ that for ω tending to infinity approaches the cup product. To see this more explicitely and in a form appropriate for the study of mirror symmetry choose *integral* smplectic forms $\omega_1, \dots, \omega_{b_2}$ spanning $H^2(M)$ and with convex hull lying inside the symplectic cone (classes of symplectic forms), and a dual basis $q_1, \dots, q_{b_2}$ of $H_2(M;\mathbb{Q})$. A third variant of the quantum product lives on $H^*(M) \otimes_{\mathbb{Q}} \mathbb{C}\{q_1, \dots, q_{b_2}\}$ by setting

$$\alpha *_\eta \beta \ := \ \sum_i \sum_{r\geq 0} \frac{1}{r!} \sum_{R\in\mathcal{R}\mathrm{C}\,(M)} \Phi^M_R(\alpha,\beta,\gamma_i \mid \eta,\dots,\eta)\gamma_i^\vee q_1^{\omega_1(R)} \cdots q_{b_2}^{\omega_{b_2}(R)}\,.$$

This should be viewed as a family of ring structures on $H^*(M)$ analytically parametrized by an open neighbourhood of $0 \in \mathbb{C}^{b_2}_{q_1,\dots,q_{b_2}}$. For $\omega = \sum_i a_i\omega_i$, $a_1, \dots, a_{b_2} \geq 0$ one retrieves the previous form of the quantum product $*_{\eta,\omega}$ by setting $q_i = e^{-a_i}$. And putting $q_i = 0$ for any i we get the ordinary cohomology ring. Note that $\mathcal{R}\mathrm{C}\,(M)$ is a submonoid of $\mathbb{N}_{\geq 0}q_1 + \dots + \mathbb{N}_{\geq 0}q_{b_2}$ and that the latter monoid depends on the choice of $\omega_1, \dots, \omega_{b_2}$.

Allowing also changes of η one expects an analytic family of ring structures defined on a neighbourhood of the origin in $H^*(M;\mathbb{C}) \times \mathbb{C}^{b_2}_q$. More precisely, since Φ is skew-symmetric, the parameter space should be taken as complex superspace, i.e. as complex space with $\mathbb{Z}_2$-graded structure sheaf. For simplicity we will ignore such questions here, i.e. restruct to the even part $H^{2*}(M;\mathbb{Z})$. It is not hard to see that moving η about some sufficiently small $\mu = \sum_i b_i\omega_i \in H^2(M,\mathbb{C})$ is equivalent to the change of coordinates $q_i \mapsto e^{b_i}q_i$. So we can eliminate any 2-cohomology from η and a more natural parameter space would be an open neighbourhood T of $0 \in H^{*\neq 2}(M;\mathbb{C}) \times \mathbb{C}^{b_2}_q$. The second factor minus the coordinate hyperplanes can then be identified with a neighbourhood of infinity of the tube domain $(H^2(M;\mathbb{R})/H^2(M;2\pi i\mathbb{Z})) + i\mathbb{R}_{\geq 0} \cdot \mathrm{conv}\{\omega_1, \dots, \omega_{b_2}\}$ via

$$\sum_\nu (e_\nu + i\,d_\nu)\omega_\nu \longmapsto \Big(e^{-d_1+ie_1}, \dots, e^{-d_{b_2}+ie_{b_2}}\Big)\,.$$

As a matter of notation we will write $*_{\eta,\{q\}}$ for the quantum product with values in $\mathbb{C}\{q_1,\ldots,q_{b_2}\}$ and $*_t = *_{\eta,q}$ for the evaluation at a particular value of q. The analogous notations in the small case are $*_{\{q\}}$ and $*_q$. The corresponding quantum cohomology rings will be denoted $QH^*_{\eta,\{q\}}(M) = (V, *_{\eta,\{q\}})$ etc. (For simplicity we will occasionally ignore the $\mathbb{Z}_2$-grading in the sequel, i.e. consider the subring generated by even classes; this is preserved by quantum multiplication.)

Particularly simple is the case of small quantum cohomology on manifolds M with the property that the existence of a non-zero GW-invariant $\Phi_R^M(\alpha_1,\ldots,\alpha_k)$ for $R \in H_2(M;\mathbb{Z}) \setminus \{0\}$ implies $c_1(M)\cdot R > 0$. We propose the term *rationally positive* for this property. Then the dimension count shows that only finitely many classes R contribute to a quantum product $\alpha *_q \beta$, and that for $R \neq 0$ the degree of $\gamma_i^\vee$ must be strictly less than $\deg\alpha + \deg\beta$ to yield a non-trivial contribution. So in this case the quantum products are just a family of inhomogeneous refinements of the cup product, algebraically parametrized by a $b_2(M)$-dimensional affine space.

2.2 Flatness

We have seen that under appropriate convergence assumptions quantum cohomology is a family of ring structures on the complex vector space $V = H^*(M;\mathbb{C})$ with structure coefficients depending *analytically* on the parameter space T. Being a family of rings means that the sheaf of sections $\mathcal{V}$ of the complex vector bundle $V \times T \to T$ has the structure of a (finite) $\mathcal{O}_T$-algebra. To $\mathcal{V}$ is associated a complex subspace $Z := \operatorname{Specan}_T \mathcal{V} \subset V \times T$ with finite projection $\pi : Z \to T$ and such that $\mathcal{V} = \pi_* Z$, cf. e.g. [Fi, 1.15]. More explicitely, the equations defining Z are nothing but the quadratic equations defining the family of algebra structures.

Now a finite morphism $\pi : Z \to T$ of complex spaces is flat iff $\pi_*\mathcal{O}_Z$ is locally free. This is obviously the case here. Conversely, given any identification $\pi_*\mathcal{O}_Z \simeq \mathcal{O}(V \times T)$, one can produce a flat family of ring structures on the complex vector space V.

We thus see that the definition of quantum cohomology by an analytic variation of the structure coefficients on a fixed vector space is equivalent to the flatness of the associated sheaf $\mathcal{V}$ of $\mathcal{O}_T$-algebras, or the flatness of the associated finite map $\pi : Z \to T$.

As another consequence of flatness, Z does not have embedded components. This follows by the unmixedness theorem (see e.g. [Ei, Cor. 18.14]) from the fact that Z as finite cover of a smooth space is Cohen Macaulay (see e.g. [Ei, Cor. 18.17]). In particular, Z is reduced iff Z is generically reduced iff $Z_t := \pi^{-1}(t)$ is a disjoint union of $\deg\pi$ simple points for some $t \in T$.

Note that from a computational point of view the family of quantum cohomology rings is given by a $\dim V \times \dim V$-matrix of analytic functions on T (multiplication table).

2.3 Grading, filtration

V decomposes according to the dimension, i.e. the grading of $H^*(M;\mathbb{Z})$. The grading is not in general respected by quantum multiplication for $t \neq 0$. An important exception is the Calabi-Yau case at $\eta = 0$. More generally, if for any φ, $c_1(M)\cdot\varphi_*[\mathbb{P}^1]$ is divisible by some index $\nu \in \mathbb{Z}$, then the small quantum cohomology rings $QH_q^*(M)$ will be $\mathbb{Z}/2\nu\mathbb{Z}$-graded.

For rationally semi-positive symplectic manifolds at least the associated filtration

$$\langle 1\rangle = V_0 \subset V_1 \subset \ldots \subset V_{\dim M} = V, \quad V_i = H^{\leq i}(M;\mathbb{C}),$$

is preserved in $QH_t^*(M)$.

2.4 Irreducible subalgebras, idempotents and eigenvalue spectrum

The following holds for any finite dimensional k-algebra R, k a field. By finite-dimensionality R is Artinian, hence (being also Noetherian) R has only finitely many prime ideals $\mathfrak{m}_1,\ldots,\mathfrak{m}_q$, which are all maximal. In particular, $\dim R = 0$. The natural morphism

$$R \longrightarrow \prod_{i=1}^{q} R_i, \quad R_i = R_{\mathfrak{m}_i}$$

is an isomorphism with a product of *local* Artinian rings $(R_i, \bar{\mathfrak{m}}_i)$. If k is algebraically closed then the inclusion $k \hookrightarrow R$ induces isomorphisms $R/\bar{\mathfrak{m}}_i \simeq k$. $R = \prod R_i$ means $\operatorname{Spec} R = \coprod_i \operatorname{Spec} R_i$. Let $e_i \in \Gamma(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$ be defined to be 1 at $\operatorname{Spec} R_i$ and 0 away from this point. The e_i form a complete set of orthogonal idempotents, i.e. $e_i^2 = e_i$, $e_ie_j = 0$ for $i \neq j$, and $\sum_i e_i = 1$. Multiplication by e_i corresponds to projection onto the i-th component in $\prod R_i$. The existence of such idempotents for products also show that the R_i can not be further decomposed, they are irreducible as algebras. In fact, if $(R,\mathfrak{m})$ is a local Artinian ring and $e^2 = e \neq 0$ there exists a unit λ with $e - \lambda \in \mathfrak{m}$, hence $(e-\lambda)^r = 0$ for some $r > 0$ by the Artinian property. Expanding and using $e^2 = e$ one sees $e((-\lambda)^r - (1-\lambda)^r) = (-\lambda)^r$. Therefore e is a unit and then $e = 1$.

Extending functions by 0, the R_i can also be viewed as subalgebras of R ($R_i = R\cdot e_i$). Then $R = \oplus R_i$ is also the simultaneous Jordan decomposition of multiplication by elements of R, viewed as endomorphisms of the k-vector space R. For simplicity let us assume k algebraically closed. For any $x \in R$ and $i \in \{1,\ldots,q\}$ there exists $\lambda_i(x) \in k$ with $e_i \cdot x - \lambda_i(x) \in \mathfrak{m}_i$, hence $R_i \subset \ker(x - \lambda_i(x))^r$. So the *eigenvalue spectrum* of multiplication by x is $\{\lambda_1(x),\ldots,\lambda_q(x)\}$. Given a presentation $R = k[X_1,\ldots,X_n]/(f_1,\ldots,f_r)$ there exist $\mu_{ij} \in k$ with $\mathfrak{m}_i = (X_1 - \mu_{i1},\ldots,X_n - \mu_{in})$, and so, writing $x = F(X_1,\ldots,X_n)$, one obtains $\lambda_i(x) = F(\mu_{i1},\ldots,\mu_{in})$. We obtain: The eigenvalues of quantum multiplication by some x is given by evaluation of x at the maximal ideals.

For quantum cohomology rings there is a subspace of elements $V_{\text{ferm}} \subset V = H^*(M;\mathbb{C})$ with completely degenerate eigenvalue spectrum 0. V_{ferm} is the subring of $H^*(M;\mathbb{C})$ generated by classes of odd degree, the "fermionic" subring. In fact, for $x \in H^{2*+1}(M;\mathbb{C})$, $\Phi_R(x, x, \gamma \mid \eta, \ldots, \eta) = 0$ for any γ, η by graded symmetry of Φ_R, i.e. $x * x = 0$. So if $x \in V_{\text{ferm}}$ is the sum of κ elements of odd degree then $x^{\kappa+1} = 0$.

2.5 The Gorenstein property

From the point of view of commutative algebra cohomology rings have a very enjoyable property, they are Gorenstein. We will see in this subsection that the same holds true for quantum cohomology rings at least if η is sufficiently small.

Let $(R, \mathfrak{m})$ be a zero-dimensional local ring. If R is a finitely generated k-algebra zero-dimensionality is equivalent to finite-dimensionality of R as k-vector space. R being Artinian the chain of ideals $R \supset \mathfrak{m} \supset \mathfrak{m}^2 \supset \ldots$ must eventually stabilize, and by Nakayama's lemma the stabilization must be the zero ideal. Let r be the largest integer with $\mathfrak{m}^r \neq 0$. Obviously, $\mathfrak{m}^r = \operatorname{Ann} \mathfrak{m}$. This module is called the *socle* of R. $(R, \mathfrak{m})$ is called *Gorenstein* iff $\operatorname{Ann} \mathfrak{m}$ can be generated by one element. Example: $R = k[X]/(X^n)$ is Gorenstein with socle (X^{n-1}). More generally, complete intersections are Gorenstein, but $R = k[X, Y, Z]/I$, $I = (X^2, Y^2, Z^2 - XY, XZ, YZ)$ is an example of a Gorenstein ring (socle $(XY) = (Z^2)$) that is not a complete intersection. Other characterizations are (cf. e.g. [Ei])

- R has finite injective dimension (in fact, R is injective as R-module)
- the dualizing module ω_R can be generated by one element (in fact,$\omega_R \simeq R$).

(Both these characterizations can be used to define higher dimensional Gorenstein rings.) A more useful criterion for the case of quantum cohomology rings is by "Poincaré-duality":

Proposition 2.1 *Let R be a local finite-dimensional k-algebra with k algebraically closed. Then R is Gorenstein iff there exists a linear form $\lambda : R \to k$ such that the bilinear form*

$$B_\lambda : R \times R \longrightarrow k, \quad (a, b) \longmapsto \lambda(a \cdot b)$$

is non-degenerate.

