

A TROPICAL VIEW ON LANDAU-GINZBURG MODELS

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Preliminary version

ABSTRACT. We fit Landau-Ginzburg models into the mirror symmetry program pursued by the last author jointly with Mark Gross. This point of view transparently brings in tropical disks of Maslov index 2 that group together virtually as broken lines, introduced in two dimensions in [Gr2]. We obtain proper superpotentials which agree on an open part with those classically known for toric varieties. Examples include LG models of non-toric del Pezzo surfaces.

CONTENTS

Introduction	1
1. Tropical data	3
2. The superpotential at t -order zero	4
3. Scattering of monomials	7
4. The superpotential via broken lines	9
5. Broken lines via tropical disks	14
6. Toric degenerations of del Pezzo surfaces	24
7. Smooth non-Fano and singular Fano toric surfaces	35
8. Three-dimensional examples	38
References	41

INTRODUCTION

Mirror symmetry has been suggested both by mathematicians [Gi] and physicists [HV] to extend from Calabi-Yau varieties to a correspondence between Fano varieties and Landau-Ginzburg models. Mathematically a Landau-Ginzburg model is a non-compact Kähler manifold with a holomorphic function, the superpotential. The majority of studies confined themselves to toric cases where the construction of the mirror is immediate. The one exception we are aware of is the work of Auroux, Katzarkov and Orlov on mirror

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symmetry for del Pezzo surfaces [AKO], where a symplectic mirror is constructed by a surgery construction.

The purpose of this paper is to fit the Fano/Landau-Ginzburg mirror correspondence into the mirror symmetry program via toric degenerations pursued by the last author jointly with Mark Gross [GS1],[GS3]. The program as it stands suggests a non-compact variety as the mirror of a variety with effective anti-canonical bundle, or rather toric degenerations of these varieties. So the main point is the construction of the superpotential. The core technical idea of *broken lines* (Definition 4.2) for the construction of the superpotential has already appeared in a different context in the two-dimensional situation in Gross' mirror correspondence for \mathbb{P}^2 [Gr2]. We replace his case-by-case study of well-definedness with a scattering computation, making it work in any dimensions.

Our main findings can be summarized as follows.

- (1) From our point of view the natural data on the Fano side is a toric degeneration of Calabi-Yau pairs as defined in [GS3], Definition 1.8. In particular, if arising from the localization of an algebraic family, the general fibre is a pair (X, D) of a complete variety X and a reduced anti-canonical divisor D . No positivity property is ever used in our construction apart from effectivity of the anti-canonical bundle.
- (2) The mirror is a toric degeneration of algebraically convex¹ non-compact Calabi-Yau varieties, together with a canonically defined holomorphic function on the total space of the degeneration.
- (3) The superpotential is proper if and only if the anti-canonical divisor D on the mirror side is irreducible (Proposition 2.2). These conditions also have clean descriptions on the underlying tropical models governing the mirror construction from [GS1],[GS3].
- (4) For toric Fano varieties our construction provides a canonical (partial) compactification of the known construction [HV].
- (5) The terms in the superpotential can be interpreted in terms of virtual numbers of tropical disks, at least in dimension two (Proposition 5.15). On the Fano side these conjecturally count holomorphic disks with boundary on a Lagrangian torus.
- (6) The natural holomorphic parameters occurring in the construction on the Fano side lie in $H^1(X, \Theta_{(X,D)})$ where $\Theta_{(X,D)}$ is the logarithmic tangent bundle. This group rules infinitesimal deformations of the pair (X, D) as expected by (1). The interpretation of the Kähler parameters and the parameters on the Landau-Ginzburg side is less clear. Note however that all parameters come from deformations of the underlying space, the superpotential does not deform independently.
- (7) Explicit computations include del Pezzo surfaces, a singular toric Fano surface, \mathbb{P}^3 and another toric Fano threefold.

Throughout we work over an algebraically closed field \mathbb{k} of characteristic 0.

¹A variety is *algebraically convex* if it possesses a proper map to an affine variety. In [GS3], Definition 1.6 this condition makes sure the fans describing the toric components of the central fiber are convex.

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1. TROPICAL DATA

Let $(\check{B}, \check{\mathcal{P}}, \check{\varphi})$ be the polarized intersection complex associated to a polarized toric degeneration of varieties with effective anti-canonical bundle, as described in [GS1] and [GS3] (“cone picture”). Equivalently, one has the discrete Legendre dual data $(B, \mathcal{P}, \varphi)$, referred to as the dual intersection complex or the fan picture of the same degeneration (or the cone picture of the mirror). While [GS1] only treated the case of trivial canonical bundle or closed B , it generalizes in a straightforward manner to the case of interest here of pairs consisting of a variety and an anti-canonical divisor (*Calabi-Yau pair*). These correspond to non-compact B with $\partial B = \emptyset$. The reasoning of [GS1] then still shows that $H^1(B, i_*\Lambda_B \otimes_{\mathbb{Z}} \mathbb{G}_m(\mathbb{k}))$ classifies the corresponding central fibers $(\check{X}_0, \mathcal{M}_{\check{X}_0})$ of toric degenerations of CY-pairs $(\check{X} \rightarrow T, \check{\mathcal{D}})$, as a log space. Conversely, under some maximal degeneracy assumption [GS3] constructs a canonical such family. Thus we understand this side of the mirror correspondence rather well.

The objective of this paper is to give a similarly canonical picture on the mirror side. The mirrors of Fano varieties are suggested to be so-called Landau-Ginzburg models (LG-models). Mathematically these are non-compact algebraic varieties with a holomorphic function, referred to as *super potential*, see [AKO],[CO],[FOOO1],[HV]. Following the general program laid out in [GS1] and [GS3], we construct LG-models via deformations of now a non-compact union of toric varieties. The superpotential is then constructed by extension from the central fiber.

Our starting point is the Legendre dual $(B, \mathcal{P}, \varphi)$ of $(\check{B}, \check{\mathcal{P}}, \check{\varphi})$ [GS1, Section 1.4]. Then \check{B} is compact with boundary and the definition of the sheaf $\Lambda_{\check{B}}$ of integral affine tangent vectors on $\check{B} \setminus \check{\Delta}$ needs to be modified to restrict to vectors tangent to ∂B . With this interpretation, $H^1(\check{B}, i_*\Lambda_{\check{B}}^* \otimes_{\mathbb{Z}} \mathbb{G}_m(\mathbb{k}))$ parametrizes normalized gluing data for non-compact log schemes with intersection complex (B, \mathcal{P}) . Let $X_0^\dagger := (X_0, \mathcal{M}_{X_0})$ be one such log scheme. We would like to apply [GS3] to exhibit X_0^\dagger as the central fiber of a toric degeneration. However, in higher dimensions there were several places in [GS3] where we assumed boundedness of the cells (in the consistency in codimension 0, in the homological argument, and in the normalization procedure). In dimension two the proof is much simpler and the problems having to do with unbounded joints and unbounded discriminant locus do not arise. We therefore restrict to $\dim B = 2$ from now on or assume the procedure of [GS3] runs through. In particular, we obtain a sequence $(\mathcal{S}_k)_{k \geq 0}$ of structures with \mathcal{S}_k consistent to order k and subsequent $\mathcal{S}_k, \mathcal{S}_{k+1}$ compatible to order k . Denote the resulting family $\mathfrak{X} \rightarrow \text{Spec } \mathbb{k}[[t]]$, and the central fiber X_0 . For unbounded ρ and $v \in \rho$ a vertex we assume $f_{\rho,v} = 1$.

Check in detail and maybe put in an appendix?

2. THE SUPERPOTENTIAL AT t -ORDER ZERO

Let $\sigma \in \mathcal{P}$ be an unbounded maximal cell. For each unbounded edge $\omega \subset \sigma$ there is a unique monomial $z^{m_\omega} \in R_{\text{id}_\sigma, \sigma}^0$ with $\text{ord}_\sigma(m_\omega) = 0$ and $-\overline{m_\omega}$ a primitive generator of $\Lambda_\omega \subset \Lambda_\sigma$ pointing in the unbounded direction of ω . Denote by $\mathcal{R}(\sigma)$ the set of such monomials m_ω . Note that in $\mathcal{R}(\sigma)$ parallel unbounded edges ω, ω' only contribute one exponent $m_\omega = m_{\omega'}$.

Now at any point of $\partial\sigma$ the tangent vector $-\overline{m_\omega}$ points into σ . Hence

$$W^0(\sigma) := \sum_{m \in \mathcal{R}(\sigma)} z^m$$

extends to a regular function on the component $X_\sigma \subset X_0$ corresponding to σ . For bounded σ define $W^0(\sigma) = 0$. Since the restrictions of the $W^0(\sigma)$ to lower dimensional toric strata agree they define a function $W^0 \in \mathcal{O}(X_0)$. This is our *superpotential at order 0*. A motivation for this definition in terms of counts of holomorphic disks will be given in Section 5.

One insight in this paper is that in studying LG-models tropically it is advisable to restrict to B with all outgoing edges parallel.

Proposition 2.1. *A necessary and sufficient condition for W^0 to be proper is the following:*

(2.1)

For any cell σ and unbounded edges $\omega, \omega' \subset \sigma$ it holds $\Lambda_\omega = \Lambda_{\omega'}$, as subspaces of Λ_σ .

Proof. It suffices to show the claimed equivalence after restriction to a non-compact irreducible component $X_\sigma \subset X_0$, that is, for $W^0(\sigma)$. If all edges are parallel, $W^0(\sigma)$ is a multiple of a monomial with compact zero locus, hence it is proper. For the converse we show that if $m_\omega \neq m_{\omega'}$ for some $\omega, \omega' \subset \sigma$ then $W^0(\sigma)$ is not proper. The idea is to look at the closure of the zero locus of $W^0(\sigma)$ in an appropriate toric compactification $X_{\tilde{\sigma}} \supset X_\sigma$. Let $\omega_0, \dots, \omega_r$ be the unbounded edges of σ and write $m_i = m_{\omega_i}$. By assumption $\text{conv}\{0, m_0, \dots, m_r\}$ has a face not containing 0 of dimension at least one. Let $H \subset \Lambda_\sigma$ be a supporting affine hyperplane of such a face. After relabeling we may assume m_0, \dots, m_s are the vertices of this face. Note that all $m_i - m_0$ are contained in the half-space $H - \mathbb{R}_{\geq 0}m_0$. Cutting σ with an appropriate translate of $H - \mathbb{R}_{\geq 0}m_0$ thus leads to an integral bounded polytope $\tilde{\sigma} \subset \sigma$ with one facet $\tau \subset \tilde{\sigma}$ not contained in a facet of σ and with $\Lambda_\tau = H - m_0$. Then $X_{\tilde{\sigma}}$ contains X_σ as the complement of the toric prime divisor $X_\tau \subset X_{\tilde{\sigma}}$. To study the closure of the zero locus of $W^0(\sigma)$ in $X_{\tilde{\sigma}}$ consider the rational function $z^{-m_0} \cdot W^0(\sigma)$ on $X_{\tilde{\sigma}}$. This rational function does not contain X_τ in its polar locus, and its restriction to the big cell of X_τ is

$$1 + \sum_{i=1}^s z^{m_i - m_0} \in \mathbb{k}[\Lambda_\tau].$$

In fact, $z^{m_i - m_0}$ for $i > s$ vanishes along X_τ . Since $s \geq 1$ this Laurent polynomial has a non-empty zero locus. This proves that unless $m_i = m_j$ for all i, j the closure of the zero locus of $W^0(\sigma)$ in $X_{\tilde{\sigma}}$ has a non-empty intersection with X_τ , and hence $W^0(\sigma)$ can not be proper. \square

Thus if one is to study LG models via our degeneration approach, then to obtain the full picture one has to impose Condition (2.1) in Proposition 2.1.

On the mirror side Condition (2.1) also has a natural interpretation. Recall from [GS3], Definition 1.8 the notion of toric degenerations of Calabi-Yau pairs $(\pi : \mathfrak{X} \rightarrow T, \mathfrak{D} \subset \mathfrak{X})$.

Proposition 2.2. *W^0 is proper if and only if $\check{\mathfrak{D}} \rightarrow T$ is a toric degeneration of Calabi-Yau varieties.*

Proof. The Legendre dual of Condition 2.1 says that $\partial\check{B} \subset \check{B}$ is itself an affine manifold with singularities to which our program applies. If this is the case then from the definition of $\check{\mathfrak{D}} \subset \check{\mathfrak{X}}$ it follows that $\check{\mathfrak{D}} \rightarrow T$ is indeed obtained by restricting the slab functions to $\check{D}_0 \subset \check{X}_0$ and run our program. The result is hence a toric degeneration of Calabi-Yau varieties.

Conversely, assume that $\rho, \rho' \subset \partial\check{B}$ are two neighboring $(n-1)$ -faces with $\Lambda_\rho \neq \Lambda_{\rho'}$ as subspaces of Λ_v for $v \in \rho \cap \rho'$. Then a study of the toric local model underlying the degeneration shows that the general fiber of $\mathfrak{D} \rightarrow T$ is not locally irreducible. Hence $\mathfrak{D} \rightarrow T$ can not be a toric degeneration. \square

Definition 2.3. A toric degeneration of Calabi-Yau pairs $(\pi : \mathfrak{X} \rightarrow T, \mathfrak{D})$ with $\mathfrak{D} \rightarrow T$ a toric degeneration of Calabi-Yau varieties is called *irreducible*.

As an example we consider the case of \mathbb{P}^2 .

Example 2.4. The standard method to construct the LG-mirror for \mathbb{P}^2 is to start from the momentum polytope $\Xi = \text{conv}\{(-1, -1), (2, -1), (-1, 2)\}$ of \mathbb{P}^2 with its anti-canonical polarization. The rays of the corresponding normal fan associated to this polytope (using inward pointing normal vectors as in [GS3]) are generated by $(-1, 0), (0, -1), (1, 1)$. Calling the monomials corresponding to the first two points x and y , respectively, we obtain the usual (non-proper) Landau-Ginzburg model on the big torus $(\mathbb{G}_m(\mathbb{k}))^2$ by the function $x + y + \frac{1}{xy}$.

To obtain a proper superpotential we need to achieve the dual of (2.1), that is, make the boundary of the momentum polytope flat in affine coordinates. To do this one has to trade the corners with singular points in the interior. The most simple choice is a decomposition $\check{\mathcal{P}}$ of $B = \Xi$ into three triangles with three singular points with simple monodromy, that is, conjugate to $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, as depicted in Figure 2.1. A minimal choice of the PL -function $\check{\varphi}$ takes values 0 at the origin and 1 on ∂B . For this choice of $\check{\varphi}$ the Legendre dual of $(\check{B}, \check{\mathcal{P}}, \check{\varphi})$ is shown in Figure 2.1 on the right. Note that the unbounded edges are indeed parallel, so each unbounded edge comes with copies of the other two unbounded edges parallel at integral distance 1.

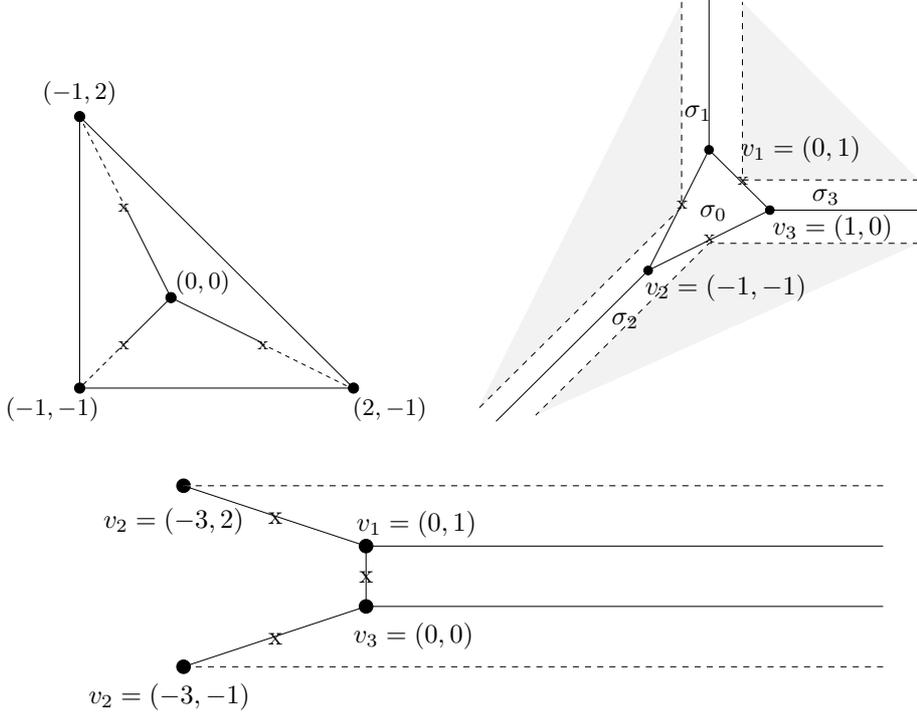


FIGURE 2.1. An intersection complex $(\check{B}, \check{\mathcal{P}})$ for \mathbb{P}^2 with straight boundary and its Legendre dual (B, φ) for the minimal polarization, with a chart on the complement of the shaded region and a chart showing the three parallel unbounded edges.

Now let us compute X_0 and $W_{\mathbb{P}^2}^0$. The polyhedral decomposition has one bounded maximal cell σ_0 and three unbounded maximal cells $\sigma_1, \sigma_2, \sigma_3$. The bounded cell is the momentum polytope of the weighted projective plane $\mathbb{P}(1, 2, 3) =: X_{\sigma_0}$. Each unbounded cell is affine isomorphic to $[0, 1] \times \mathbb{R}_{\geq 0}$, the momentum polytope of $\mathbb{P}^1 \times \mathbb{A}^1 =: X_{\sigma_i}$, $i = 1, 2, 3$. These glue together by torically identifying pairs of \mathbb{P}^1 's and \mathbb{A}^1 's as prescribed by the polyhedral decomposition to yield X_0 . Clearly $W_{\mathbb{P}^2}^0$ vanishes identically on the compact component X_{σ_0} . Each of the unbounded components has two parallel unbounded edges, leading to the pull-back to $\mathbb{P}^1 \times \mathbb{A}^1$ of the toric coordinate function of \mathbb{A}^1 , say z_i for the i -th copy. Thus $W_{\mathbb{P}^2}^0|_{X_{\sigma_i}} = z_i$ for $i = 1, 2, 3$. These functions are clearly compatible with the toric gluings.

Remark 2.5. An interesting feature of the degeneration point of view is that the mirror construction respects the finer data related to the degeneration such as the monodromy representation of the affine structure. In particular, this poses a question of uniqueness of the Landau-Ginzburg mirror. For the anti-canonical polarization such as the chosen one in the case of \mathbb{P}^2 , the tropical data $(\check{B}, \check{\mathcal{P}})$ is essentially unique, see Theorem 6.4 for

a precise statement. For larger polarizations (thus enlarging \check{B}) there are certainly many more possibilities. For example, as an affine manifold with singularities one can perturb the location of the singular points transversely to the invariant directions over the rational numbers and choose an adapted integral polyhedral decomposition after appropriate rescaling. It is not clear to us if all $(\check{B}, \check{\mathcal{P}})$ leading to \mathbb{P}^2 can be obtained by this procedure.

3. SCATTERING OF MONOMIALS

A central tool in [GS3] are scattering diagrams. The purpose of this section is to study the propagation of monomials through scattering diagrams. Assume \mathcal{S}_k is a structure that is consistent to order k and let j be a joint of \mathcal{S}_k . Let $\mathfrak{D} = (\mathfrak{r}_i, f_c)$ be the associated scattering diagram, for some vertex $v \in \sigma_j$ and $\omega \in \mathcal{P}$ with $\sigma_j \subset \omega$, as explained in Construction 3.4 of [GS3]. For an exponent m_0 with $\bar{m}_0 \in \Lambda_v \setminus \Lambda_j$ we wish to define the scattering of the monomial z^{m_0} , which we think of traveling along the ray $-\mathbb{R}_{\geq 0}\bar{m}_0$ into the origin of $\mathcal{Q}_{j,\mathbb{R}}^v \simeq \mathbb{R}^2$. In a scattering diagram monomials travel along *trajectories*. These are defined in exactly the same way as rays ([GS3], Definition 3.3), but will have an additive meaning.