Proof. Let $x, y \in \operatorname{Ann} \mathfrak{m}$. Then $z = \lambda(x)y - \lambda(y)x \in \operatorname{Ann} \mathfrak{m}$ and $\lambda(z) = 0$. But any $a \in R$ may be written $a = a_0 + \bar{a}$ with $a_0 \in k$, $\bar{a} \in \mathfrak{m}$, and so $\lambda(a \cdot z) = a_0 \lambda(z) = 0$ for any $a \in R$. By non-degeneracy of B_λ this shows $z = 0$, i.e. $\dim_k \operatorname{Ann} \mathfrak{m} = 1$.

Conversely, choose any linear form that is nonzero on the socle. $\diamond$

For cohomology rings of connected, oriented manifolds M the socle is $H^{\dim M}(M)$, i.e. generated by the class of a normalized volume form Ω. There is a distinguished choice λ^{top} for the linear form by imposing $\ker \lambda^{\text{top}} = H^{<\dim M}(M; k)$ and $\lambda^{\text{top}}[\Omega] = 1$. $B_0 := B_{\lambda^{\text{top}}}$ is the classical Poincaré-duality pairing on the cohomology ring.

Let $R = QH_t^*(M)$ for some $t \in T$, $R = \prod_i R_i$ the decomposition into local rings. λ^{top} induces the bilinear form

$$B_t : R \times R \longrightarrow \mathbb{C}, \quad (x, y) \longmapsto \lambda^{\mathrm{top}}(x *_t y)\,.$$

We claim that in the small case $t = (0, q)$, B_t coincides with B_0 via the natural identification of $\mathbb{C}$-vector spaces $R = V = H^*(M;\mathbb{C})$. In fact, let $\{\gamma_i\}$, $\{\gamma_i^\vee\}$ be dual homogeneous bases of $H^*(M;\mathbb{C})$ with $\gamma_0 = 1$. Then $\lambda^{\mathrm{top}}(\gamma_i^\vee) \neq 0$ iff $i = 0$. For α, $\beta \in H^*(M;\mathbb{C})$ we get

$$B_t(\alpha, \beta) \;=\; \lambda^{\mathrm{top}}(\alpha *_q \beta) \;=\; \Phi_0(\alpha, \beta, 1) \;=\; B_0(\alpha, \beta)$$

since $\Phi_R(\alpha, \beta, 1) = 0$ for any $R \neq 0$. In particular, B_t is non-degenerate. And the decomposition $R = \prod_i R_i$ into local rings makes B_t block-diagonal. Thus $B_t|_{R_i}$ is non-degenerate for any i, hence R_i is Gorenstein.

Recall that a zero-dimensional scheme (complex space) Z is called Gorenstein iff $\mathcal{O}_{Z,z}$ is Gorenstein for any $z \in Z$.

Proposition 2.2 *There exists an open neighbourhood V of $0 \in \mathbb{C}_q^{b_2} \cap T$ such that $Z_t = \mathrm{Spec}\, QH_t^*(M)$ is Gorenstein for all $t \in V$.*

Proof. For nonzero η we can rely on the general result that the property of a fiber of a finite morphism $\pi : Z \to T$ to be Gorenstein is open. This follows from [Ha, V.9.6] and [GrMa] (cf. also [BiFl]). $\diamond$

2.6 Presentations

We have already noted that since we deal with ring structures on finite dimensional vector spaces, quantum cohomology is in principle given by a $n \times n$-matrix of holomorphic fundtions on T, $n = \dim_{\mathbb{C}} H^*(M;\mathbb{C})$. However, associativity in combination with the classical relations in the cohomology ring impose strong conditions among the entries of this matrix. A much more convenient way of describing quantum cohomology rings, that (in the small case) clearly shows the independent information contained in GW-invariants, is by generators and relations.

Proposition 2.3 [SiTi2][AsSa] *Let $H^*(M;\mathbb{C}) \simeq \mathbb{C}[X_1, \ldots, X_n]/(f_1, \ldots, f_r)$ be a presentation with X_i corresponding to $\gamma_i \in H^{d_i}(M;\mathbb{C})$ and f_i homogeneous with respect to weights d_i on X_i. Then the homomorphism*

$$\mathbb{C}[X_1, \ldots, X_n] \otimes_{\mathbb{C}} \mathbb{C}\{q_1, \ldots, q_{b_2}\} \longrightarrow QH_{\{q\}}^*(M)$$

sending a monomial $X_1^{i_1} \ldots X_n^{i_n}$ to $\gamma_1 * \ldots * \gamma_1 * \ldots * \gamma_n$ (γ_ν occuring i_ν-times) is surjective with kernel $(\hat{f}_1, \ldots, \hat{f}_r)$, $\hat{f}_i \equiv f_i$ modulo $(q_1, \ldots, q_{b_2})$.

In particular, there is a presentation

$$QH_{\{q\}}^*(M) \;\simeq\; \mathbb{C}[X_1, \ldots, X_n] \otimes_{\mathbb{C}} \mathbb{C}\{q_1, \ldots, q_{b_2}\}/(\hat{f}_1, \ldots, \hat{f}_r)\,. \qquad \diamond$$

The proof is by an essentially trivial induction on powers of q_i. Despite its simplicity the theorem can be extremely powerful in applications by drastically reducing the number of GW-invariants to be determined, cf. the cases of $M = G(k,n)$ below and $\mathcal{N}_g$ in Chapter 3.

There is one source of confusion in the application of the theorem: The meaning of a monomial $X_1^{i_1} \dots X_n^{i_n}$, $i_1 + \dots + i_n > 1$, as element of the fixed complex vector space $V = H^*(M;\mathbb{C})$ changes with q. Only linear terms correspond directly, while higher degree monomials in the presentation of $QH_q^*(M)$ have to be interpreted as quantum products.

For a non-trivial example consider the moduli space $\mathcal{N}_2$ to be studied in Chapter 3. A subring $H_I^*(\mathcal{N}_2) \subset H^*(\mathcal{N}_2)$ has the presentation $\mathbb{C}[\alpha,\beta,\gamma]/(\alpha^2+\beta, \alpha\beta+\gamma, \alpha\gamma) \simeq \mathbb{C}[\alpha]/(\alpha^4)$ with $\deg\alpha = 2$, $\deg\beta = 4$, $\deg\gamma = 6$. So up to a constant, $\alpha^3 \equiv \gamma$ both represent the volume form. $H_I^*(\mathcal{N}_2)$ is respected by quantum multiplication. One obtains the presentation $QH(\mathcal{N}_2)^{\mathrm{inv}} = \mathbb{C}[\alpha,\beta,\gamma]/(\alpha^2+\beta-8q, \alpha\beta+\gamma+8\alpha q, \alpha\gamma) \simeq \mathbb{C}[\alpha]/(\alpha^4 - 16\alpha^2 q)$ ($b_2(\mathcal{N}_2) = 1$). Now γ still corresponds to a multiple of the volume form, but $\alpha^3 = 16\alpha q + \gamma$ differs from this by a quantum correction involving α.

So while the coefficients of the quantum corrections to f_i are completely independent (any such will give a flat deformation of the cup product structure), a single coefficient might involve several GW-invariants. It is usually non-trivial to explicitely find this expression, cf. the case of $\mathcal{N}_g$ in Chapter 4.

As was pointed out to the author by F.-O. Schreyer, it is a standard fact in deformation theory of singularities that deformations of the relations at $t = 0$ form a complete set of relations at sufficiently small $t \neq 0$. This is just a reflection of flatness of the family $\pi : Z \to T$, cf. [Ar]. So as with the Gorenstein property, for sufficiently small η there are presentations of $QH_\eta^*(M)$ of the form $\mathbb{C}[X_1,\dots,X_n] \otimes_{\mathbb{C}} \mathbb{C}\{q_1,\dots,q_{b_2}\}/(\hat{f}_1^\eta,\dots,\hat{f}_r^\eta)$, but the $\hat{f}_i^\eta$ modulo $(q_1,\dots,q_{b_2})$ will only be arbitrary perturbations of f_i.

2.7 GW-invariants

If the (small) quantum cohomology ring $QH_q^*(M)$ is given by a multiplication table with respect to a basis $\{\gamma_i\}_{i=0,\dots,N}$ of $H^*(M)$ with γ_N the volume form and $\deg\gamma_i < \dim M$ for any $i < N$, one can easily determine any GW-invariant $\Phi_R(\alpha_1,\dots,\alpha_k)$ by taking the coefficient of $q_1^{\omega_1(R)} \dots q_{b_2}^{\omega_{b_2}(R)} \cdot \gamma_N$ in $\alpha_1 * (\alpha_2 * \dots * (\alpha_{k-1} * \alpha_k) \dots)$ (similarly for the large case $\eta \neq 0$). If $QH_q^*(M)$ is given in the form $\mathbb{C}[X_1,\dots,X_n] \otimes_{\mathbb{C}} \mathbb{C}\{q_1,\dots,q_{b_2}\}/(\hat{f}_1,\dots,\hat{f}_r)$ of Proposition 2.3 more efficient methods are available at least in two cases.

1) $\pi^{-1}(H^2(M;\mathbb{C}) \cap T) \subset Z$ is reduced, i.e. for generic $q = (q_1,\dots,q_{b_2}) \in H^2(M;\mathbb{C}) \cap T$, $QH_q^*(M)$ is a semi-simple algebra, i.e. decomposes into a product of reduced algebras. Write $\mathrm{Spec}\,QH_q^*(M) = \{P_0,\dots,P_N\} \subset \mathbb{C}^n_{X_1,\dots,X_n}$. Then since any linear form on $QH_q^*(M)$ is a linear combination of evaluations at P_ν there are $\lambda_\nu \in \mathbb{C}$,

$\nu = 1, \dots, N$, with

$$\sum_{R \in \mathcal{R}\mathrm{C}\,(M)} \Phi_R(\gamma_{i_1}, \dots, \gamma_{i_k}) q_1^{\omega_1(R)} \dots q_{b_2}^{\omega_{b_2}(R)} \;=\; \sum_{\nu=0}^{N} \lambda_\nu X_{i_1}(P_\nu) \cdot \ldots \cdot X_{i_k}(P_\nu)\,.$$

This formula depends only on the coordinates of the P_ν!

2) The quantum corrections are all of lower degree. This is the case iff M is rationally positive (cf. 2.1). Then at any $q \in T \cap \mathbb{C}_q^{b_2}$, $(\hat{f}_1^q, \dots, \hat{f}_r^q)$ form a Gröbner basis with respect to the partial monomial order given by weights as follows:

$$X_1^{i_1} \dots X_n^{i_n} \;>\; X_1^{j_1} \dots X_n^{j_n} \quad \text{iff } \sum_\nu i_\nu > \sum_\nu j_\nu\,.$$

In fact, $\mathrm{in}(\hat{f}_1^q, \dots, \hat{f}_r^q) = (f_1, \dots, f_r) = (\mathrm{in}\,\hat{f}_1^q, \dots, \mathrm{in}\,\hat{f}_r^q)$. Let $>_m$ be any monomial order refining $>$. Then there is a unique minimal monomial $\Omega = X_1^{j_1} \dots X_n^{j_n}$ of top degree. The GW-invariant $\sum_{R \in \mathcal{R}\mathrm{C}\,(M)} \Phi_R(\gamma_{i_1}, \dots, \gamma_{i_k}) q_1^{\omega_1(R)} \dots q_{b_2}^{\omega_{b_2}(R)}$ is the coefficient of Ω in the reduction of $X_{i_1} \dots X_{i_k}$ modulo the Gröbner basis. This is an effective method readily accessible to computer algebra programs.