Definition 3.1. A *trajectory* in $\mathcal{Q}_{j,\mathbb{R}}^v$ is a triple $(\mathfrak{t}, m_{\mathfrak{t}}, a_{\mathfrak{t}})$, where $m_{\mathfrak{t}}$ is a monomial on a maximal cell $\sigma \ni v$ with $\pm \bar{m}_{\mathfrak{t}} \in \bar{\sigma}$ and $m \in P_x$ for any $x \in j \setminus \Delta$, $\mathfrak{t} = \pm \mathbb{R}_{\geq 0}\bar{m}$, and $a_{\mathfrak{t}} \in \mathbb{k}$. The trajectory is called *incoming* if $\mathfrak{t} = \mathbb{R}_{\geq 0}\bar{m}$, and *outgoing* if $\mathfrak{t} = -\mathbb{R}_{\geq 0}\bar{m}$. By abuse of notation we often suppress $m_{\mathfrak{t}}$ and $a_{\mathfrak{t}}$ when referring to trajectories.

Here is the generalization of the central existence and uniqueness result for scattering diagrams ([GS3], Proposition 3.9) incorporating trajectories.

Proposition 3.2. *Let \mathfrak{D} be the scattering diagram defined by \mathcal{S}_k for $j \in \text{Joints}(\mathcal{S}_k)$, $g : \omega \rightarrow \sigma_j$ and $v \in \omega$. Let $(\mathbb{R}_{\geq 0}\bar{m}_0, m_0, 1)$ be an incoming trajectory and $\sigma \supset j$ a maximal cell with $\bar{m}_0 \in \bar{\sigma}$. For $\bar{m} \in \mathcal{Q}_{j,\mathbb{R}}^v \setminus \{0\}$ denote by*

$$\theta_{\bar{m}} : R_{g,\sigma'}^k \rightarrow R_{g,\sigma}^k$$

the ring isomorphism defined by \mathfrak{D} for a path connecting $-\bar{m}$ to $-\bar{m}_0$, where σ' is a maximal cell with $-\bar{m} \in \bar{\sigma}'$.

Then there is a set of outgoing trajectories \mathfrak{T} such that

$$(3.1) \quad z^{m_0} = \sum_{\mathfrak{t} \in \mathfrak{T}} \theta_{\bar{m}_{\mathfrak{t}}} (a_{\mathfrak{t}} z^{m_{\mathfrak{t}}})$$

holds in $R_{g,\sigma}^k$. Moreover, \mathfrak{T} is unique if $a_{\mathfrak{t}} \neq 0$ for all $\mathfrak{t} \in \mathfrak{T}$ and if $m_{\mathfrak{t}} \neq m_{\mathfrak{t}'}$ whenever $\mathfrak{t} \neq \mathfrak{t}'$.

Proof. The proof is by induction on $l \leq k$. We first discuss the case that σ_j is a maximal cell, that is, $\text{codim } \sigma_j = 0$. Then \mathfrak{D} has only rays, no cuts. In particular, any $\theta_{\bar{m}}$ is an automorphism of $R_{g,\sigma}^k$ that is the identity modulo $I_{g,\sigma}^{>0}$. Thus for $l = 0$, (3.1) forces one outgoing trajectory $(-\mathbb{R}_{\geq 0}\bar{m}_0, m_0, 1)$ if $\text{ord}_{\sigma_j}(m_0) = 0$ or none otherwise. For the

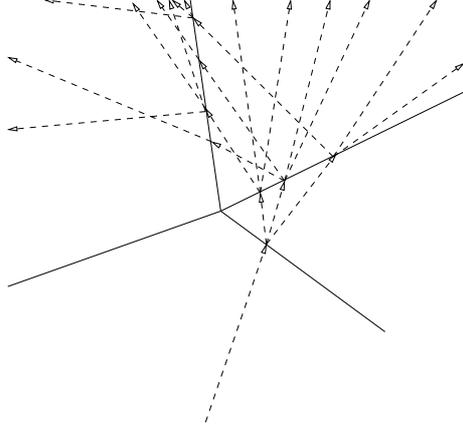


FIGURE 3.1. Scattering diagram with perturbed trajectories (cuts and rays solid, perturbed trajectories dashed).

induction step assume (3.1) holds in $R_{g,\sigma}^{l-1}$. Then in $R_{g,\sigma}^l$ the difference of the two sides of (3.1) is a sum of monomials az^m with $\text{ord}_{\sigma_j}(m) = l$. Since $l > 0$ and since there are no cuts (these represent slabs containing j), it holds $\theta_{\overline{m}}(az^m) = az^m$. Thus after adding appropriate trajectories $(-\mathbb{R}_{\geq 0}\overline{m}, m, a)$ with $\text{ord}_{\sigma_j}(m) = l$ to \mathfrak{T} , Equation (3.1) holds in $R_{g,\sigma}^l$. This is the unique minimal choice of \mathfrak{T} .

Under the presence of cuts we have several rings $R_{g,\sigma'}^k$ for various maximal cells $\sigma' \supset j$. This possibly brings in denominators that are powers of $f_{\rho,v}$ for cells $\rho \supset j$ of codimension one. In this case we show existence by a perturbation argument. To this end consider first the most simple scattering diagram in codimension one consisting of only two cuts $\mathbf{c}_{\pm} = (\pm \mathbf{c}, f_{\rho,v})$ dividing \mathcal{Q} into two halfplanes $\overline{\sigma}_{\pm}$ and with the same attached function. The signs are chosen in such a way that $\overline{m}_0 \in \overline{\sigma}_-$. Let $\theta : R_{g,\sigma_-}^k \rightarrow R_{g,\sigma_+}^k$ be the isomorphism defined by a path from $\overline{\sigma}_-$ to $\overline{\sigma}_+$ and let $n \in \Lambda_{\rho}^{\perp} \subset \Lambda_v^*$ be the primitive integral vector that is positive on σ_- . Then $\langle m_0, n \rangle \geq 0$ and

$$\theta(z^{m_0}) = f_{\rho,v}^{\langle m_0, n \rangle} \cdot z^{m_0}.$$

Expanding yields the finite sum

$$\theta(z^{m_0}) = \sum_{\langle m, n \rangle \geq 0} a_m z^m = \theta \left(\sum_{\langle m, n \rangle \geq 0} a_m \theta^{-1}(z^m) \right) = \theta \left(\sum_{\langle m, n \rangle \geq 0} \theta_{\overline{m}}(a^m z^m) \right),$$

for some $a_m \in \mathbb{k}$. This equals θ applied to the right-hand side of (3.1) for the set of trajectories

$$\mathfrak{T} := \{(-\mathbb{R}_{\geq 0}\overline{m}, m, a_m) \mid \langle m, n \rangle \geq 0\}.$$

Hence existence is clear in this case. In the general case we work with perturbed trajectories as suggested by Figure 3.1. More precisely, a perturbed trajectory is a trajectory with the origin shifted. There will be one unbounded perturbed incoming trajectory, a translation of $(-\mathbb{R}_{\geq 0}\overline{m}_0, m_0, a)$, and a number of perturbed outgoing trajectories, each the result of scattering of other trajectories with rays or cuts. At each intersection point of

a trajectory with a ray or cut, the incoming and outgoing trajectories at this point fulfill an equation analogous to (3.1). Similar to Construction 4.5 in [GS3] with our additive trajectories replacing the multiplicative s -rays², there is then an asymptotic scattering diagram with trajectories obtained by taking the limit $\lambda \rightarrow 0$ of rescaling the whole diagram by $\lambda \in \mathbb{R}_{>0}$. Any choice of perturbed incoming trajectory determines a unique minimal scattering diagram with perturbed trajectories. Moreover, for a generic choice of perturbed incoming trajectory the intersection points of trajectories with rays or cuts are pairwise disjoint, and they are in particular different from the origin. Hence the perturbed diagram can be constructed uniquely by induction on $l \leq k$. Taking the associated asymptotic scattering diagram with trajectories now establishes the existence in the general case.

Next we show uniqueness, first for $\text{codim } \sigma_j = 2$. In this case, any monomial z^m in $\mathbb{k}[\Lambda_{\sigma_j}]$ fulfills $\text{ord}_{\sigma_j}(m) = 0$, and all $\theta_{\bar{m}}$ extend to $\mathbb{k}(\Lambda_{\sigma_j})$ -algebra automorphisms of $R_{g,\sigma}^k \otimes_{\mathbb{k}[\Lambda_{\sigma_j}]} \mathbb{k}(\Lambda_{\sigma_j})$. Hence we can deduce uniqueness as in codimension 0 by taking the factors a of trajectories to be polynomials with coefficients in $\mathbb{k}(\Lambda_{\sigma_j})$. Thus we combine all trajectories \mathfrak{t} with the same $\bar{m}_{\mathfrak{t}}$ and the same $\text{ord}_{\sigma_j}(m_{\mathfrak{t}})$. It is clear that such generalized trajectories can be split uniquely into proper trajectories with all m distinct, showing uniqueness in this case.