2.8 Residue formulas

The projection λ^{top} onto the top dimensional part is a trace map $\pi_* \mathcal{O}_Z \to \mathcal{O}_T$, $\pi : Z \to T$ the quantum deformation of $\mathrm{Spec}\, H^*(M; \mathbb{C})$. If $H^*(M; \mathbb{C})$ has a presentation as complete intersection $\mathbb{C}[X_1, \dots, X_n]/(f_1, \dots, f_n)$ then Z is the fiber over 0 of a holomorphic map $f : T \times \mathbb{C}^n \to \mathbb{C}^n$ with π the restriction of the projection onto π. In this case there is another trace map given by higher dimensional residues. For a holomorphic map $f : \mathbb{C}^n \to \mathbb{C}^n$ with $f^{-1}(0)$ finite the residue at $a \in f^{-1}(0)$ of $F \in \mathcal{O}_{\mathbb{C}^n, a}$ is

$$\mathrm{res}_f(a; F) \;:=\; \frac{1}{(2\pi i)^k} \int_{\Gamma_a^\varepsilon} \frac{F}{f_1 \cdot \ldots \cdot f_n} dX_1 \dots dX_n$$

with $\Gamma_a^\varepsilon = \{x \in U(a) \mid |f_i(x)| = \varepsilon\}$ for ε sufficiently small, $U(a)$ a neighbourhood of a with $f^{-1}(0) \cap U(a) = \{a\}$. The total residue of the germ F of a holomorphic function along $f^{-1}(0)$ is

$$\mathrm{Res}_f(F) \;:=\; \sum_{a \in f^{-1}(0)} \mathrm{res}_f(a; F)\,.$$

Quite generally, Res_f vanishes on the ideal generated by $f_1, \dots, f_n$. In a homogeneous situation more is true:

Proposition 2.4 [SiTi1][CaDiSt] *1) Let $R = \mathbb{C}[X_1, \dots, X_n]/(f_1, \dots, f_n)$ be Artinian with f_i homogeneous with respect to weights d_i on X_i. Put $N := \sum_i \deg f_i - \sum_i d_i$, $J = \det(\partial f_i / \partial X_j)$. Then*

$$R = R_{<N} \oplus \mathbb{C} \cdot J, \quad R_{<N} \subset \ker \mathrm{Res}_f, \quad \mathrm{Res}_f(J) = \dim_{\mathbb{C}} R\,.$$

2) The analogous conclusions hold true for $R' = \mathbb{C}[X_1, \ldots, X_n]/(f_1+g_1, \ldots, f_n+g_n)$ *with* $\deg g_i < \deg f_i$. $\diamond$

The reason for the vanishing on polynomials F of low degree is the vanishing of the global residue of the associated meromorphic differential form σ on a convenient compactification of $\mathbb{C}^n$. The condition of regularity of σ along infinity bounds this argument to polynomials with $\deg F < N$. $\mathrm{Res}_f(J)$ counts the number of sheets of $f : \mathbb{C}^N \to \mathbb{C}^N$ near 0. And in the homogenous case (1) a theorem of Macaulay says that a homogeneous F of degree $> N$ is contained in $(f_1, \ldots, f_N)$.

The point of view of [CaDiSt] is to use the proposition in connection with Gröbner bases as in the previous section to produce fast algorithms for the computation of higher dimensional residues.

Applied to complete intersection cohomology rings $H^*(M;\mathbb{C}) = \mathbb{C}[X_1, \ldots, X_n]/(f_1, \ldots, f_n)$ with X_i, f_i homogeneous we see that $N = \sum_i \deg f_i - \sum_i d_i$ is top-dimensional and $H^{\dim M}(M;\mathbb{C})$ is spanned by the Jacobian J of $f_1, \ldots, f_n$. (2) of the proposition may be applied to small quantum cohomology rings provided M is rationally positive. In fact, the total residue coincides with λ^{top} up to a constant. We obtain

Corollary 2.5 *Let* M *be rationally positive and assume the cohomology ring be presented as* $H^*(M;\mathbb{C}) = \mathbb{C}[X_1, \ldots, X_n]/(f_1, \ldots, f_n)$ *with* X_i *corresponding to* $\gamma_i \in H^{d_i}(M;\mathbb{C})$*,* f_i *homogeneous. Let* $\hat{f}_1, \ldots, \hat{f}_n \in \mathbb{C}[X_1, \ldots, X_n] \otimes_{\mathbb{C}} \mathbb{C}\{q_1, \ldots, q_{b_2}\}$ *be the quantum deformations of* $f_1, \ldots, f_n$ *as in Proposition 2.3.*

Then there exists a constant $c \in \mathbb{C}$ *such that for* $q = (q_1, \ldots, q_{b_2}) \in \mathbb{C}^{b_2}_q \cap T$ *and* $i_\nu \in \{1, \ldots, n\}$*,* $\nu = 1, \ldots, k$

$$\sum_{R \in \mathcal{R}\mathrm{C}\,(M)} \Phi_R(\gamma_{i_1}, \ldots, \gamma_{i_k}) q_1^{\omega_1(R)} \ldots q_{b_2}^{\omega_{b_2}(R)} \;=\; \mathrm{Res}_{\hat{f}^q}(X_{i_1} \cdot \ldots \cdot X_{i_k})$$

where $\hat{f}^q = (\hat{f}^q_1, \ldots, \hat{f}^q_n) : \mathbb{C}^n \to \mathbb{C}^n$. $\diamond$

The constant c can be fixed by evaluating a polynomial representing the class of a normalized volume form.

2.9 The generalized Vafa-Intriligator formula

In the situation of Corollary 2.5 a more explicit formula can be derived if 0 is a regular value of $\hat{f}^q$. In fact, if $f : \mathbb{C}^n \to \mathbb{C}^n$ has a simple isolated zero in $a \in \mathbb{C}^n$ then

$$\mathrm{res}_f(a; F) \;=\; \frac{F(a)}{J(a)}, \quad J = \det\left(\frac{\partial f_i}{\partial X_j}\right).$$

Therefore,

Proposition 2.6 *(Generalized Vafa-Intriligator formula) In the situation of Corollary 2.5 assume* 0 *is a regular value of* $\hat{f}^q$, $q = (q_1, \ldots, q_{b_2}) \in \mathbb{C}_q^{b_2} \cap T$, *i.e. the fiber of* $\pi : Z \to T$ *over* q *is reduced. Write* $J = \det(\partial f_i / \partial X_j)$.

Then there exists a constant $c \in \mathbb{C}$ *such that for any* $i_\nu \in \{1, \ldots, n\}$, $\nu = 1, \ldots, k$

$$\sum_{R \in \mathcal{RC}\,(M)} \Phi_R(\gamma_{i_1}, \ldots, \gamma_{i_k}) q_1^{\omega_1(R)} \ldots q_{b_2}^{\omega_{b_2}(R)} \;=\; c \cdot \sum_{P \in (\hat{f}^q)^{-1}(0)} J(P)^{-1} \cdot X_{i_1}(P) \ldots X_{i_k}(P)\,.$$

$\diamond$

Note that this formula is a special case of the one given in 2.6,(1), but with the λ_ν explicitely determined up to one constant c.

I would also like to mention that a similar formula with J^{g-1} replacing J^{-1} computes GW-invariants for genus g, cf. [SiTi1, Prop.4.4].

Example 2.7 Everything we have presented here works perfectly well for the Graßmannians $G(k, n)$ and was actually developed in [SiTi1] to treat this case. The cohomology ring has a presentation

$$H^*(G(k, n)) \;=\; \mathbb{C}[X_1, \ldots, X_k]/(f_{n-k+1}, \ldots, f_n)$$

with X_i corresponding to the i-th Chern class of the universal bundle S and f_j corresponding to the expression in X_i of the j-th Segre class of S. In particular $\deg f_j = j$ in agreement with the formula for the dimension in Proposition 2.4. $G(k, n)$ being Fano of index n, a dimension count shows that the only possible quantum contributions are to f_n, and these are just by lines, irreducible classes in $H_2(M; \mathbb{Z})$. The corresponding moduli space is smooth of the expected dimension and the quantum contribution can actually be figured out by linear algebra to be $(-1)^k$. With q corresponding to the positive generator of $H_2(G(k, n); \mathbb{Z}) \simeq \mathbb{Z}$ we get

$$QH^*_{\{q\}}(G(k, n)) \;=\; \mathbb{C}[X_1, \ldots, X_k]/(f_{n-k+1}, \ldots, f_n + (-1)^k q)\,.$$

For any $q \neq 0$ the spectrum of $QH^*_q(G(k, n))$ splits into $\binom{n}{k} = \dim H^*(G(k, n))$ simple points. In this case Proposition 2.6 is the classical Vafa-Intriligator formula [In], which is hereby established as a formula actually computing GW-invariants of $G(k, n)$. $\diamond$

3 Quantum cohomology of $\mathcal{N}_g$

A promising task, both in view of relations to gauge theory [Do] and in its own right, is the computation of the (small for the time being) quantum cohomology ring of $\mathcal{N}(\Sigma, 2, L)$, the moduli space of stable bundles E over a Riemann surface Σ of genus $g \geq 2$, $\mathrm{rk} E = 2$, $\det E = L$, L a line bundle of *odd* degree. $\mathcal{N}(\Sigma, 2, L)$ is a compact Kähler manifold of dimension $3g - 3$ with $c_1(\mathcal{N}(\Sigma, 2, L)) = 2\alpha$, α the ample generator of $\mathrm{Pic}\mathcal{N}(\Sigma, 2, l) \simeq \mathbb{Z}$. In particular, $\mathcal{N}(\Sigma, 2, L)$ is a Fano manifold of index 2. The underlying symplectic manifold depends only on g and will be denoted $\mathcal{N}_g$.

As a differentiable manifold $\mathcal{N}_g$ can be identified via the Narasimhan-Seshadri theorem with the space of isomorphism classes of representations

$$\rho : \pi_1(\Sigma \setminus \{P\}) \longrightarrow SU(2), \quad \rho(\prod_{i=1}^{g} A_i B_i A_i^{-1} B_i^{-1}) = -I \, ,$$

modulo the diagonal action of $SU(2)$ by conjugation, P some fixed point on Σ. Here A_i, B_i are a set of canonical generators of $\pi_1(\Sigma \setminus \{P\})$, a free group on $2g$ generators. From this point of view one obtains a well-defined symplectic action of the mapping class group of Σ on $\mathcal{N}_g$.

3.1 The cohomology ring

$H^*(\mathcal{N}_g)$ is generated by $\alpha \in H^2(\mathcal{N}_g)$, $\psi_i \in H^3(\mathcal{N}_g)$ $(i = 1, \ldots, 2g)$, $\beta \in H^4(\mathcal{N}_g)$, which can be obtained as Künneth components of $c_2(\mathrm{End}\,\mathcal{U})$, $\mathcal{U} \downarrow \Sigma \times \mathcal{N}_g$ the universal bundle. The subring $H_I^*(\mathcal{N}_g)$ left invariant by the action of the mapping class group, which factorizes over the action of $Sp(2g, \mathbb{Z})$ on $\psi_1, \ldots, \psi_{2g}$, is generated by α, β and $\gamma := -2\sum_i \psi_i \psi_{i+g}$. Another way to define $H_I^*(\mathcal{N}_g)$ is as subring of algebraic classes for *generic* choice of Σ [BaKiNe]. The ring structure on $H^*(\mathcal{N}_g)$ is easily expressed in terms of the ring structure on $H_I^*(\mathcal{N}_{g'})$ for $g' \leq g$ [KiNe, Prop.2.5]. The action of the mapping class group being symplectic, $H_I^*(\mathcal{N}_g)$ is respected by quantum multiplication; the full quantum cohomology ring $QH^*(\mathcal{N}_g)$ is determined by the invariant part $QH_I^*(\mathcal{N}_g)$ as in the classical case. We may and will thus concentrate on the subring generated by α, β, γ.

There is a beautiful recursive description of the relation ideal of $H_I^*(\mathcal{N}_g)$, found independently by several people [Za],[Br],[KiNe],[SiTi2].

Theorem 3.1 *Set $f_r = 0$ for $r < 0$, $f_0 = 1$ and inductively for $r > 0$*

$$f_{r+1} = \alpha f_r + r^2 \beta f_{r-1} + 2r(r-1)\gamma f_{r-2}.$$

Then for $g \geq 2$

$$H_I^*(\mathcal{N}_g) \;=\; \mathbb{Q}[\alpha, \beta, \gamma]/I_g, \quad I_g = (f_g, f_{g+1}, f_{g+2})\, . \qquad \diamond$$

It is also easy to determine the initial ideal of $\mathcal{I}_g$ with respect to the reverse lexicographic order on α, β, γ with weights 1,2,3, namely [SiTi2, Prop.4.2]

$$\mathrm{in}\, I_g \;=\; \{\alpha^a \beta^b \gamma^c \mid a + b + c \geq g\}\, .$$

Thus a basis for $V = H_I^*(\mathcal{N}_g; \mathbb{C})$ is given by the monomials $\alpha^a \beta^b \gamma^c$ with $a+b+c < g$. The socle is spanned by γ^{g-1}.

It is amusing to see how how far one can get in a proof of the recursion formula just from the fact that

> *Any $f \in I_g$ of minimal degree is a multiple of the Mumford relation f_g,* $\mathrm{in}\, f_g = \alpha^g$

together with a simple geometric argument assuming some non-degeneracy. f_g can be obtained by a Grothendieck-Riemann-Roch computation of a $2g$-th Chern class of a $(2g-1)$-bundle. For the geometric input note that contraction of the g-th handle $\Sigma_g \to \Sigma_{g-1}$ induces the differentiable embedding

$$\iota : \mathcal{N}_{g-1} \longrightarrow \mathcal{N}_g, \quad (A_i, B_i)_{i=1,\ldots,g-1} \longmapsto (A_i, B_i)_{i=1,\ldots,g} ,$$

with $A_g = B_g = I$. It turns out that $\iota_*[\mathcal{N}_{g-1}]$ is Poincaré-dual to $\gamma_g := -\psi_g\psi_{2g}$ [Th2]. Under $\iota^* : H^*(\mathcal{N}_g) \to H^*(\mathcal{N}_{g-1})$, α, β, γ map to the respective classes in $H^*(\mathcal{N}_{g-1})$ We get the following inclusions of ideals:

$$\gamma I_{g-1} \ \subset \ I_g \ \subset \ I_{g-1} . \tag{$*$}$$

The first inclusion is by Poincaré-duality using the action of the mapping class group to make γ_g invariant, observing $\gamma = -2\sum_i \psi_i\psi_{i+g}$.