Finally, for uniqueness in codimension one we can not argue just with ord_{σ_j} because there are monomials z^m with $\text{ord}_{\sigma_j}(m) = 0$ but $\bar{m} \neq 0$. Instead we look closer at the effect of adding trajectories. By induction it suffices to study the insertion of trajectories $(-\mathbb{R}_{\geq 0}\bar{m}, m, a)$ with $\text{ord}_{\sigma_j}(m) = l$ for each m and such that (3.1) continues to hold. Then

$$0 = \sum_m \theta_{\bar{m}}^{-1}(a_m z^m) = \sum_{i=0}^l \sum_{\langle m,n \rangle=i} \theta_{\bar{m}}^{-1}(a_m z^m) = \sum_{i=0}^l f_{\rho,v}^{-i} \sum_{\langle m,n \rangle=i} a_m z^m.$$

Since all monomials in $f_{\rho,v}$ have vanishing ord_{σ_j} and only monomials z^m with the same value of $\langle m,n \rangle$ can cancel, this equation implies

$$f_{\rho,v}^{-i} \sum_{\langle m,n \rangle=i} a_m z^m = 0.$$

Multiplying by $f_{\rho,v}^i$ thus shows $\sum_{\langle m,n \rangle=i} a_m z^m = 0$ in $R_{g,\sigma}^l$, and hence $a_m = 0$ for all m . This proves uniqueness also in codimension one. \square

4. THE SUPERPOTENTIAL VIA BROKEN LINES

The easiest way to define the superpotential in full generality is by the method of broken lines. Broken lines have been introduced by Mark Gross for $\dim B = 2$ in his work on mirror symmetry for \mathbb{P}^2 [Gr2]. We assume we are given a locally finite scattering diagram \mathcal{S}_k for a polarized LG-model $(B, \mathcal{P}, \varphi)$ that is consistent to order k . The notion of broken

²For technical reasons s -rays were not asked to be piecewise affine. In the present situation it is enough to restrict to piecewise affine objects.

lines is based on the transport of monomials by changing chambers of \mathcal{S}_k . Recall from [GS3], Definition 2.22, that a chamber is the closure of a connected component of $B \setminus |\mathcal{S}_k|$.

Definition 4.1. Let \mathbf{u}, \mathbf{u}' be neighbouring chambers of \mathcal{S}_k , that is, $\dim(\mathbf{u} \cap \mathbf{u}') = n - 1$. Let az^m be a monomial defined at all points of $\mathbf{u} \cap \mathbf{u}'$ and assume without loss of generality that \bar{m} points from \mathbf{u}' to \mathbf{u} . Let $\tau := \sigma_{\mathbf{u}} \cap \sigma_{\mathbf{u}'}$ and

$$\theta : R_{\text{id}\tau, \sigma_{\mathbf{u}}}^k \rightarrow R_{\text{id}\tau, \sigma_{\mathbf{u}'}}^k$$

be the gluing isomorphism changing chambers. Then if

$$(4.1) \quad \theta(az^m) = \sum_i a_i z^{m_i}$$

we call any summand $a_i z^{m_i}$ with $\text{ord}_{\sigma_{\mathbf{u}'}}(m_i) \leq k$ a *result of transport of az^m from \mathbf{u} to \mathbf{u}'* .

Note that since the change of chamber isomorphisms commute with changing strata, the monomials $a_i z^{m_i}$ in Definition 4.1 are defined at all points of $\mathbf{u} \cap \mathbf{u}'$.

Definition 4.2. (Cf. [Gr2], Definition 4.9.) A *broken line* for \mathcal{S}_k is a proper continuous map

$$\beta : (-\infty, 0] \rightarrow B$$

with image disjoint from any joints of \mathcal{S}_k , along with a sequence $-\infty = t_0 < t_1 < \dots < t_{r-1} \leq t_r = 0$ for some $r \geq 0$ with $\beta(t_i) \in |\mathcal{S}_k|$, and for $i = 1, \dots, r$ monomials $a_i z^{m_i}$ defined at all points of $\beta([t_{i-1}, t_i])$ (for $i = 1, \beta((-\infty, t_1])$), subject to the following conditions.

- (1) $\beta|_{(t_{i-1}, t_i)}$ is a non-constant affine map with image disjoint from $|\mathcal{S}_k|$, hence contained in the interior of a unique chamber \mathbf{u}_i of \mathcal{S}_k , and $\beta'(t) = -\bar{m}_i$ for all $t \in (t_{i-1}, t_i)$. Moreover, if $t_r = t_{r-1}$ then $\mathbf{u}_r \neq \mathbf{u}_{r-1}$.
- (2) $a_1 = 1$ and there exists a (necessarily unbounded) $\omega \in \mathcal{P}^{[1]}$ with $\bar{m}_1 \in \Lambda_\omega$ primitive and $\text{ord}_\omega(m_1) = 0$.
- (3) For each $i = 1, \dots, r - 1$ the monomial $a_{i+1} z^{m_{i+1}}$ is a result of transport of $a_i z^{m_i}$ from \mathbf{u}_i to \mathbf{u}_{i+1} (Definition 4.1).

The *type* of β is the tuple of all \mathbf{u}_i and m_i . By abuse of notation we suppress the data t_i, a_i, m_i when talking about broken lines, but introduce the notation

$$a_\beta := a_r, \quad m_\beta := m_r.$$

For $p \in B$ the set of broken lines β with $\beta(0) = p$ is denoted $\mathfrak{B}(p)$.

Remark 4.3. 1) If all unbounded edges are parallel (2.1) then the condition $\bar{m}_1 \in \Lambda_\omega$ in (2) follows from (1).

2) A broken line β is determined uniquely by specifying its endpoint $\beta(0)$ and its type. In fact, the coefficients a_i are determined inductively from $a_1 = 1$ by Equation (4.1).

According to Remark 4.3,(2) the map $\beta \mapsto \beta(0)$ identifies the space of broken lines of a fixed type with a subset of \mathbf{u}_r . This subset is the interior of a polyhedron:

Proposition 4.4. *For each type (\mathbf{u}_i, m_i) of broken lines there is an integral, closed, convex polyhedron Ξ , of dimension n if non-empty, and an affine immersion*

$$\Phi : \Xi \longrightarrow \mathbf{u}_r,$$

so that $\Phi(\text{Int } \Xi)$ is the set of endpoints $\beta(0)$ of broken lines β of the given type.

Proof. This is an exercise in polyhedral geometry left to the reader. For the statement on dimensions it is important that broken lines are disjoint from joints. \square

Remark 4.5. A point $p \in \Phi(\partial\Xi)$ still has a meaning as an endpoint of a piecewise affine map $\beta : (-\infty, 0] \rightarrow B$ together with data t_i and $a_i z^{m_i}$, defining a *degenerate broken line*. For this not to be a broken line $\text{im}(\beta)$ has to intersect a joint. By convexity of the chambers this comprises the case that there exists $t \in (-\infty, 0] \setminus \{t_0, \dots, t_r\}$ with $\beta(t) \in |\mathcal{S}_k|$, or even that β maps a whole interval to $|\mathcal{S}_k|$. Note also the possibility that $t_{i-1} = t_i$ for some $i \in \{2, \dots, r-1\}$, but then $\beta(t_{i-1}) = \beta(t_i)$ is contained in a joint. All other conditions in the definition of broken lines are closed.

The set of endpoints $\beta(0)$ of degenerate broken lines of a given type is the $(n-1)$ -dimensional polyhedral subset $\Phi(\partial\Xi) \subset \mathbf{u}$. The set of degenerate broken lines *not transverse* to each joint of \mathcal{S}_k is polyhedral of smaller dimension.

Any finite structure \mathcal{S}_k involves only finitely many slabs and walls, and each polynomial coming with each slab or wall carries only finitely many monomials. Hence broken lines for $|\mathcal{S}_k|$ exist only for finitely many types. The following definition is therefore meaningful.

Definition 4.6. A point $p \in B$ is called *general* (for the given structure \mathcal{S}_k) if it is not contained in $\Phi(\partial\Xi)$, for any Φ as in Proposition 4.4.

Recall from [GS3], §2.6 that \mathcal{S}_k defines a k -th order deformation of X_0 by gluing the sheaf of rings defined by $R_{g, \sigma_{\mathbf{u}}}^k$, with $g : \omega \rightarrow \tau$ and \mathbf{u} a chamber of \mathcal{S}_k with $\omega \cap \mathbf{u} \neq \emptyset$, $\tau \subset \sigma_{\mathbf{u}}$. Given a general $p \in \mathbf{u}$ we can now define the *superpotential* up to order k locally as an element of $R_{g, \sigma_{\mathbf{u}}}^k$ by

$$(4.2) \quad W_{g, \mathbf{u}}^k(p) := \sum_{\beta \in \mathfrak{B}(p)} a_{\beta} z^{m_{\beta}}.$$

The existence of a canonical extension W^k of W^0 to X_k follows once we check that (i) $W_{g, \mathbf{u}}^k(p)$ is independent of the choice of a general $p \in \mathbf{u}$ and (ii) the $W_{g, \mathbf{u}}^k(p)$ are compatible with changing strata or chambers ([GS3], Construction 2.24). This is the content of the following two lemmas.

Lemma 4.7. *Let \mathbf{u} be a chamber of \mathcal{S}_k and $g : \omega \rightarrow \tau$ with $\omega \cap \mathbf{u} \neq \emptyset$, $\tau \subset \sigma_{\mathbf{u}}$. Then $W_{g, \mathbf{u}}^k(p)$ is independent of the choice of $p \in \mathbf{u}$.*

Proof. By Proposition 4.4 the set $A = \Phi(\partial\Xi) \subset \mathbf{u}$ of non-general points is a finite union of nowhere dense polyhedra. Moreover, since all Φ in this Proposition are local affine isomorphisms, for each path $\gamma : [0, 1] \rightarrow \mathbf{u} \setminus A$ and broken line β_0 with $\beta_0(0) = \gamma(0)$ there

exists a unique family β_s of broken lines with endpoints $\beta_s(0) = \gamma(s)$ and with the same type as β_0 . Hence $W_{g,u}^k(p)$ is locally constant on $\mathfrak{u} \setminus A$.

To pass between the different connected components of $\mathfrak{u} \setminus A$ consider the set $A' \subset A$ of endpoints of degenerate broken lines that are not transverse to the joints of \mathcal{S}_k . More precisely, for each type of broken line, the set of endpoints intersecting a given joint defines a polyhedral subset of \mathfrak{u} of dimension at most $n - 1$. Then A' is the union of $n - 2$ -cells of these polyhedral subsets, for any joint and any type of broken lines. Since $\dim A' = n - 2$ we conclude that $\mathfrak{u} \setminus A'$ is path-connected. It thus suffices to study the following situation. Let $\gamma : [-1, 1] \rightarrow \text{Int } \mathfrak{u} \setminus A'$ be an affine map with $\gamma(0)$ the only point of intersection with A . Let $\bar{\beta}_0 : (-\infty, 0] \rightarrow B$ be the underlying map of a degenerate broken line with endpoint $\gamma(0)$. The point is that $\bar{\beta}_0$ may arise as a limit of several different types of broken lines with endpoints $\gamma(s)$ for $s \neq 0$. The lemma follows once we show that the contributions to $W_{g,u}^k(\gamma(s))$ of such broken lines for $s < 0$ and for $s > 0$ coincide. Note we do not claim a bijection between the sets of broken lines for $s < 0$ and for $s > 0$, which in fact needs not be true.