Let us assume that inductively we know $I_r = (f_r, f_{r+1}, f_{r+2})$ with

$$f_{r+1} \ = \ \alpha f_r + \lambda_r \beta f_{r-1} + \mu_r \gamma f_{r-2}$$

for some $\lambda_r, \mu_r \in \mathbb{Q} \setminus \{0\}$, $r \le g-1$. To set up an induction it is more convenient to work with the unique set of generators $f_r^1 = f_r, f_r^2, f_r^3$ of I_r with $\operatorname{in} f_r^1 = \alpha^r$, $\operatorname{in} f_r^2 = \alpha^{r-1}\beta$, $\operatorname{in} f_r^3 = \alpha^{r-1}\gamma$. The recursion for f_r as given above is then equivalent to

$$f_{r+1}^1 \ = \ \alpha f_r^1 + \lambda_r f_r^2, \quad f_{r+1}^2 \ = \ \beta f_r^1 + \nu_r f_r^3, \quad f_{r+1}^3 \ = \ \gamma f_r^1 ,$$

with $\nu_r = \mu_r/\lambda_r$. Let us assume we know these relations inductively for $r < g$ with some $\lambda_r, \nu_r \in \mathbb{Q} \setminus \{0\}$. Again as in [SiTi2, Prop. 4.2] one shows

$$\operatorname{in} I_r \ = \ \left\langle \alpha^a \beta^b \gamma^c \middle| a + b + c \ge r \right\rangle .$$

Now $f_{g+1}^1 \in I_g^{(g+1)} = \langle \alpha f_g^1, f_g^2 \rangle$ (the $(g+1)$-homogeneous part of I_g) and since $\operatorname{in} f_{g+1}^1 = \alpha^{g+1}$ we get $f_{g+1}^1 = \alpha f_g^1 + \lambda_g f_g^2$. Similarly, $f_{g+1}^2 \in I_g^{(g+2)} = \langle \alpha^2 f_g^1, \beta f_g^1, \alpha f_g^2, f_g^3 \rangle$, so eliminating $\alpha^2 f_g^1$ and assuming the coefficient of βf_g^1 to be non-zero, $f_{g+1}^2 = \beta f_g^1 + \mu_g \alpha f_g^2 + \nu_g f_g^3$. Finally, $\gamma f_g^1 \in I_{g+1}$, and γf_g^1 has the right initial term $\alpha^g \gamma$, so $f_{g+1}^3 = \gamma f_g^1$.

To summarize we see that the only thing really astonishing here is the vanishing of μ_g. Note that non-zero μ_g would be compatible with our input as one sees by taking a linear combination of β and α^2 as generator in degree 2. β is distinguished among such generators by the property that $\beta^g = 0$, a fact directly related to the vanishing of the g-th Chern class of $\mathcal{N}_g$.

3.2 Quantum recursion relations

By heuristical comparison with the Donaldson series of $T \times \Sigma$, T a torus, a group of physicists derived the eigenvalue spectrum of quantum multiplication by α, β, γ, the last of which being completely degenerate by the remark at the end of section 2.4

[BeJoSaVa]. From the eigenvalue spectrum they were able to show that the quantum deformations q_g^1, q_g^2, q_g^3 of f_g^1, f_g^2, f_g^3 obey a very similar recursion formula, only with β replaced by $\beta \pm 8$. The following convention will be used throughout the present chapter: Since $\mathrm{Pic}(\mathcal{N}_g)$ has rank 1 we only have one quantum parameter q, which in this Fano case is nothing but a homogenizing parameter for the relation ideal and may thus be set to 1. q_g^i are then really inhomogeneous refinements of f_g^i.

The starting point for our approach was the following observation.

Proposition 3.2 *Let $J_g = (q_g^1, q_g^2, q_g^3) \subset \mathbb{C}[\alpha,\beta,\gamma]$ with q_g^1, q_g^2, q_g^3 $\mathbb{Z}/2\mathbb{Z}$-graded inhomogeneous refinements of f_g^1, f_g^2, f_g^3 ($\deg\alpha = 1$, $\deg\beta = 2$, $\deg\gamma = 3$). Then the existence of a recursion*

$$\begin{aligned} q_{g+1}^1 &= \alpha q_g^1 + g^2 q_g^2 \\ q_{g+1}^2 &= (\beta + c_{g+1}) q_g^1 + \frac{2g}{g+1} q_g^3 \\ q_{g+1}^3 &= \gamma q_g^1 \end{aligned}$$

for some $c_{g+1} \in \mathbb{Q}$ is equivalent to the inclusions

$$\gamma J_g \subset J_{g+1} \subset J_g\,. \qquad \text{(q}*\text{)}$$

Proof. Let us first assume (q$*$) given. By $J_{g+1} \subset J_g$ and since q_g^1 is a relation of minimal degree, $q_{g+1}^1 \in \langle \alpha q_g^1, q_g^2 \rangle$. Looking at the homogeneous parts of top-degree and comparing with the classical recursion yields $q_{g+1}^1 = \alpha q_g^1 + g^2 q_g^2$. As for q_{g+1}^2, by the $\mathbb{Z}/2\mathbb{Z}$-grading, this must be a linear combination of $\alpha^2 q_g^1$, βq_g^1, αq_g^2, q_g^3 and q_g^1. Of these only q_g^1 is of lower degree. Hence its coefficient c_{g+1} can not be determined by comparison of the top-degree parts. Finally, $\gamma q_g^1 \in J_g$ and has the same homogeneous part as $f_{g+1}^3 = \gamma f_g^1$. Comparing initial terms this must be q_{g+1}^3. Finally, $(q_{g+1}^1, q_{g+1}^2, q_{g+1}^3) \subset J_{g+1}$ and both these ideals have the same initial terms and so must coincide.

For the converse, only $\gamma q_g^2, \gamma q_g^3 \in J_{g+1}$ require attention. But using the recursion, both are multiples of γq_g^1 modulo J_{g+1}, and $\gamma q_g^1 = q_{g+1}^3 \in J_{g+1}$. $\diamond$

In the next step we translate (q$*$) into a recursion for GW-invariants. Let us introduce the following notation: For a polynomial $F = \sum f_{abc}\alpha^a\beta^b\gamma^c \in \mathbb{Q}[\alpha,\beta,\gamma]$ we write

$$\langle F \rangle_g \;:=\; \sum_R f_{abc} \Phi_R^{\mathcal{N}_g}\Big(\underbrace{\alpha,\ldots,\alpha}_{a}, \underbrace{\beta,\ldots,\beta}_{b}, \underbrace{\gamma\ldots,\gamma}_{c}\Big)$$

for the "expectation value" of F.

Lemma 3.3 *Let $J_g \subset \mathbb{C}[\alpha,\beta,\gamma]$ be the kernel of $\mathbb{C}[\alpha,\beta,\gamma] \to QH_I^*(\mathcal{N}_g)$. Then the following are equivalent:*

1. $\gamma J_g \subset J_{g+1} \subset J_g$
2. $\langle F \cdot \gamma \rangle_g = 2g \langle F \rangle_{g-1}\ \forall\, F \in \mathbb{C}[\alpha,\beta,\gamma]$.

3. For any $R \in H_2(\mathcal{N}_{g-1};\mathbb{Z})$, $\varphi_1,\ldots,\varphi_k \in H^*_I(\mathcal{N}_g)$

$$\Phi^{\mathcal{N}_g}_{\iota_* R}(\varphi_1,\ldots,\varphi_k,\gamma_i) \;=\; \Phi^{\mathcal{N}_{g-1}}_R(\iota^*\varphi_1,\ldots,\iota^*\varphi_k)$$

where $\iota : \mathcal{N}_{g-1} \subset \mathcal{N}_g$ is the embedding induced by contraction of a handle in such a way that $\iota_*[\mathcal{N}_{g-1}]$ is Poincaré-dual to $\gamma_i = -\psi_i\psi_{i+g}$. $\diamond$

We will establish (3) of this list up to some problem involving curves with components in some bad locus in section 3.5, and express in section 3.3 the missing coefficients c_g in terms of GW-invariants of lines (indecomposable classes $R \in H_2(\mathcal{N}_g;\mathbb{Z})$). The case $g = 2$ has a description as complete intersection of two quadrics in $\mathbb{P}^5$ and can thus be treated directly, cf. [Do]. We obtain

Theorem 3.4 [SiTi3] *Set $q_i = 0$ for $i < 0$, $q_0 = 1$ and define inductively for $r > 0$*

$$q_{r+1} = \alpha q_r + r^2(\beta + c_r)q_{r-1} + 2r(r-1)\gamma q_{r-2}$$

with

$$c_r \;=\; \frac{(-1)^r}{(r-1)!4^{r-1}}\Phi^{\mathcal{N}_r}_l(\underbrace{\alpha,\ldots,\alpha}_{r-1},\underbrace{\beta,\ldots,\beta}_{r})\,,$$

$l \in H_2(\mathcal{N}_g;\mathbb{Z})$ the positive generator. Assume Conjeture 3.6 holds. Then for $g \geq 2$

$$QH^*_I(\mathcal{N}_g) \;=\; \mathbb{C}[\alpha,\beta,\gamma]/J_g, \quad J_g = (q_g, q_{g+1}, q_{g+2})\,. \qquad \diamond$$

Note that to agree with the formula predicted in [BeJoSaVa] we would need

$$\Phi^{\mathcal{N}_g}_l(\underbrace{\alpha,\ldots,\alpha}_{g-1},\underbrace{\beta,\ldots,\beta}_{g}) \;=\; (-1)^g(g-1)!4^{g-1}\cdot 8\,.$$

These GW-invariants are certainly computable by standard means to study moduli spaces of stable sheaves over $\mathbb{P}^1\times\Sigma$, but I have not yet carried out the computations. The verification of this number would also serve as an interesting check on the connection to gauge theory.

3.3 The coefficients c_r

To determine the coefficient c_{g+1} in the recursion we start from the equality

$$(\beta + c_{g+1})q^1_q(\alpha,\beta,\gamma) + \frac{2g}{g+1}\gamma q^1_g(\alpha,\beta,\gamma) \;=\; 0 \qquad (*)$$

in $QH^*_I(\mathcal{N}_{g+1})$ (Prop. 3.2). Now recall that a vector space basis for $QH^*_I(\mathcal{N}_{g+1})$ can be given by the monomials $\alpha^a\beta^b\gamma^c$, $a+b+c \leq g$ (interpreted as (a+b+c)-fold quantum product at $q = 1$!). The only monomial occuring in $(*)$ that is not of this form is $\beta\alpha^g$ in βq^1_g. In other words, $(*)$ can also be interpreted as an expression for the quantum product $\beta * \alpha * \ldots * \alpha$ (α occuring g-times) in terms of this basis. Moreover, the monomial α^g occurs only in the term $c_{g+1} q_g^1$, with coefficient c_{g+1}.