Since $\gamma^{-1}(A) = \{0\}$ any broken line β with endpoint $\gamma(s_0)$ for $s_0 \neq 0$ extends uniquely to a family of broken lines β_s for $s \in [-1, 0)$ or $s \in (0, 1]$. In particular, β has a unique limit $\lim \beta := \lim_{s \rightarrow 0} \beta_s$, a possibly degenerate broken line. For $s \neq 0$ denote by \mathfrak{B}_s the space of broken lines β with endpoint $\gamma(s)$ and such that the map underlying $\lim \beta$ equals $\bar{\beta}_0$. Since $\bar{\beta}_0$ is the underlying map of a degenerate broken line, $\mathfrak{B}_s \neq \emptyset$ for some sufficiently small s , hence also for all s of the same sign, by unique continuation. Possibly by changing signs in the domain of γ we may thus assume $\mathfrak{B}_s \neq \emptyset$ for $s < 0$. We have to show

$$(4.3) \quad \sum_{\beta \in \mathfrak{B}_{-1}} a_\beta z^{m_\beta} = \sum_{\beta \in \mathfrak{B}_1} a_\beta z^{m_\beta}.$$

The central observation is the following. Let $J \subset B$ be the union of the joints of \mathcal{S}_k intersected by $\text{im } \bar{\beta}_0$. Let $x := \bar{\beta}_0(t)$ for $t \ll 0$ be a point far off to $-\infty$. Thus x lies in one of the unbounded cells of \mathcal{P} and $\bar{\beta}$ is asymptotically parallel to an unbounded edge. Let $U \subset B$ be a local affine hyperplane intersecting $\bar{\beta}_0$ transversely at x . Then by transversality with J there is a local affine hyperplane $U \subset B$ containing x such that the images of the degenerate broken lines of types contained in any \mathfrak{B}_s lie in an affine hyperplane $H \subset U$ ($\dim H = n - 2$). Moreover, locally around $\text{im } \bar{\beta}_0$ the images of degenerate broken lines of the considered types separate B into two connected components. It follows that the broken lines in \mathfrak{B}_s for $s < 0$ intersect U only on one side of H , and for $s > 0$ only on the other. Choose one family of broken lines $\beta_s^0 \in \mathfrak{B}_s$, say for $s < 0$, of fixed type, and denote $x_s \in U$ the point of intersection of $\text{im}(\beta_s^0)$ with U . Clearly, $\lim_{s \rightarrow 0} x_s = x$. Let $V \subset \mathfrak{u}$ be a local affine hyperplane transverse to $\bar{\beta}_0$ and containing $\text{im } \gamma$. Then if U is chosen sufficiently small, for each $\beta_s \in \mathfrak{B}_s$, $s < 0$, there is a unique β'_s of the same type as β_s with $\text{im}(\beta'_s) \cap U = \{x_s\}$ and with $\beta'_s(0) \in V$. Said differently, β'_s is the unique deformation of

β_s with the same asymptotic as β_s^0 and with endpoint on V . Now the β'_s are all broken lines of the considered types intersecting U in the same point x_s .

The full set of β'_s can be constructed as follows. Start with the broken line ending at x_s and of type (\mathbf{u}_1, m_1) . This broken line can be continued until it hits a wall or slab, where it splits into several broken lines, one for each summand in (4.1). Iterating this process leads to the infinite set of all broken lines with asymptotic given by m_1 and running through x_s . The β'_s are the subset of the considered types, that is, with the unique deformation for $s \rightarrow 0$ having underlying map $\bar{\beta}_0$ and endpoint on V .

From this point of view it is clear that at each joint j intersected by $\text{im } \bar{\beta}_0$ the β'_s compute a scattering of monomials as considered in §3. In fact, the union of the β'_s with the same incoming part az^m near j induce a scattering diagram with perturbed trajectories as considered in the proof of Proposition 3.2. Thus the corresponding sum of monomials leaving a neighbourhood of j can be read off from the right-hand side of (3.1) in this proposition, applied to the incoming trajectory $(\mathbb{R}_{\geq 0} \bar{m}, m, a)$.

We conclude that $\sum_{\beta \in \mathfrak{B}_s} a_\beta z^{m_\beta}$ for $s < 0$ computes the transport of z^{m_1} along $\bar{\beta}_0$. This transport is defined by applying (3.1) instead of (4.1) at each joint intersected by $\bar{\beta}_0$. The same argument holds for $s > 0$, thus proving (4.3). \square

Remark 4.8. The proof of Lemma 4.7 really shows that the scattering of monomials introduced in §3 allows to replace the condition that broken lines have image disjoint from joints by transversality with joints. In the following we refer to these as *generalized broken lines*.

By Lemma 4.7 we are now entitled to define

$$W_{g,\mathbf{u}}^k := W_{g,\mathbf{u}}^k(p)$$

for any general choice of $p \in \text{Int } \mathbf{u}$.

Lemma 4.9. *The $W_{g,\mathbf{u}}^k$ are compatible with changing strata and changing chambers.*

Proof. Compatibility with changing strata follows trivially from the definitions. As for changing from a chamber \mathbf{u} to a neighbouring chamber \mathbf{u}' ($\dim \mathbf{u} \cap \mathbf{u}' = n-1$) the argument is similar to the proof Lemma 4.7. Let $g : \omega \rightarrow \tau$ be such that $\omega \cap \mathbf{u} \cap \mathbf{u}' \neq \emptyset$, $\tau \subset \sigma_{\mathbf{u}} \cap \sigma_{\mathbf{u}'}$ and

$$\theta : R_{g,\mathbf{u}}^k \longrightarrow R_{g,\mathbf{u}'}^k$$

be the corresponding change of chamber isomorphism. We have to show $\theta(W_{g,\mathbf{u}}^k) = W_{g,\mathbf{u}'}^k$.

Let $A \subset \mathbf{u} \cup \mathbf{u}'$ be the endpoints of degenerate broken lines. Consider a path $\gamma : [-1, 1] \rightarrow \mathbf{u} \cap \mathbf{u}'$ connecting general points $\gamma(-1) \in \mathbf{u}$, $\gamma(1) \in \mathbf{u}'$ and with $\gamma^{-1}(\mathbf{u} \cap \mathbf{u}') = \{0\}$. We may also assume that $\gamma(s) \in A$ at most for $s = 0$, and that any degenerate broken line with endpoint $\gamma(0)$ is transverse to joints. For $s \neq 0$ we then consider the space \mathfrak{B}_s of broken lines β_s with endpoint $\gamma(s)$ and with deformation for $s \rightarrow 0$ a fixed underlying map of a degenerate broken line $\bar{\beta}_0$. By transversality of $\bar{\beta}_0$ with the set of joints the

limits of families β_s , $s \rightarrow 0$, group into generalized broken lines (Remark 4.8). Each such generalized broken line β has as endpoint $p_0 := \gamma(0)$, but viewed as an element either of \mathbf{u} or of \mathbf{u}' . We call this chamber the *reference chamber* of β . Generalized broken lines with reference chambers \mathbf{u} and \mathbf{u}' contribute to $W_{g,\mathbf{u}}^k(p_0) = W_{g,\mathbf{u}}^k$ and $W_{g,\mathbf{u}'}^k(p_0) = W_{g,\mathbf{u}'}^k$, respectively. Moreover, m_β is either tangent to $\mathbf{u} \cap \mathbf{u}'$ or points properly into \mathbf{u} or into \mathbf{u}' . We claim that θ maps each of the three types of contributions to $W_{g,\mathbf{u}}^k$ to the three types of contributions to $W_{g,\mathbf{u}'}^k$. Then $\theta(W_{g,\mathbf{u}}^k) = W_{g,\mathbf{u}'}^k$ and the proof is finished.

Let us first consider the set of degenerate broken lines β with m_β tangent to $\mathbf{u} \cap \mathbf{u}'$. Then changing the reference chamber from \mathbf{u} to \mathbf{u}' defines a bijection between the considered generalized broken lines with endpoint p_0 and reference cell \mathbf{u} and those with reference cell \mathbf{u}' . Note that in this case β has to intersect a joint, so this statement already involves the arguments from the proof of Proposition 4.7. Because $\theta(a_\beta z^{m_\beta}) = a_\beta z^{m_\beta}$ this proves the claim in this case.

Next assume m_β points from $\mathbf{u} \cap \mathbf{u}'$ into the interior of \mathbf{u} . This means that β approaches p_0 from the interior of \mathbf{u} . If we want to change the reference chamber to \mathbf{u}' we need to introduce one more point $t_{r+1} := t_r = 0$ and chamber $\mathbf{u}_{r+1} := \mathbf{u}'$. According to Equation (4.1) in Definition 4.1 the possible monomials $a_{r+1} z^{m_{r+1}}$ are given by the summands in $\theta(az^m) = \sum_i a_{r+1,i} z^{m_{r+1,i}}$. Thus for each summand we obtain one generalized broken line with reference cell \mathbf{u}' . Clearly, this is exactly what is needed for compatibility with θ of the respective contributions to the local superpotentials.

By symmetry the same argument works for generalized broken lines β with reference cell \mathbf{u}' and m_β pointing from $\mathbf{u} \cap \mathbf{u}'$ into the interior of \mathbf{u}' , and θ^{-1} replacing θ . Inverting θ means that a number of generalized broken lines with reference cell $\mathbf{u}_{r+1} = \mathbf{u}$ and two points $t_{r+1} = t_r$ (and necessarily $\mathbf{u}_r = \mathbf{u}'$), one for each summand of $\theta^{-1}(a_\beta z^{m_\beta})$, combine into a single generalized broken line with reference cell $\mathbf{u}_r = \mathbf{u}'$. This process is again compatible with applying θ to the respective contributions to the local superpotentials. This finishes the proof of the claim, which was left to complete the proof of the lemma. \square

5. BROKEN LINES VIA TROPICAL DISKS

We now aim for an alternative construction of the potential W in terms of tropical disks.

5.1. Tropical disks. Our definition of tropical disks depends only on the integral affine geometry of B and not on its polyhedral decomposition \mathcal{P} . As usual let $i : \Delta \rightarrow B$ denote the inclusion of the singular locus of the integral affine structure and let Λ_B be the sheaf of integral tangent vectors. Assume B is non-compact and on the complement U of some orientable compact subset of B , $\Gamma(U, i_* \Lambda)$ has rank one. Then there exists a unique primitive vector field in $\Gamma(U, i_* \Lambda)$ pointing away from U . We assume the semiflow of this vector field is complete and call its orbits the *asymptotic rays*. This is the situation met in Proposition 2.1.

Definition 5.1. Let Γ be a tree with root vertex V_{root} . Denote by $\Gamma^{[1]}, \Gamma^{[0]}, \Gamma_{\text{leaf}}^{[0]}$ the set of edges, vertices, and leaf vertices (univalent vertices different from the root vertex), respectively. We allow unbounded edges, that is, edges adjacent to only one vertex, defining a subset $\Gamma_{\infty}^{[1]} \subset \Gamma^{[1]}$. Let $w : \Gamma^{[1]} \rightarrow \mathbb{N} \setminus \{0\}$ be a weight function.

Let $x \in B \setminus \Delta$. A *tropical disk bounded by x* is a proper, locally injective, continuous map

$$h : (|\Gamma|, \{V_{\text{root}}\}) \rightarrow (B, \{x\})$$

with the following properties.

- (1) $h^{-1}(\Delta) = \Gamma_{\text{leaf}}^{[0]}$.
- (2) For every edge $E \in \Gamma^{[1]}$ the image $h(E \setminus \partial E)$ is a locally closed integral affine submanifold of $B \setminus \Delta$ of dimension one.
- (3) If $V \in \Gamma^{[0]}$ there is a primitive integral vector $m \in i_* \Lambda_{B, h(V)}$ extending to a local vector field tangent to $h(E)$ and pointing away from $h(V)$. Define the *tangent vector of h at V along E* as $\bar{m}_{V, E} := w(E) \cdot m$.
- (4) For every $V \in \Gamma^{[0]} \setminus \Gamma_{\text{leaf}}^{[0]}$ the following *balancing condition* holds:

$$\sum_{\{E \in \Gamma^{[1]} \mid V \in E\}} \bar{m}_{V, E} = 0.$$

- (5) The image of an unbounded edge is an asymptotic ray.

Two disks $h : |\Gamma| \rightarrow B$, $h' : |\Gamma'| \rightarrow B$ are identified if $h = h' \circ \phi$ for a homeomorphism $\phi : |\Gamma| \rightarrow |\Gamma'|$ respecting the weights.

The *Maslov index* of h is defined as $\mu(h) := 2 \sum_{E \in \Gamma_{\infty}^{[1]}} w(E)$. □

Note that for a tropical disk $h^*(i_* \Lambda_B)$ is a trivial local system. In particular, there is a unique parallel transport of tangent vectors along h .

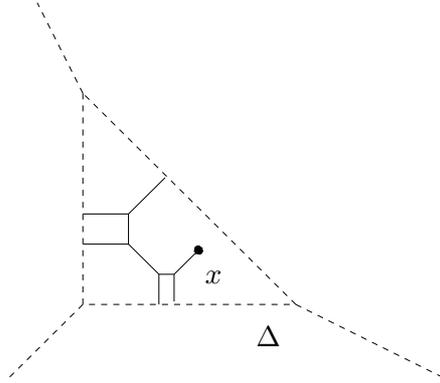


FIGURE 5.1. A tropical Maslov index zero disk bounding x belonging to a moduli space of dimension 5. The dashed lines indicate a part of the discriminant locus.

Example 5.2. Suppose $\dim B = 3$ and Δ bounds an affine two simplex σ with $T_x\sigma$ contained in the image of $i_*\Lambda_{B,x}$ for all $x \in \Delta$. Such a situation occurs in toric degenerations of local Calabi-Yau threefolds, for example the total space of $K_{\mathbb{P}^2}$. Then any point $x \in \sigma \setminus \Delta$ bounds a family of tropical Maslov index zero disks of arbitrary dimension, as illustrated in Figure 5.1.

So far, our definition of tropical disks only depends on $|\Gamma|$ and not on its underlying graph Γ . A distinguished choice of Γ is by assuming that there are no divalent vertices. At an interior vertex $V \in \Gamma^{[0]}$ (that is, neither the root vertex nor a leaf vertex) the rays $\mathbb{R}_{\geq 0} \cdot \bar{m}_{E,V}$ of adjacent edges E define a fan $\Sigma_{h,V}$ in $\Lambda_{B,hV} \otimes_{\mathbb{Z}} \mathbb{R}$. Denote by $\Sigma_{h,V}^0$ the parallel transport along h of $\Sigma_{h,V}$ to V_{root} . The *type* of h consists of the weighted graph (Γ, w) along with the $\Sigma_{h,V}^0$, $V \in \Gamma^{[0]} \setminus \Gamma_{\infty}^{[0]}$. For $x \in B \setminus \Delta$ and $\bar{m} \in \Lambda_{B,x}$ denote by $\mathcal{M}_{\mu}(\bar{m})$ the moduli space of tropical disks of Maslov index μ and root tangent vector \bar{m} . It comes with a natural stratification by type: A stratum consists of disks of fixed type, and the boundary of a stratum is reached when the image of an interior edge contracts to a vertex of higher valency.

From now on assume B is equipped with a compatible polyhedral structure \mathcal{P} as defined in [GS1], §1.3. It is then natural to adapt Γ to \mathcal{P} by appending Definition 5.1 as follows:

- (5) For any $E \in \Gamma^{[1]}$ there exists $\tau \in \mathcal{P}$ with $h(\text{Int } E) \subset \text{Int}(\tau)$, and if $V \in E$ is a divalent vertex then $h(V) \subset \partial\tau$.

In other words, we insert divalent vertices precisely at those points of $|\Gamma|$ where h changes cells of \mathcal{P} locally. Note however that we still consider the stratification on $\mathcal{M}_{\mu}(\bar{m})$ defined with all divalent vertices removed.

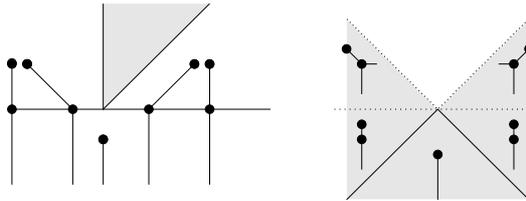


FIGURE 5.2. Disks near Δ (left) and their moduli cell complex (right).

Example 5.3. As it stands the type does not define a good stratification of the moduli space of tropical disks. For each vertex $V \in \Gamma$ mapping to a codimension one cell $\rho \in \mathcal{P}$ we also need to specify the connected component of $\rho \setminus \Delta$ containing $h(V)$ (that is, specify a reference vertices $v \in \rho$). This is illustrated in Figure 5.2. Here the dotted lines in the right picture correspond to generalized tropical disks, fulfilling all but (1) in Definition 5.1.

Tropical disks are closely related to broken lines as follows. We place ourselves in the context of §4. In particular, we assume given a structure \mathcal{S}_k that is consistent to order k .

Lemma 5.4 (Disk completion). *As a map, any broken line is the restriction of a Maslov index two disk $h : |\Gamma| \rightarrow B$ to the smallest connected subset of $|\Gamma|$ containing the root vertex and the (unique) unbounded leaf. The restriction of h to the closure of the complement of this subset consists of Maslov index zero disks,*

Proof. We continue to use the terminology of [GS3]. First, verify that any projected exponent \bar{m} attached to a point p of a wall or slab in \mathcal{S}_k is the root tangent vector of a Maslov index zero disk h rooted in $h(V_{\text{root}}) = p$. Clearly, this is true for \mathcal{S}_0 . Note that by simplicity the exponents of a slab function $f_{\rho,v}$ are root tangent vectors of Maslov index zero disks with only one edge. Assume inductively this holds as well for \mathcal{S}_l , $0 \leq l \leq k$, and show the claim for walls in $\mathcal{S}_{l+1} \setminus \mathcal{S}_l$ arising from scattering. We must show that the exponents of the outgoing rays are generated by those of the incoming rays or cuts. But if there existed an additional exponent, it would be preserved by any product with log automorphisms attached to the rays or cuts, as up to higher orders the latter are multiplications by polynomials with non trivial constant term. This contradicts consistency.

In particular, if $p = \beta(t_i)$ is a break point of a broken line β then t_i can be turned into a balanced trivalent vertex by attaching a Maslov index zero disk h with root tangent vector \bar{m} equal to the projected relative exponent taken from the unique wall or slab containing p . \square

We call any tropical disk as in the lemma a *disk completion* of the broken line. The disk completion is in general not unique due to the following reasons:

- (1) First, Example 5.2 shows that tropical Maslov index zero disks may come in families of arbitrarily high dimension.
- (2) Even if the moduli space of tropical Maslov index zero disks is of expected finite dimension, there may be joints with different incoming root tangent vectors.
- (3) Still, there may exist several Maslov index zero disks with the same root tangent vector, for example a closed geodesic of different winding numbers.

We now take care of these issues.

5.2. Virtual tropical disks. Example 5.2 illustrates that for $\dim B \geq 3$, tropical disks whose image are contained in a union of slabs lead to an unbounded dimension of the moduli space of tropical Maslov index 2μ disks. In order to get enumerative invariants which recover broken lines we need a virtual count of tropical disks. Throughout we assume B is oriented.

Suppose Δ is straightened as in [GS1], Remark 1.49, that is, Δ hence defines a finite subcomplex Δ^\bullet of the barycentric refinement of a polyhedral decomposition \mathcal{P} of B . Note that the simplicial structure of Δ refines the natural stratification of Δ given by monodromy type. Let Δ_{\max} denote the set of maximal cells of Δ^\bullet together with an orientation, chosen once and for all. Each $\tau \in \Delta_{\max}$ is contained in a unique $(n-1)$ -cell

$\rho \in \mathcal{P}$. Then monodromy along a small loop about τ defines a monodromy transvection vector $m_\tau \in \Lambda_\rho$, where the signs are fixed by the orientations via some sign convention. In view of the orientations of τ and B we can then also choose a maximal cell $\sigma_\tau \supset \tau$ unambiguously.