To determine c_{g+1} we may now use the diagonalization of the classical intersection pairing of Zagier [Za, Thm.3]. He sets for $r, s, t \geq 0$

$$\xi_{r,s,t} \;=\; \sum_{l=0}^{\min\{r,s\}} \binom{r+s-l}{r} \beta^{s-l} \frac{(2\gamma)^{l+t}}{l!t!} \frac{f_{r-l}}{(r-l)!}\,.$$

$\xi_{r,s,t}$ has initial term $\alpha^r \beta^s \gamma^t$ and so $\{\xi_{r,s,t} \mid r+s+t \leq g\}$ form a basis for $QH_I^*(\mathcal{N}_{g+1})$. Moreover, the $\xi_{r,s,t}$ diagonalize the intersection pairing:

$$\langle \xi_{r,s,t} \cdot \xi_{r',s',t'} \rangle_{H^*(\mathcal{N}_{g+1})} = \begin{cases} (-1)^{r+s} 4^g \dfrac{(g+1)!}{(r+s+a)r!s!t!t'!} & , \begin{array}{l} r' = s, s' = r \\ t' = g-r-s-t \end{array} \\ 0 & , \text{ else} \end{cases}$$

where $\langle\,.\,\rangle_{H^*(\mathcal{N}_{g+1})} = \lambda^{\mathrm{top}}(\,.\,)$ is the classical projection to the normalized volume form. But for polynomials of degree up to the dimension this is the same as our quantum projection $\langle\,.\,\rangle_{g+1}$. So $\xi_{r,s,t} \in \mathbb{C}[\alpha,\beta,\gamma]$, $r+s+t \leq g$, viewed as elements of $QH_I^*(\mathcal{N}_{g+1})$, are a basis diagonalizing the quantum intersection product. And the only $\xi_{r,s,t}$ containing the monomial α^g is $\xi_{g,0,0} = f_g/g! = \alpha^g/g! + \ldots$. The dual basis element to $\xi_{g,0,0}$ is

$$\xi_{0,g,0} \cdot (-1)^g 4^{-g} = (-1)^g 4^{-g} \beta^g\,.$$

Using the expansion of $\beta\alpha^g$ from $(*)$ in terms of $\xi_{r,s,t}$ we obtain

$$\begin{aligned} c_{g+1} &= c_{g+1} \left\langle \frac{\alpha^g}{g!} \cdot \frac{(-1)^g}{4^g} \cdot \beta^g \right\rangle_{g+1} = \frac{(-1)^g}{g!4^g} \langle -\beta\alpha^g\beta^g \rangle_{g+1} \\ &= \frac{(-1)^{g+1}}{g!4^g} \Phi_l^{\mathcal{N}_{g+1}}(\underbrace{\alpha,\ldots,\alpha}_{g}, \underbrace{\beta,\ldots,\beta}_{g+1})\,. \end{aligned}$$

3.4 A degeneration of $\mathcal{N}_g$

To establish the decisive geometric input Lemma 3.3,3 for $g \geq 3$ we use an algebraic degeneration of $\mathcal{N}_g$ constructed by Gieseker [Gi]. For a family $\pi : \mathcal{X} \to S$ of curves of genus g with a 1-dimensional smooth parameter scheme S, $\mathcal{X}$ smooth, π smooth away from $0 \in S$ and $\mathcal{X}_0 = \pi^{-1}(0)$ irreducible and smooth up to one node, he has constructed a proper flat family $\mathcal{W} \to S$ with

$\mathcal{W}$ smooth
for $s \neq 0$: $\mathcal{W}_s = \mathcal{N}(\mathcal{X}_s, 2, 1)$, the moduli space of
stable 2-bundles on $\mathcal{X}_s$ of degree 1
$\mathcal{W}_0 \subset \mathcal{W}$ an irreducible, reduced divisor with normal crossings.

$\mathcal{W}_0$ is also a moduli space of 2-bundles of degree 1, but the curve might have to be changed. To explain this let C be the normalization of $\mathcal{X}_0$, a smooth curve of genus

$g-1$, with P, $Q \in C$ corresponding to the double point of $\mathcal{X}_0$. Write C_i, $i \geq 0$, for the nodal curve obtained from C by joining P and Q by a chain of rational curves $R_1, \ldots, R_i$. In particular $C_0 = \mathcal{X}_0$. For a vector bundle E over C_i write $\tilde{E}$ for the pull-back of E to C and $E_{R_i} = E|_{R_i}$. $\mathcal{W}_0$ is the moduli space of 2-bundles E of degree 1 on C_i for some $i \leq 2$ subject to the stability condition

- $i = 0$: $\tilde{E}$ is unstable or $\tilde{E}$ has a line subbundle of degree 0 (type $\mathrm{I_s}$ and $\mathrm{I_u}$)
- $i = 1$: $\tilde{E}$ is semistable and $E_{R_1} = \mathcal{O} \oplus \mathcal{O}(1)$ (type $\mathrm{II_1}$)
  or $\tilde{E}$ is stable and $E_{R_1} = \mathcal{O}(1) \oplus \mathcal{O}(1)$ (type $\mathrm{II_s}$)
- $i = 2$: $\tilde{E}$ is stable and $E_{R_1} = E_{R_2} = \mathcal{O} \oplus \mathcal{O}(1)$, but the $\mathcal{O}(1)$ sub-line bundles of E_{R_1}, E_{R_2} are not glued together (type III)

$\mathcal{W}$ is a fine moduli space of this extended notion of stable bundles on the fibers of π: A universal 2-bundle $\mathcal{E}$ is defined over a blow-up $\mathcal{Y}$ (the universal curve) of $\mathcal{W} \times_S \mathcal{X}$.

There is an explicit description of $\mathcal{W}_0$ as modification of $\mathbb{P}(\mathrm{Hom}(\tilde{\mathcal{U}}_P, \tilde{\mathcal{U}}_Q) \oplus \mathcal{O})$, where $\tilde{\mathcal{U}} \downarrow C \times \mathcal{N}(C, 2, 1)$ is the universal bundle and $\tilde{\mathcal{U}}_P = \tilde{\mathcal{U}}|_{\{P\} \times \mathcal{N}(C,2,1)}$, $\tilde{\mathcal{U}}_Q = \tilde{\mathcal{U}}|_{\{Q\} \times \mathcal{N}(C,2,1)}$. Under this correspondence a dense open set in $\mathcal{W}_0$ is identified with bundles of type $\mathrm{I_u}$ by gluing a stable 2-bundle E on C at P and Q via a non-degenerate homomorphism $E_P \to E_Q$.

This is not quite the form appropriate for us because the determinant line bundle has not been fixed so far. This has been carried out in an unpublished part of Thaddeus' thesis [Th1, §6,Thm.1]. The picture is as follows: Let $L \in \mathrm{Pic}(\mathcal{X})$ be of degree 1 on $\mathcal{X}_s$ for any s. Let $\mathcal{V} \subset \mathcal{W}$ be the reduced subscheme having closed points $[E] \in \mathcal{W}_s$ with $H^0(\mathcal{Y}_{[E]}, \Lambda^2 E^\vee \otimes \hat{L}_s) \neq 0$ where $\hat{L}$ is the pull-back of L to the universal curve $\mathcal{Y}$ ($\mathcal{V}$ is expressable as the support of a sheaf). For $s \in S \setminus \{0\}$, $\mathcal{V}_s = \mathcal{N}(\mathcal{X}_s, 2, L_s)$, and $\mathcal{V} \to S$ is flat. For an explicit description of $\mathcal{V}_0$ let $\mathcal{U} \downarrow C \times \mathcal{N}_{g-1}$ be the universal bundle, where we wrote $\mathcal{N}_{g-1} = \mathcal{N}(C, 2, \tilde{L}_0)$, $\tilde{L}_0$ the pull-back of L_0 to C. One irreducible component of $\mathcal{V}_0$ is the $\mathbb{P}^3$-bundle $P := \mathbb{P}\mathrm{Hom}(\mathcal{U}_P, \mathcal{U}_Q)$. P parametrizes bundles of type $\mathrm{II_2}$ (generically) and III. The latter correspond to the divisor $Q \subset P$ corresponding to degenerate homomorphisms. Fiberwise this is a quadric in $\mathbb{P}^3$, so Q is a $\mathbb{P}^1 \times \mathbb{P}^1$-bundle. There is a 2-fold covering $S \to P$ branched along Q. Note that $S \setminus Q$ can be identified with the SL_2-bundle of homomorphisms $E_P \to E_Q$ compatible with the gluing datum given on $\tilde{L}_0$ by L_0. Since the descent of a stable bundle $\tilde{E}$ on C to E on C_0 is stable, $S \setminus Q$ parametrizes bundles of type $\mathrm{I_s}$, while an open set in Q can be identified with bundles of type $\mathrm{II_2}$.

The set of bundles of type $\mathrm{II_2}$ being irreducible, S and P must be glued along Q to build up a model for $\mathcal{V}_0$. But the identification map $(Q \subset S) \to (Q \subset P)$ is not the identity, and can actually not be globally defined. So we need to modify S somewhere on Q. In fact, bundles of type $\mathrm{I_u}$ lie over the locus in $\mathcal{N}_{g-1}$ of bundles $\tilde{E}$ that are extensions

$$0 \longrightarrow M \xrightarrow{i} \tilde{E} \xrightarrow{q} M^{-1} \otimes \tilde{L}_0 \longrightarrow 0 \qquad (*)$$

for some $M \in \mathrm{Pic}^0(C) =: J_{g-1}$. There exists a vector bundle F over J_{g-1} of rank $g-1$ with $\mathbb{P}(F)$ parametrizing such extensions modulo equivalence. And there is a closed embedding $Z := \mathbb{P}(F) \hookrightarrow Q$ by sending an extension $(*)$ to the ray of degenerate homomorphisms

$$\tilde{E}_P \xrightarrow{q_P} (M^{-1} \otimes \tilde{L}_0) \;\simeq\; M_Q \xrightarrow{i_Q} \tilde{E}_Q\,.$$

Note that $\dim Z = 2g - 3$, i.e. $\operatorname{codim}_S Z = g \geq 3$. Let $\tilde{S}$ be the blow up of S in Z, $\tilde{Z} \subset \tilde{S}$ the strict transform. It turns out that $\tilde{Z} = \mathbb{P}(F) \times_{\mathcal{J}_{g-1}} \mathbb{P}(F')$ for some $(g-1)$-bundle F' over $\mathcal{J}_{g-1}$ and that $\tilde{Z}$ can be partially contracted in $\tilde{S}$ along the direction of $\mathbb{P}(F)$ to give a space $\bar{S}$ with the image $\bar{Z}$ of $\tilde{Z}$ of dimension $2g-2$. The birational transformation $S \leftrightarrow \bar{S}$ is a flip along Z in the sense of Mori theory.

The attaching map $(Q \subset S) \to (Q \subset P)$ now induces an isomorphism of the strict transform of Q in $\bar{S}$ with $Q \subset P$, and so this strict transform will also be called Q.

Proposition 3.5 [Gi][Th1] $\qquad \mathcal{V}_0 = \bar{S} \coprod_Q P.$ $\diamond$

3.5 The recursive formula for GW-invariants

To compute $\Phi^{\mathcal{N}_g}_{\iota_* R}(\varphi_1, \ldots, \varphi_k, \gamma_i)$ for $\iota : \mathcal{N}_{g-1} \hookrightarrow \mathcal{N}_g$, $R \in H_2(\mathcal{N}_{g-1}; \mathbb{Z})$, $\varphi_1, \ldots, \varphi_k \in H^*_I(\mathcal{N}_g)$, γ_i Poincaré-dual to $\iota_*[\mathcal{N}_{g-1}]$, we start with $\Phi^{\mathcal{N}_{g-1}}_R(\iota^*\varphi_1, \ldots, \iota^*\varphi_k)$ and try to argue backwards. We keep the notations of the previous subsection.

Recall the universal bundle $\mathcal{U} \downarrow \mathcal{N}_{g-1} \times C$. The choice of a differentiable isomorphism $\mathcal{U}_P \simeq \mathcal{U}_Q$ induces a differentiable section s of $P' = \mathbb{P}\mathrm{Hom}(\mathcal{U}_P, \mathcal{U}_Q)$. We proceed as in Proposition 1.1 and represent (multiples of) $\iota^*\varphi_i$ by submanifolds $A_i \subset \mathcal{N}_{g-1}$, choose compatible almost complex structures J on $\mathcal{N}_{g-1}$ and $\tilde{J}$ on P' in such a way that

$$\Phi^{\mathcal{N}_{g-1}}_R(\iota^*\varphi_1, \ldots, \iota^*\varphi_k) \;=\; \sharp\left\{\psi : \mathbb{P}^1 \to \mathcal{N}_{g-1} \;\middle|\; \bar{\partial}_J \psi = 0,\; \psi_*[\mathbb{P}^1] = R,\; \psi(t_i) \in A_i \;\right\}$$

for pairwise disjoint $t_i \in \mathbb{P}^1$, and such that there are unique $\tilde{J}$-holomorphic liftings $\tilde{\psi}$ for ψ in the set of the right-hand side, $\tilde{\psi}_*[\mathbb{P}^1] = s_* R$, $\psi(t_{k+1}) \in s(\mathcal{N}_{g-1})$. Since $Q \cdot s_* R = 0$, we also get, with a little more care, $\mathrm{im}\,\tilde{\psi} \cap Q \neq \emptyset$ for any such lift. Since $S \to P$ is branched only along Q and s naturally lifts to S the same statements hold true with P replaced by S. We will keep the notation $\tilde{\psi}$ for the corresponding unique lifts of the ψ. Let $K \subset S$ be a compact neighbourhood of Q disjoint from $\mathrm{im}\,\tilde{\psi}$ for any $\tilde{\psi}$. Extend $\tilde{J}$ to an almost complex structure on a desingularization $\tilde{\mathcal{V}}$ of $\mathcal{V}$ keeping the map to the basis $\tilde{\mathcal{V}} \to S$ holomorphic. One can check the following:

- There are submanifolds $\tilde{A}_i \subset \tilde{V}$ transversal to $\tilde{\mathcal{V}}_s = \mathcal{V}_s$ for $s \neq 0$ and to $S \setminus K$, such that
  - $\tilde{A}_i \cap \mathcal{V}_s$ is Poincaré-dual to φ_i
  - $\tilde{A}_i \cap (S \setminus K) = q^{-1}(A_i) \cap (S \setminus K)$ where $q : S \to \mathcal{N}_{g-1}$ is the bundle projection

- There exists an extension $\tilde{s} : S \times \mathcal{N}_{g-1} \hookrightarrow \tilde{\mathcal{V}}$ of the embedding $s : \mathcal{N}_{g-1} \hookrightarrow S \setminus K \subset \mathcal{V}_0$. For any $s \neq 0$, $\tilde{s}_s$ is Poincaré-dual to $\gamma_i \in H^6(\mathcal{V}_s \simeq \mathcal{N}_g)$
- The deformation problem $\psi^s : \mathbb{P}^1 \to \tilde{\mathcal{V}}_s$ with the incidence conditions $\psi^s(t_i) \in q^{-1}(\tilde{A}_i)$, $\psi^s(t_{k+1}) \in \operatorname{im} s$, $\operatorname{im} \psi^s \subset \tilde{\mathcal{V}}_s$ is unobstructed at the lifts $\tilde{\psi}$ of the ψ.