For each $\tau \in \Delta_{\max}$ let w_τ be the choice of a partition of $|w_\tau| \in \mathbb{N}$ (with $w_\tau = \emptyset$ for $|w_\tau| = 0$). To separate leaves of tropical disks we will now locally replace Δ by a branched cover. We can then consider deformations of a disk h whose leaves end on that cover instead of Δ , with weights and directions prescribed by the partitions $\mathbf{w} := (w_\tau | \tau \in \Delta_{\max})$.

Deformations of Δ . We first define a deformation of the barycentric refinement Δ_{bar} of Δ as a polyhedral subset of B . For each $\tau \in \Delta_{\max}$, denote by $\mathfrak{s}_\tau \subset \sigma_\tau$ the 1-cell connecting the barycenter b_τ of τ to the barycenter of σ_τ . Note that $\Lambda_{\mathfrak{s}_\tau} \otimes_{\mathbb{Z}} \mathbb{R}$ intersects $i_* \Lambda_{b_\tau} \otimes_{\mathbb{Z}} \mathbb{R} = \text{span}(\Lambda_\tau, m_\tau)$ transversely. Moving the barycenter of the barycentric refinement τ_{bar} of τ along \mathfrak{s}_τ while fixing $\partial \tau_{\text{bar}}$ now defines a piecewise linear deformation τ_s of τ over $s \in \mathfrak{s}_\tau$ as a polyhedral subset of σ_τ . Thus we obtain a deformation $\{\Delta_s | s \in S\}$ of Δ over the cone $S := \coprod_\tau \mathfrak{s}_\tau$. It is trivial as deformation of cell complexes, as parallel transport in direction \mathfrak{s}_τ in each cell σ_τ induces an isomorphism of cell complexes $\Delta_{\text{bar}} \cong \Delta_s$.

For an infinitesimal point of view let $i_\tau : \tau \rightarrow \sigma$ be the inclusion. Consider the preimage of the deformation of $\tau \subset \Delta$ under the natural inclusion $\sigma_\tau \hookrightarrow i_\tau^* T\sigma_\tau$. For $s := (s_1, \dots, s_{\text{length } w_\tau}) \in \mathfrak{s}_\tau^{\text{length } w_\tau}$

$$\tau_s^{\mathbf{w}} := \bigcup_{k=1}^{\text{length } w_\tau} \tau_{s_k} \subset i_\tau^* T\sigma_\tau$$

is then a $\text{length } w_\tau$ -fold branched cover of τ via the natural projection $\pi : i_\tau^* T\sigma_\tau \rightarrow \tau$.

Note that $\tau_s^{\mathbf{w}} = \emptyset$ if $|w_\tau| = 0$ and $\tau_s^{\mathbf{w}} \subset \tau_{s'}^{\mathbf{w}'}$ if $\text{length } w_\tau \leq \text{length } w'_\tau$ and if the first $\text{length } w_\tau$ entries of s and s' agree. We make $\tau_s^{\mathbf{w}}$ into a weighted cell complex by equipping each cell of $\tau_s^{\mathbf{w}}$ with the weight defined by the partition w_τ . Finally, set $S^{\mathbf{w}} := \prod_\tau \mathfrak{s}_\tau^{\text{length } w_\tau}$ and $\Delta_s^{\mathbf{w}} := \bigcup_\tau \tau_{s_\tau}^{\mathbf{w}}$, where $s \in S^{\mathbf{w}}$. We still call $\Delta_s^{\mathbf{w}}$ a deformation of Δ , as for $\epsilon \rightarrow 0$, $\tilde{\Delta}_{\epsilon s}^{\mathbf{w}}$ converges to Δ as *weighted* complexes in an obvious sense.

Deformations of tropical disks. We now want to define a virtual tropical disk as an infinitesimal deformation \tilde{h} of a tropical disk h such that the leaves of \tilde{h} end on $\Delta_s^{\mathbf{w}}$ as prescribed by \mathbf{w} .

The idea is that for small $\epsilon > 0$ and suitable environments $U_v \subset T_v B$ of 0, $v = h(V)$ the images of internal vertices, the rescaled exponential map $\exp|_{\bigcup_v (\frac{1}{\epsilon} U_v)} \circ \epsilon \text{id}_{T(B \setminus \Delta)}$ maps the union of the tropical curves \tilde{h}_V onto the image of a tropical disk $\tilde{h}_\epsilon : \tilde{\Gamma} \rightarrow B$ with leaves emanating from $\Delta_{\epsilon s}^{\mathbf{w}}$. By choosing $\epsilon > 0$ sufficiently small, the image of \tilde{h}_ϵ is contained in an arbitrary small neighborhood of the image of h . Thus \tilde{h}_ϵ indeed defines a deformation of h . Conversely, for ϵ sufficiently small, \tilde{h}_ϵ determines \tilde{h} uniquely. Hence in order to simplify language and visualization, we may and will identify a virtual curve \tilde{h} with its “images” \tilde{h}_ϵ in B for small $\epsilon > 0$.

Definition 5.5. Let $h : |\Gamma| \rightarrow B$ be a tropical disk not intersecting $|\Delta^{[\dim B - 3]}|$. A *virtual tropical disk* \tilde{h} of intersection type \mathbf{w} deforming h consists of:

- (1) For each interior vertex $V \in \Gamma^{[0]}$ a possibly disconnected genus zero ordinary tropical curve $\tilde{h}_V : \tilde{\Gamma}_V \rightarrow T_v B$ with respect to the fan $\Sigma_{h,V}$. This means that $\tilde{\Gamma}_V$ is a possibly disconnected graph with simply connected components and without di- and univalent vertices, the map \tilde{h}_V satisfies conditions (2)–(4) of Definition 5.1, while instead of (5) the unbounded edges are parallel displacements of rays of $\Sigma_{h,V}$.
- (2) A cover \tilde{h}_E of each edge E of h by weighted parallel sections of the normal bundle $h|_E^* TB/Th(E)$. For each edge E adjacent to an interior vertex V , we require that the inclusion defines a weight-respecting bijection between the cosets of $\tilde{h}_E^{-1}(V)$ over V and rays of \tilde{h}_V in direction E . Moreover, the intersection defines a weight-preserving bijection between the cosets of $\{\tilde{h}_E^{-1}(V) \mid h(V) \in \tau, E \ni V\}$ over the leaf vertices in τ and branches of $\tau_s^{\mathbf{w}}$.
- (3) A virtual root position, that is a point $\tilde{h}_{V_{\text{root}}}(\widetilde{V_{\text{root}}})$ in $T_{h(V_{\text{root}})}B$ such that $\tilde{h}(\widetilde{V_{\text{root}}}) + Th(E) = \tilde{h}_E^{-1}(V_{\text{root}})$, where E is the root edge.

We denote by $\mathcal{M}_{2\mu}(\Delta_s^{\mathbf{w}}, h)$ the moduli space of virtual Maslov index 2μ disks of intersection type \mathbf{w} deforming h . In order to exclude the phenomena in Example 5.2, we now restrict to sufficiently general tropical disks. For such tropical disks a local deformation of the constraints on $\tilde{h}(\tilde{\Gamma}^{[0]})$ lifts to a local deformation of \tilde{h} preserving the type. Formally, we define:

Definition 5.6. Let $s \in S^{\mathbf{w}}$ and $\mu \in \{0, 1\}$. A virtual tropical disk $\tilde{h} \in \mathcal{M}_{2\mu}(\Delta_s^{\mathbf{w}}, h)$ is *sufficiently general* if:

- (1) \tilde{h} has no internal vertices of valency higher than three,
- (2) all intersections of \tilde{h} with the codimension one cells of \mathcal{P} are transverse intersections at divalent vertices outside $|\mathcal{P}^{[\dim B - 2]}|$,
- (3) there exists a subspace $L \subset T_{h(V_{\text{root}})}B$ of dimension $1 - \mu$ and an open cone $C^{\mathbf{w}} \subset S^{\mathbf{w}}$ containing s such that the natural map

$$(5.1) \quad \pi \times \text{ev}_{\widetilde{V_{\text{root}}}} : \bigcup_{s \in C^{\mathbf{w}}} \mathcal{M}_{\mu}(\Delta_s^{\mathbf{w}}, h) \rightarrow C^{\mathbf{w}} \times (T_{h(V_{\text{root}})}B)/L$$

is open.

$\Delta_s^{\mathbf{w}}$ is in *general position* if for all Maslov index zero disks h the complement of the set $\mathcal{M}_0(\Delta_s^{\mathbf{w}}, h)^{\text{gen}}$ of sufficiently general disks in $\mathcal{M}_0(\Delta_s^{\mathbf{w}}, h)$ is nowhere dense.

Lemma 5.7. *Given \mathbf{w} , the space of non-general position deformations of Δ is nowhere dense in S . For general position, $M_{\mu}(\Delta_s^{\mathbf{w}}, h)$ is of expected dimension $\dim B + \mu - 1$.*

Proof. (Sketch) Consider a generalized class of tropical disks by forgetting the leaf constraints, allowing for edge contraction and replacing condition (5) by assuming that the graph contains no divalent vertices. Fix a type with a trivalent graph Γ , then any

such disk is determined by the position of x and the length of the $N := |\Gamma^{[1]}| - \mu = 2|\Gamma_{\text{leaf}}^{[0]}| - 1 - \mu$ bounded edges, such that the inverse map restricts to an open embedding $\bigcup_{s \in S^{\mathbf{w}}} \mathcal{M}_\mu(\Delta_s^{\mathbf{w}}) \rightarrow B \times \mathbb{R}_{\geq 0}^N$ in obvious identifications dictated by Γ . The statement now follows from the observation that the map (5.1) expressed in $\mathbb{R}_{\geq 0}^N$ is piecewise linear, and any violation of stability defines a subset of a finite union of hyperplanes. In particular, the dimension follows by noting that the positions of the leaf vertices define constraints of codimension $2(|\Gamma_{\text{leaf}}^{[0]}| - \mu)$. \square

Remark 5.8. A stratum of $\mathcal{M}_0(\overline{m})$ admits a natural affine structure. Hence a disk $h \in \mathcal{M}_0(\overline{m})^{[k]}$ belonging to a k -dimensional stratum naturally comes with the k -dimensional subspace of induced infinitesimal vertex deformations

$$j_V(h)^{[k]} := T_{h \text{ ev}_V}(T_h \mathcal{M}_0^{[k]}(\overline{m})) \subset T_{h(V)}B.$$

Likewise, infinitesimal deformations of a sufficiently general Maslov index zero disk \tilde{h} give rise to *virtual joints*, that is the codimension two subsets defined by restricting the deformation family to the vertices. Such virtual joints converge to some codimension two space $j_V(\tilde{h}) \in T_{h(V)}B$, as $s \rightarrow 0 \in C^{\mathbf{w}}$. Moreover, if the limiting disk h of \tilde{h} belongs to a $(\dim B - 2)$ -stratum of $\mathcal{M}_0(\overline{m})$ such that (5.1) extends to an open map at $0 \in \partial C^{\mathbf{w}}$, then $j_V(h)^{[\dim B - 2]} = j_V(\tilde{h})$. This may be used to define stability for tropical disks.