We thus see that we can produce, for any $s \in S \setminus \{0\}$, a finite subset of pseudoholomorphic maps $\psi : \mathbb{P}^1 \to \mathcal{N}(\mathcal{X}_s, 2, L_s)$ contributing to the requested GW-invariant $\Phi^{\mathcal{N}_g}_{\iota_* R}(\varphi_1, \ldots, \varphi_k, \gamma_i)$ precisely $\Phi^{\mathcal{N}_{g-1}}_{R}(\iota^*\varphi_1, \ldots, \iota^*\varphi_k)$ curves (counted with signs).

Contrary to what we initially believed, there is no homological argument prohibiting curves with components in $\bar{Z}$ in the limit $s \to 0$ of a sequence of pseudoholomorphic curves in $\mathcal{N}(\mathcal{X}_s, 2, L_s)$ with the required incidence conditions. So we can only prove the recursion formula under the additional input that

Conjecture 3.6 *Trees of rational curves with components in $\bar{Z}$ do not contribute to the GW-invariant under study.*

4 Quantum cohomology of toric manifolds (after Batyrev)

We are going to discuss [Bt]. In preparing these notes [MoPl] has also been helpful.

4.1 Toric manifolds

It will be most convenient for us to describe a toric manifold M as quotient of $\mathbb{C}^n \setminus F$, F an algebraic subset of the coordinate hyperplanes, by a diagonal action of $(\mathbb{C}^*)^{n-d}$, $d = \dim M$. The easiest example of this is of course $\mathbb{P}^d = (\mathbb{C}^{d+1} \setminus \{0\})/\mathbb{C}^*$.

How precisely F and the action are defined is encoded in a *fan* Σ, which is a collection of cones $\{\sigma\}$ in a d-dimensional vector space $N_\mathbb{R} = N \otimes_\mathbb{Z} \mathbb{R}$, $N \simeq \mathbb{Z}^d$ a lattice. One restricts to cones that are spanned by finitely many lattice vectors (*rational polyhedral cones*) and that meet some hyperplane in N only at the origin (*strong convexity*). We write $\sigma = \langle v_1, \ldots, v_k \rangle$. It is also required that the cones fit together, i.e. $\sigma, \sigma' \in \Sigma$ imply $\sigma \cap \sigma' \in \Sigma$, and that any face of some $\sigma \in \Sigma$ is also an element of Σ.

Many of the properties of the associated toric variety, the construction of which we will give only in the smooth case below, are directly reflected in properties of the fan. To produce a non-singular variety of dimension d, for example, the condition is that the defining vectors of maximal cones (i.e. which are not faces of others) form a lattice basis, while completeness is equivalent to $N_\mathbb{R} = \bigcup_{\sigma \in \Sigma} \sigma$. We will fix such a regular, complete fan Σ during the following duscussion. Let $v_1, \ldots, v_n$ be the defining lattice vectors (generators of one-dimensional cones).

$F = F(\Sigma)$ is now defined as union of all $V(z_{i_1}, \ldots, z_{i_k}) \subset \mathbb{C}^n$ such that $\{v_{i_1}, \ldots, v_{i_k}\}$ is *not* contained in any cone $\sigma \in \Sigma$. Note that $\operatorname{codim} F \geq 2$. For the $(\mathbb{C}^*)^{n-d}$-

action let $R = R(\Sigma) \subset \mathbb{Z}^n$ be the kernel of the linear map

$$\mathbb{Z}^n \longrightarrow N \simeq \mathbb{Z}^d, \quad \lambda = (\lambda_1, \ldots, \lambda_n) \longmapsto \sum_i \lambda_i v_i \,.$$

Being surjective by regularity, R is a free $\mathbb{Z}$-module of rank $n-d$. Any $\lambda \in R$ generates a $\mathbb{C}^*$-action on $\mathbb{C}^n \setminus F$ by

$$\mathbb{C}^* \times (\mathbb{C}^n \setminus F) \longrightarrow \mathbb{C}^n \setminus F, \quad (t; z_1, \ldots, z_n) \longmapsto (t^{\lambda_1} z_1, \ldots, t^{\lambda_n} z_n)\,,$$

and these commute among each other. Choosing a basis $\lambda^1, \ldots, \lambda^{n-d}$ for R this amounts to our $(\mathbb{C}^*)^{n-d}$-action on $\mathbb{C}^n \setminus F$. This action is free and there always exists a good algebraic quotient, i.e. the quotient in the category of locally ringed spaces is a scheme with closed points being the set of orbits. In fact, there is a covering of $\mathbb{C}^n \setminus F$ by the $(\mathbb{C}^*)^{n-d}$-invariant affine open sets

$$U_\sigma \;=\; \mathbb{C}^n \setminus \bigcup_{v_j \notin \sigma} V(z_j) \simeq \mathbb{C}^d \times (\mathbb{C}^*)^{n-d}, \quad \sigma \in \Sigma \text{ a } d\text{-dimensional cone.}$$

If $\sigma = \{v_1, \ldots, v_d\}$ and $u_1, \ldots, u_d \in N^\vee$ is a dual basis to $v_1, \ldots, v_d$ then

$$w_1 := z_1 \cdot z_{d+1}^{\langle v_{d+1}, u_1\rangle} \cdot \ldots \cdot z_n^{\langle v_n, u_1\rangle} \;,\; \ldots \;,\; w_d := z_d \cdot z_{d+1}^{\langle v_{d+1}, u_d\rangle} \cdot \ldots \cdot z_n^{\langle v_n, u_d\rangle}$$

freely generate the invariant ring $\mathbb{C}[z_1, \ldots, z_n, z_{d+1}^{-1}, \ldots, z_n^{-1}]^{(\mathbb{C}^*)^{n-d}}$. Therefore the quotient

$$\pi : \mathbb{C}^n \setminus F \longrightarrow M \;=\; M(\Sigma) \;:=\; (\mathbb{C}^n \setminus F)/(\mathbb{C}^*)^{n-d}$$

exists and is a smooth algebraic variety. The quotient map π has the local sections

$$\pi(U_\sigma) \longrightarrow \mathbb{C}^n \setminus F, \quad (w_1, \ldots, w_d) \longrightarrow z = (w_1, \ldots, w_d, 1, \ldots, 1)$$

(if $\sigma = \{v_1, \ldots, v_d\}$ as above) and is thus an algebraic principal $(\mathbb{C}^*)^{n-d}$-bundle. Note also that the quotient of $\mathbb{C}^n \setminus V(z_1 \cdot \ldots \cdot z_n) = (\mathbb{C}^*)^n$ gives a dense "big cell" $(\mathbb{C}^*)^d \subset M$, a d-dimensional torus. Hence the name "toric variety" or "torus embedding" for M. The complement of the big cell consists of n divisors $Z_1, \ldots, Z_n$, the images of the coordinate hyperplanes $V(z_i)$.

4.2 The cohomology ring, c_1, H_2, and the Kähler cone

The cohomology classes associated to the n divisors Z_i (also denoted Z_i) generate the cohomology ring of M. These generators are subject to two kinds of relations: Linear ones

$$\sum_i \langle m, v_i \rangle Z_i \;=\; 0$$

coming from the $(\mathbb{C}^*)^{n-d}$-invariant global rational functions $\prod_i z_i^{\langle m, v_i\rangle}$ for any $m \in N^\vee \simeq \mathbb{Z}^d$, the dual lattice, and nonlinear ones

$$Z_{i_1} \ldots Z_{i_k} \;=\; 0 \;\Leftrightarrow\; V(z_{i_1}) \cap \ldots \cap V(z_{i_k}) \subset F \;\Leftrightarrow\; \begin{array}{l} \{v_{i_1}, \ldots, v_{i_k}\} \text{ is not cont-} \\ \text{ained in a single } \sigma \in \Sigma \end{array} \;,$$

of evident geometric origin. (The number of generators of non-linear relations can be reduced by restricting to so-called primitive sets $\{v_{i_1}, \ldots, v_{i_k}\}$ [Bt], but we won't need that.) Let us denote the ideals generated by the linear and nonlinear relations $\Lambda = \Lambda(\Sigma)$ and $\mathrm{SR} = \mathrm{SR}(\Sigma)$ respectively (SR stands for *Stanley-Reisner* ideal).

Fact 1. $H^*(M;\mathbb{Z}) \;=\; \mathbb{C}[Z_1, \ldots, Z_n]/(\mathrm{SR} + \Lambda)$. $\diamond$

It is often better not to reduce the set of generators by the linear relations. For example,

$$c_1(M) \;=\; Z_1 + \ldots + Z_n$$

is more easily expressed in terms of all Z_i.

It can also be shown that intersections of the Z_i as cycles or Chow classes generate the integral homology or (what is the same here) Chow groups. For our purposes a more convenient description of the second homology is however as dual lattice of $H^2(M;\mathbb{Z})$. For $A \in H_2(M;\mathbb{Z})$ let $\lambda_i = Z_i \cdot A \in \mathbb{Z}$. Since $H^2(M;\mathbb{Z}) = \mathbb{C}^n_{Z_1,\ldots,Z_n}/(\sum_i \langle m, v_i\rangle Z_i \mid m \in N^\vee)$ we get

Fact 2. $H_2(M;\mathbb{Z}) \;=\; R \;=\; \{\lambda \in \mathbb{Z}^n \mid \sum_i \lambda_i v_i = 0\}$. $\diamond$

In this description the dual pairing evaluates λ on Z_i to λ_i.

To describe the Kähler cone $\mathcal{K}_M$ one uses the notion of piecewise linear functions on Σ, which are continuous functions $\omega : N_\mathbb{R} \to \mathbb{R}$ with $\omega|_\sigma$ linear for any $\sigma \in \Sigma$. Notation: $\mathrm{PL} = \mathrm{PL}(\Sigma)$. Obviously, $\mathrm{PL} \simeq \mathbb{R}^n$ via $\omega \mapsto (\omega(v_1), \ldots, \omega(v_n))$. By the description of $H^2(M)$ the surjection $PL \to H^2(M;\mathbb{R})$ sending ω to $\sum_i \omega(v_i) Z_i$ induces an isomorphism of $H^2(M;\mathbb{R})$ with PL modulo linear forms, i.e. $\mathrm{PL}/N^\vee_\mathbb{R} \simeq H^2(M;\mathbb{R})$. By abuse of notation we write ω for both the piecewise linear function and its class in H^2.

Fact 3. *Under this isomorphism $\omega \in \mathrm{PL}$ corresponds to a class in $\mathcal{K}_M$ iff ω is convex, i.e.*

$$\omega(x+y) \;\le\; \omega(x) + \omega(y) \quad \forall\, x, y \in N_\mathbb{R}\,,$$

and ω is a Kähler class (i.e. $\omega \in \mathcal{K}^o_M$, the interior) iff $\omega|_{\sigma+\sigma'}$ is not linear for any two different d-dimensional cones $\sigma, \sigma' \in \Sigma$. $\diamond$

4.3 Moduli spaces of maps $\mathbb{P}^1 \to M$

Let us now try to get our hands on the moduli spaces of stable rational curves in M. Essential building blocks of these moduli spaces are spaces $\mathcal{C}^0_\lambda(M)$ of morphisms

$$\varphi : \mathbb{P}^1 \longrightarrow M$$

with $\varphi_*[\mathbb{P}^1] = \lambda$ for some fixed n-tuple $\lambda \in R = H_2(M;\mathbb{Z})$, i.e. $\lambda_i = \deg \varphi^* \mathcal{O}_M(Z_i)$. By renumbering we may assume $\lambda_1, \ldots, \lambda_l < 0$ and $\lambda_{l+1}, \ldots, \lambda_n \ge 0$. In this subsection we will give a fairly explicit description of $\mathcal{C}^0_\lambda(M)$ for any λ.

We start with the following simple but crucial observation:

Lemma 4.1 *Let $\pi : \mathbb{P}^1 \to M$ and $Z_{i_1}, \ldots, Z_{i_l}$ be the maximal set of divisors with* $\mathrm{im}\varphi \subset Z_{i_1} \cap \ldots \cap Z_{i_k}$. *Then $\{Z_i \mid i \neq i_\nu \, \forall \nu\}$ spans $H^2(M)$ and $\varphi_*[\mathbb{P}^1] \in H_2(M;\mathbb{Z})$ is determined by* $\deg \varphi^{-1}(Z_i)$, $i \neq i_\nu$.

Proof. By assumption $Z_{i_1} \cap \ldots \cap Z_{i_k} \neq \emptyset$. The definition of $F \subset \mathbb{C}^n$ thus implies the existence of $\sigma \in \Sigma$ with $\{v_{i_1}, \ldots, v_{i_k}\} \subset \sigma$, whence $v_{i_1}, \ldots, v_{i_k}$ are linearly independent. Letting $m_1, \ldots, m_d \in M$ be a dual basis to $v_{i_1}, \ldots, v_{i_k}$ we may express $Z_{i_1}, \ldots, Z_{i_k}$ by Z_i, $i \neq i_\nu$, since

$$\sum_i \langle m_\mu, v_i \rangle \;=\; Z_{i_\mu} + \sum_{i \neq i_\nu} \langle m_\mu, v_i \rangle$$

is in the linear relation ideal Λ. $\diamond$

In other words, any irreducible curve $C \subset M$ (rational or not) is transversal to some divisors $Z_{j_1}, \ldots, Z_{j_{n-k}}$, the intersection numbers of which with C determine the homology class $[C]$.

We want to produce elements of $\mathcal{C}^0_\lambda(M)$ as extensions φ_f of $\varphi^0_f = \pi \circ f$ with

$$f = (0, \ldots, 0, f_{l+1}, \ldots, f_n) : \mathbb{A}^1 \longrightarrow \mathbb{C}^n \setminus F$$

with $f_i \in \mathbb{C}[t]$, $\deg f_i \leq \lambda_i$, $\pi : \mathbb{C}^n \setminus F \to M$ the quotient map. To make φ^0_f well-defined on all of $\mathbb{A}^1$ we restrict to such f with $f^{-1}(F) = \emptyset$. To discuss the extension we change coordinates to $u = t^{-1}$ and write

$$\varphi^0_f \;=\; \pi \circ (0, \ldots, 0, u^{\lambda_{l+1}} f_{l+1}(u^{-1}), \ldots, u^{\lambda_n} f_n(u^{-1}))\,,$$

where we used the $\mathbb{C}^*$-action on $\mathbb{C}^n \setminus F$ associated to λ. If $\deg f_i = \lambda_i$ for all i, the limit $u \to 0$ is $\pi(0, \ldots, 0, a_{l+1}, \ldots, a_n)$ with a_i the leading coefficient of f_i. In particular, $\varphi_f(\infty) \notin Z_{l+1} \cup \ldots Z_n$ and $\deg \varphi_f^{-1}(Z_i) = \lambda_i$ for $i > l$. So in view of Lemma 4.1, $\varphi_f \in \mathcal{C}^0_\lambda(M)$. If some f_i have lower degree, one also expects intersections with Z_i at ∞, but in nastier cases $\deg \varphi_f^{-1}(Z_i)$ might be less than λ_i, i.e. φ_f would not have the right homology. Nevertheless one can show, using the local trivializations of π over $\pi(U_\sigma)$ (cf. section 4.1), that this happens only on an algebraic subset of the space of f. Thus

$$V \;:=\; \Big\{ f = (0, \ldots, 0, f_{l+1}, \ldots, f_n) \mid f^{-1}(F) \neq \emptyset,\, \varphi_f \in \mathcal{C}^0_\lambda(M) \Big\}$$

is a Zariski-open subset of $\mathbb{C}^{\Sigma_{i>l}(\lambda_i+1)}$. Note that $V \neq \emptyset$ iff there exists $\sigma \in \Sigma$ with $v_1, \ldots, v_l \in \sigma$. We get a map

$$\Phi : V \longrightarrow \mathcal{C}^0_\lambda(M), \quad f \longmapsto \varphi_f\,,$$

which is clearly a morphism of complex schemes. There is a free action of $(\mathbb{C}^*)^{n-d}$ on V, coming from the diagonal action on $\mathbb{C}^n \setminus F$, under which Φ is invariant. As free action of a reductive group on an affine variety $V = \mathrm{Spec}\, A$, the geometric invariant theory quotient $V/\!\!/(\mathbb{C}^*)^{n-d} = \mathrm{Spec}\,(A^{(\mathbb{C}^*)^{n-d}})$ is an orbit quotient $V/(\mathbb{C}^*)^{n-d}$. Φ induces a morphism

$$\bar{\Phi} : V/(\mathbb{C}^*)^{n-d} \longrightarrow \mathcal{C}^0_\lambda(M)\,.$$

Lemma 4.2 $\bar{\Phi}$ *is bijective.*

Proof. Let $\varphi \in \mathcal{C}^0_\lambda(M)$. Then either $\mathrm{im}\varphi \subset Z_i$ or $\varphi^{-1}(Z_i)$ is a divisor on $\mathbb{P}^1$ of degree λ_i. $\varphi^{-1}(Z_i) \subset \mathbb{A}^1 = \mathbb{P}^1 \setminus \{\infty\}$ thus determines all zeros of f_i. Moreover, the leading coefficients can be determined by $\varphi(t)$ uniquely up to the action of $(\mathbb{C}^*)^{n-d}$ for any $t \in \mathbb{P}^1$. This shows that $\bar{\Phi}$ is injective.

Conversely, we have found $f \in V$ with $\varphi^{-1}(Z_i) = \varphi_f^{-1}(Z_i)$, $\varphi(t) = \varphi_f(t)$, and have to show $\varphi = \varphi_f$. We use the local section of π over $\pi(U_\sigma)$ (cf. section 4.1) with $f(t) \in U_\sigma$. Renumbering $v_{l+1}, \ldots, v_n$ we may assume $\sigma = \langle v_1, \ldots, v_d \rangle$. Then $\pi(U_\sigma) = M \setminus \bigcup_{i=d+1}^n Z_i$ has coordinates $w_1, \ldots, w_d$ and a section

$$\Lambda : \pi(U_\sigma) \longrightarrow \mathbb{C}^n \setminus F, \quad (w_1, \ldots, w_d, 1, \ldots, 1)\,.$$

From the expression of w_i in terms of z_i together with $\varphi^{-1}(Z_i) = \varphi_f^{-1}(Z_i)$, we see that for any i, $z_i(\Lambda \circ \varphi)$ and $z_i(\Lambda \circ \varphi_f)$ are rational functions on $\mathbb{P}^1$ that either both vanish identically ($\mathrm{im}\varphi \subset Z_i$), or have the same zero and polar divisor. Taking into account $\varphi(t) = \varphi_f(t) \in U_\sigma$ this shows $\Lambda \circ \varphi = \Lambda \circ \varphi_f$ and hence $\varphi = \varphi_f$. $\diamond$

We may also characterize now classes of rational curves.

Proposition 4.3 *The nef cone of M (the dual cone of $\mathcal{K}_M$) is*

$$\mathrm{NE}\,(M) = \{\lambda \in H_2(M;\mathbb{Z}) \mid \exists \sigma \in \Sigma : \lambda_i < 0 \Rightarrow v_i \in \sigma\}\,,$$

and such classes can be represented by irreducible rational curves.

Proof. This follows from the lemma together with the characterization of emptyness of V. $\diamond$

$\mathrm{NE}\,(M)$ is thus a union of $\sharp\{\sigma \in \Sigma \mid \dim\sigma = d\}$ quadrants $(\mathbb{N}_{\geq 0})^{n-d} \subset H_2(M;\mathbb{Z}) \simeq \mathbb{Z}^{n-d}$, and in particular $\mathrm{NE}\,(M)$ is polyhedral. Extremal rays of $\mathrm{NE}\,(M)$ are of the form: $\lambda_i = 1$ for some i, $\lambda_j = 0$ for $j \neq i$, $j \notin \sigma = \langle v_{i_1}, \ldots, v_{i_d} \rangle$, and $\lambda_{i_1}, \ldots, \lambda_{i_d}$ fixed by the requirement $\sum \lambda_i v_i = 0$.

To determine the scheme-theoretic structure of $\mathcal{C}^0_\lambda(M)$ we compute the embedding dimension of $\mathcal{C}^0_\lambda(M)$ at $\varphi : \mathbb{P}^1 \to M$, i.e. the dimension of the Zariski tangent space $H^0(\mathbb{P}^1, \varphi^* T_M)$. Pulling back the generalized Euler sequence

$$0 \longrightarrow \mathcal{O}_M^{n-d} \longrightarrow \mathcal{O}_M(Z_1) \oplus \ldots \oplus \mathcal{O}_M(Z_n) \longrightarrow T_M \longrightarrow 0$$

we obtain

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^1}^{n-d} \longrightarrow \mathcal{O}_{\mathbb{P}^1}(\lambda_1) \oplus \ldots \oplus \mathcal{O}_{\mathbb{P}^1}(\lambda_n) \longrightarrow \varphi^* T_M \longrightarrow 0\,.$$

Recall our convention that $\lambda_i < 0$ precisely for $i \leq l$. We get

$$\begin{aligned} h^0(\mathbb{P}^1, \varphi^* T_M) &= \sum_{i=l+1}^{n} (\lambda_i + 1) - (n-d) = \sum_{i=l+1}^{n} \lambda_i + d - k =: h^0(\lambda) \\ h^1(\mathbb{P}^1, \varphi^* T_M) &= \sum_{i=1}^{l} (-\lambda_i - 1) = \sum_{i=1}^{l} |\lambda_i| - l =: h^1(\lambda), \end{aligned}$$

and for the expected dimension

$$d(M, \lambda, 0, 3) = d + \sum_{i=1}^{n} \lambda_i =: d(\lambda).$$

It is remarkable that the embedding dimension of $\mathcal{C}^0_\lambda(M)$ is constant. This is already a strong hint towards smoothness of $\mathcal{C}^0_\lambda(M)$. In fact,

Proposition 4.4 *$\mathcal{C}^0_\lambda(M)$ is either empty or a smooth rational variety of dimension $h^0(\lambda)$. Moreover, $\Phi : V \to \mathcal{C}^0_\lambda(M)$ is a $(\mathbb{C}^*)^{n-d}$-bundle.*

Proof. By the lemma, $\mathcal{C}^0_\lambda(M)$ is irreducible of dimension $h^0(\lambda)$, which being the embedding dimension at any φ proves half of the claim. Rationality and local triviality of Φ follow from the open embeddings of $\{\varphi \in \mathcal{C}^0_\lambda(M) \mid \varphi(t) \in U_\sigma\}$ into $\mathbb{C}^{h^0(\lambda)}$ that one obtains from the construction of f in the lemma upon using the section $\Lambda : \pi(U_\sigma) \to U_\sigma$ to fix the leading coefficients. $\diamond$

In concluding this section I would like to remark that by enlarging V one may construct (not in general unique) toric varieties containing $\mathcal{C}^0_\lambda(M)$, that do not have an immediate interpretation as space of maps.

4.4 Moduli spaces of stable curves and GW-invariants

Moduli spaces $\mathcal{C}_{\lambda,k}(M)$ of stable k-pointed rational curves in M can be stratifed according to their combinatorial type. The combinatorial type of a stable rational curve $(C, \mathbf{x}, \varphi)$ is given by the number of irreducible components $C_1, \ldots, C_m$, their intersection pattern, the homology classes $\varphi_*[C_i] = \lambda^i$, and by saying on which components the various marked points lie, cf. [BeMa] for a description in terms of associated trees.

Quite generally, easiest are cases of indecomposable classes $\lambda \in \mathrm{NE}\,(M)$, i.e. extremal classes. Then no unstable components (bubbles) may occur and general fibers of $\mathcal{C}_{\lambda,k}(M) \to \mathcal{M}_{0,k}$ are isomorphic to the smooth $h^0(\lambda)$-dimensional variety $\mathcal{C}^0_\lambda(M)$. $\mathcal{C}^0_\lambda(M)$ now being compact it is its own toric compactification. In case all λ_i are non-negative, $h^0(\lambda) = d(\lambda)$ and GW-invariants constructed from $\mathcal{C}_{\lambda,k}(M)$ are enumerative in the strong sense of Proposition 1.3.

In other cases, say $\lambda_1, \ldots, \lambda_l < 0$, by Corollary 1.6, one just has to work out the obstruction bundle. Let us do this over a fixed $(\mathbb{P}^1, (t_1, \ldots, t_k)) \in \mathcal{M}_{0,k}$ without non-trivial automorphisms (this will suffice for small quantum cohomology). The universal curve is simply $p : \mathbb{P}^1 \times \mathcal{C}^0_\lambda(M) \to \mathcal{C}^0_\lambda(M)$ with evaluation map

$$\mathrm{ev} : \mathbb{P}^1 \times \mathcal{C}^0_\lambda(M) \longrightarrow M, \quad (t, \varphi) \longmapsto \varphi(t) .$$

In view of the generalized Euler sequence we just have to compute $R^1 p_* \mathrm{ev}^* \mathcal{O}(Z_i)$. Let $W^0_{l+1}, \ldots, W^{\lambda_{l+1}}_{l+1}, \ldots, W^0_n, \ldots, W^{\lambda_n}_n$ be the toric divisors on $\mathcal{C}^0_\lambda(M)$, i.e. W^i_ν corresponds to the vanishing of the ν-th coefficient of f_i. Note that $\mathrm{Pic}(\mathcal{C}^0_\lambda(M))$ is generated by $n-d$ of $W^0_{l+1}, \ldots, W^0_n$. But $\mathrm{Pic}(\mathbb{P}^1 \times \mathcal{C}^0_\lambda(M)) = \mathrm{Pic}(\mathbb{P}^1) \times \mathrm{Pic}\mathcal{C}^0_\lambda(M)$, so from

$$\mathrm{ev}^* \mathcal{O}(Z_i)|_{\{0\} \times \mathcal{C}^0_\lambda(M)} \;=\; \mathcal{O}(W^0_i) , \quad i > l ,$$

we deduce $\mathrm{ev}^* \mathcal{O}(Z_i) = \mathrm{pr}_1^* \mathcal{O}_{\mathbb{P}^1}(\lambda_i) \otimes p^* \mathcal{O}_{\mathcal{C}^0_\lambda(M)}(W^0_i)$ for $i > l$. For $i \le l$ we write $Z_i = \sum_{j>l} a^j_i Z_j$ to get

$$R^1 p_* \mathrm{ev}^* \mathcal{O}(Z_i) \;=\; \mathcal{O}_{\mathcal{C}^0_\lambda(M)} \Big(\sum_{j>l} a^j_i W^0_j \Big)^{\oplus -\lambda_i - 1} , \quad i \le l .$$

Therefore,

$$\begin{aligned} [\![\mathcal{C}^0_\lambda(M)]\!] &= c_{h^1(\lambda)} \Big(\bigoplus_{i \le l} R^1 p_* \mathrm{ev}^* \mathcal{O}(Z_i) \Big) \cap [\mathcal{C}^0_\lambda(M)] \\ &= \sum_J a^{j^1_1}_1 \ldots a^{j^1_{-\lambda_1 - 1}}_1 \ldots a^{j^l_1}_l \ldots a^{j^l_{-\lambda_l - 1}}_1 W^0_{j^1_1} \ldots W^0_{j^l_{-\lambda_l - 1}} . \end{aligned}$$

So again, such GW-invariants are amenable to toric computations.

The situation gets more involved as soon as we drop the assumption of indecomposability of λ. Then the moduli space splits into several irreducible components labeled by the generic combinatorial type they parametrize. If $\lambda \in \mathrm{NE}(M)$ is such that for any decomposition $\lambda = \lambda_1 + \ldots + \lambda_r$ in $\mathrm{NE}(M)$, $\lambda^j_i \ge 0$, then any irreducible component of $\mathcal{C}^0_\lambda(M)$ will still have the expected dimension, and the associated GW-invariants can be computed from (any) toric compactifications of $\mathcal{C}^0_\lambda(M)$. By Corollary 1.6 and the discussion above one can still get away with toric computations as long as the dimension of the "bad part" of $\mathcal{C}_{\lambda,0,k}(M)$, which is where the combinatorial type is not locally constant, is of dimension less than $d(M, \lambda, 0, k)$. For the general case it will be necessary to work out the irreducible components of $\mathcal{C}_{\lambda,k}(M)$, say over generic $(\mathbb{P}^1, \mathbf{x}) \in \mathcal{M}_{0,k}$ (hope: smooth, toric), and the way they intersect (hope: normal crossing of high codimension).

4.5 Batyrev's quantum ideals

At the time of the writing of [Bt] there was no rigorous definition of quantum cohomology available. In loc.cit. Batyrev mostly discussed the properties of a certain deformation of the (non-linear) relation ideal SR and with hindsight justified his choice by an intersection theoretic computation on space $\mathcal{C}^0_\lambda(M)$ with all $\lambda_i > 0$.

$\mathcal{C}^0_\lambda(M)$ having the expected dimension in this case there was no need to discuss a compactification. Here we want to make a statement of when his result really computes the (small) quantum cohomology of M.

Batyrev fixes a Kähler class $\omega = \sum_i d_i Z_i$ and defines three quantum ideals:

1. A natural deformation of $\mathrm{SR} = \left(\prod_{j\in J} Z_j\right)_J$ (over all $J = \{j_1, \ldots, j_k\}$ with $\{v_{i_1}, \ldots, v_{i_k}\} \not\subset \sigma \ \forall \sigma \in \Sigma$): For any such J, let $\sigma = \langle v_{i_1}, \ldots, v_{i_d}\rangle$ be a cone containing $v_{j_1}+\ldots+v_{j_k}$. Then $v_{j_1}+\ldots+v_{j_k} = c_1 v_{i_1}+\ldots+c_d v_{i_d}$, $c_\nu \geq 0$ with the non-vanishing terms on the right-hand side uniquely determined. Moreover, $R_J = v_{j_1} + \ldots + v_{j_k} - c_1 v_{i_1} - \ldots - c_d v_{i_d} \in \mathrm{NE}\,(M)$ (cf. Proposition 4.3). With $E_\omega(J) := \exp(-\omega(R_J))$ Batyrev sets
$$\mathrm{SR}_\omega \ := \ \Big(\prod_{j\in J} Z_j - Z_{i_1}^{c_1}\cdot\ldots\cdot Z_{i_d}^{c_d}\cdot E_\omega(J)\Big)_J .$$

2. An auxiliary ideal
$$Q_\omega \ := \ \Big(Z_1^{a_1}\cdot\ldots\cdot Z_n^{a_n}\cdot E_\omega(a) - Z_1^{b_1}\cdot\ldots\cdot Z_n^{b_n}\cdot E_\omega(b)\Big)_{a_i, b_i\in\mathbb{Z}_{\geq 0}, \sum(a_i-b_i)v_i = 0},$$
where $E_\omega(\lambda) := \sum_i d_i a_i$.

3. An ideal more amenable to computations of intersection numbers:
$$B_\omega \ := \ \Big(Z_1^{\lambda_1}\cdot\ldots\cdot Z_n^{\lambda_n} - E_\omega(\lambda)\Big)_{\lambda_i\in\mathbb{Z}_{\geq 0}, \sum \lambda_i v_i = 0}.$$

Note that SR_ω and B_ω could alternatively be defined without a choice of ω by taking coefficients in $\mathrm{NE}\,(M)$ (notation: $\mathrm{SR}_{\mathrm{NE}}$, B_{NE}), but Q_ω could not.

Proposition 4.5 (Batyrev) $\mathrm{SR}_\omega \ = \ Q_\omega \ = \ B_\omega$.

This is a little hidden in [Bt], so for the reader's convenience let us include the proof here.

Proof. The inclusions $\mathrm{SR}_\omega \subset Q_\omega \supset B_\omega$ are trivial. As for $Q_\omega \subset B_\omega$, for a, b with $\sum(a_i - b_i)v_i = 0$ choose λ, μ with $\lambda_i, \mu_i \geq 0$ such that $a - b = \lambda - \mu$. Let us abbreviate $Z^a = Z_1^{a_1}\cdot\ldots\cdot Z_n^{a_n}$ etc. Then in $\mathbb{C}[Z_1, \ldots, Z_n]/B_\omega$, $Z^\mu = E_\omega(\mu)$, $Z^\lambda = E_\omega(\lambda)$ and so
$$\begin{aligned} E_\omega(\mu)\Big(Z^a\cdot E(a) - Z^b\cdot E(b)\Big) \ &= \ Z^{a+\mu}\cdot E(a) - Z^b\cdot E(b+\mu) \\ = \ Z^{b+\lambda}\cdot E(a) - Z^b\cdot E(b+\mu) \ &= \ Z^b\Big(E(\lambda+\mu) - E(b+\mu)\Big) \ = \ 0 . \end{aligned}$$

It remains to be shown that $B_\omega \subset \mathrm{SR}_\omega$. To do this Batyrev introduces a weight order on $\mathbb{C}[Z_1, \ldots, Z_n]$ by setting $\mathrm{wt}(Z_i) = d_i$. Since ω is strongly convex
$$\mathrm{wt}\Big(\prod_{j\in J} Z_j\Big) \ > \ \mathrm{wt}\Big(Z_{i_1}^{c_1}\cdot\ldots\cdot Z_{i_d}^{c_d}\Big),$$

which shows $\mathrm{in}\,\mathrm{SR}_\omega = \mathrm{SR}$. Trivially, $\mathrm{in}\,B_\omega = (Z_1^{\lambda_1} \ldots Z_n^{\lambda_n})_{\sum \lambda_i v_i = 0}$. And if $\sum \lambda_i v_i = 0$ then $\{v_i \mid \lambda_i \neq 0\}$ can not be contained in a single cone. Thus $Z_1^{\lambda_1} \ldots Z_n^{\lambda_n} \in \mathrm{SR}$, i.e. $\mathrm{in}\,B_\omega \subset \mathrm{SR} = \mathrm{in}\,\mathrm{SR}_\omega$ and hence $B_\omega = \mathrm{SR}_\omega$. $\diamond$

In view of Proposition 2.3 it thus suffices to show

$$Z_1^{\lambda_1} * \ldots * Z_n^{\lambda_n} \;=\; [\lambda] \quad \text{in } QH^*_{\mathcal{R}\mathrm{C}}(M)$$

for any $\lambda \in H_2(M;\mathbb{Z})$, $\lambda_i \geq 0\,\forall i$. In terms of GW-invariants this means

Proposition 4.6

$$QH^*_{\mathcal{R}\mathrm{C}}(M) \;=\; \mathbb{C}[Z_1,\ldots,Z_n] \otimes_{\mathbb{Z}} \mathbb{Z}[\![\mathcal{R}\mathrm{C}(M)]\!]/\Big(\Lambda(M) + \mathrm{SR}_{\mathrm{NE}}(M)\Big)$$

iff for any $\lambda \in H_2(M;\mathbb{Z})$, $\lambda_i \geq 0$, $i_1,\ldots,i_n \geq 0$ *and* $\mu \in \mathrm{NE}(M)$

$$\Phi_\mu(\underbrace{Z_1,\ldots,Z_1}_{\lambda_1},\ldots,\underbrace{Z_n,\ldots,Z_n}_{\lambda_n}, Z_1^{i_1}\ldots Z_n^{i_n}) \;=\; \begin{cases} \deg Z_1^{i_1}\ldots Z_n^{i_n} & ,\mu=\lambda,\ \sum i_\nu = d \\ 0 & ,\text{else.}\end{cases}$$

$\diamond$

If all $\mu_i \geq 0$ the contributions of *irreducible* curves are in fact easy to work out (this is essentially Batyrev's computation): Let $\varphi : \mathbb{P}^1 \to M$ map distinct $t_1,\ldots,t_{\Sigma(\lambda_i+1)} \in \mathbb{P}^1$ to the cycles $Z_1, Z_1, \ldots$ in Φ_μ. By Lemma 4.1, φ is transversal to at least $n-d$ divisors, say $Z_{d+1},\ldots,Z_n$. Since at least μ_i distinct points map to Z_i this shows $\mu_i = \deg \varphi^*\mathcal{O}(Z_i) \geq \lambda_i$ for $i > d$. But from the dimension count

$$\sum_i \mu_i + d \;=\; \sum_i \lambda_i + \sum_\nu i_\nu\,,$$

and hence $\mu_i = \lambda_i$ for all i and $\sum i_\nu = d$. Thus $Z_1^{i_1}\ldots Z_n^{i_n}$ is zero-dimensional. With the description of $\mathcal{C}^0_\lambda(M)$ the contribution is seen to be $\deg Z_1^{i_1}\ldots Z_n^{i_n}$.

Cases with some $\mu_i < 0$ and contributions from reducible curves seem to be much harder to control, as we saw in the last subsection. Nevertheless, according to a recent paper of Givental, where he employed equivariant GW-invariants, Batyrev's prediction seems to be correct if M is Fano [Gl].

Finally, Batyrev points out that while SR_ω has finitely many generators but B_ω has infinitely many, only the latter manifestly shows that quantum cohomology (at least in nice cases) is expected to depend on the generating vectors $v_1,\ldots,v_n$ only, not on the specific structure of the fan. This is because different choices of fan structures on $v_1,\ldots,v_n$ correspond to flips in the sense of Mori theory. Independence under flips is not true for the ordinary cohomology *ring*!

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