5.3. Structures via virtual tropical disks. We now relate the counting of virtual Maslov index zero disks with the structures of [GS3]. Let \mathfrak{B} be the set of connected components of the codimension one cells of \mathcal{P} when Δ is removed. If $\mathfrak{b} \in \mathfrak{B}$ contained in $\rho \in \mathcal{P}^{[n-1]}$ and $v \in \rho$ is a vertex contained in \mathfrak{b} Denote by $f_{\mathfrak{b}} := f_{\rho, v}$ the order zero slab function attached to $\mathfrak{b} \in \mathfrak{B}$ via the log structure. Then $f_{\mathfrak{b}} \in \mathbb{k}[C_{\mathfrak{b}}]$ where $C_{\mathfrak{b}}$ is the monoid generated by one of the two primitive invariant τ -transverse vectors $\pm m_{\tau}$ for each positively oriented $\tau \in \Delta^{[max]}$ with $\overline{\mathfrak{b}} \cap \tau \neq \emptyset$.

Let $k \in \mathbb{N}$. Define the order k scattering parameter ring by

$$R^k := \mathbb{k}[t_{\tau} \mid \tau \in \Delta_{\max}] / \mathcal{I}_k, \quad \mathcal{I}_k := (t_{\tau}^{k+1} \mid \tau \in \Delta_{\max}),$$

and let \widehat{R} be its completion as $k \rightarrow \infty$.

As $f_{\mathfrak{b}}$ has a non-trivial constant term, we can take its logarithm as in [GPS]

$$(5.2) \quad \log f_{\mathfrak{b}} = \sum_{\overline{m} \in C_{\mathfrak{b}}} \text{length}(\overline{m}) a_{\mathfrak{b}, \overline{m}} z^{\overline{m}} \in k[[C_{\mathfrak{b}}]],$$

defining virtual multiplicities $a_{\mathfrak{b}, \overline{m}} \in k$. We consider $\log f_{\mathfrak{b}}$ as an element of $k[[C_{\mathfrak{b}}]] \otimes_k \widehat{R}$ via the completion of the inclusions

$$\iota_k : \mathbb{k}[C_{\mathfrak{b}}] \rightarrow \mathbb{k}[C_{\mathfrak{b}}] \otimes_k R^k, \quad z^{m_{\tau \mathfrak{b}}} \mapsto z^{m_{\tau \mathfrak{b}}} t_{\tau \mathfrak{b}}.$$

Definition 5.9. Attach the following numbers to a sufficiently general virtual tropical disk $\tilde{h} \in \mathcal{M}_\mu(\Delta_s^{\mathbf{w}}, h)^{gen}$:

(1) The virtual multiplicity of a vertex $V \in \tilde{\Gamma}^{[0]}$ of \tilde{h} is

$$\text{vmult}_V(\tilde{h}) := \begin{cases} a_{\mathfrak{b}, \bar{m}} & \text{if } V \text{ is univalent, } \pi(\tilde{h}(V)) \in \mathfrak{b} \\ s(\bar{m}) & \text{if } V \text{ is divalent} \\ |\bar{m} \wedge \bar{m}'|_{j_V(\tilde{h})} & \text{if } V \text{ is trivalent} \end{cases}$$

where \bar{m} denotes the tangent vector of \tilde{h} at V in the direction leading to the root, \bar{m}' a complementary tangent vector of \tilde{h} at V , $a_{\bar{m}}$ the coefficients in (5.2), $s : \Lambda_{\tilde{h}(V)} B \rightarrow k^\times$ the change of stratum function at $\tilde{h}(V)$, and the last expression the quotient density on $T_{\tilde{h}(V)} B / j_V(\tilde{h})$ induced from the natural density on $B \setminus \Delta$. Explicitly,

$$|\bar{m} \wedge \bar{m}'|_{j_V(\tilde{h})} := |\bar{m} \wedge \bar{m}' \wedge \bar{j}_1 \wedge \dots \wedge \bar{j}_{n-2}|,$$

where $\{\bar{j}_1, \dots, \bar{j}_{n-2}\}$ are generators of $j_V(\tilde{h}) \cap \Lambda_{B, h(V)}$ (cf. Remark 5.8).

(2) The virtual multiplicity $\text{vmult}(\tilde{h})$ of \tilde{h} is the total product

$$\text{vmult}(\tilde{h}) := \frac{1}{|\text{Aut}(\mathbf{w})|} \cdot \prod_{V \in \tilde{\Gamma}^{[0]}} \text{vmult}_V(\tilde{h})$$

where \mathbf{w} is the intersection type of \tilde{h} , $|\text{Aut}(\mathbf{w})|$ is the product of the automorphisms³ of the partitions w_τ over all $\tau \in \Delta_{\max}$

(3) The t -order of \tilde{h} is the sum of the changes in the t -order of the tangent vectors \bar{m}_V at divalent vertices V under changing the adjacent maximal strata σ_V^\pm , that is

$$\text{ord } \tilde{h} := \sum_{\substack{V \in \Gamma^{[1]} \\ \tilde{h}(V) \in \sigma_V^+ \cap \sigma_V^- \in \mathcal{P}^{[n-1]}}} \left| \left\langle d\varphi|_{\sigma_V^-} - d\varphi|_{\sigma_V^+}, \bar{m}_V \right\rangle \right|.$$

Remark 5.10. Intuitively, the t -order may be considered as a combinatorial analogue of the symplectic area of a holomorphic disk.

Note that the virtual multiplicity of a sufficiently general tropical disk depends only on its type. Moreover, we have:

Lemma 5.11. *The virtual multiplicity of a (type of) sufficiently general tropical disk \tilde{h} of intersection type \mathbf{w} deforming a Maslov index zero disk h is independent of the choice $s \in S^{\mathbf{w}}$ of the general position deformation Δ_s .*

Proof. We only give a very rough sketch here: If the type only changes by the number of divalent vertices, the claim follows immediately from the definition of φ as a continuous and piecewise linear function. In dimension two, the result then reduces to a standard one, cf. [?]. In higher dimension, the only remaining unstable *hyperplanes* consist of disks with a four-valent vertex. Here the independence of their stable deformations reduces to the dimension two case, as the virtual multiplicity is invariant under splitting each edge of Γ , acting with the stabilizer $SL(n, \mathbb{Z})_{j_V(h)}$ on each fan $\Sigma_{h, V}$, and regluing formally. \square

³ that is the number of permutations of the entries of w_τ that do not change the partition

We are now ready for our central definitions: Denote by $\#\mathcal{M}_0(\mathbf{w}, \bar{m}, \ell)^{gen}$ the number of types of sufficiently general virtual tropical disks with intersection type \mathbf{w} and t -order ℓ which deform a tropical Maslov index zero disk with root tangent vector $\bar{m} \in \Lambda_{B \setminus \Delta, x}$, counted with virtual multiplicity. Alternatively, for $\mu \in \{0, 1\}$ we can define $\#\mathcal{M}_0(\mathbf{w}, \bar{m}, \ell)^{gen}$ by counting the corresponding disks themselves, but specifying the virtual root as follows: The virtual root is $0 \in T_x B$ if $\mu = 1$, and belongs to a line in $T_{h(V_{\text{root}})} B$ transverse to $j_{V_{\text{root}}}(\tilde{h})$ if $\mu = 0$.

Let $\sigma \in \mathcal{P}^{[\max]}$, and $P_{v, \sigma}$ the associated monoid at $v \in \sigma$ which is determined by φ as in [GS3], Construction 2.17. Define the *counting function* to order k in $x \in \sigma$ by

$$(5.3) \quad \log f_{\sigma, x} := \sum_{\substack{\bar{m} \in \Lambda_{x, B}, \\ \ell \leq k}} \sum_{\mathbf{w}} \text{length}(\bar{m}) \#\mathcal{M}_0(\mathbf{w}, \bar{m}, \ell)^{gen} z^{\bar{m}} t^\ell \prod_{\tau} t_\tau^{|w_\tau|},$$

which is an element of the ring $\mathbb{k}[P_{v, \sigma}] \otimes_k R^k$.

Conjecture 5.12. *For each $k \in \mathbb{N}$, the counting polynomial (5.3) modulo (t^{k+1}) stabilizes in \mathbf{w} and then lifts to the rings $R_{g, \sigma}^k$ via $t_\tau \mapsto 1$. Thus the sets*

$$(5.4) \quad \mathbf{u}^k[x] := \overline{\{y \in \sigma \setminus \partial\sigma \mid \log f_{\sigma, y} = \log f_{\sigma, x} \neq 0 \in R_{g, \sigma}^k\}},$$

are either empty or define polyhedral subsets of codimension at least one. Together with their intersections, they define a polyhedral decomposition of each $\sigma \in \mathcal{P}^{[\max]}$ which induces a refinement \mathcal{P}^ℓ of $\mathcal{P}^{[n-1]}$. Thus the stabilized counting polynomials lifted to $R_{g, \sigma}^k$ define change of chamber morphisms

$$z^m \mapsto z^m f_{\sigma, x}^{\langle n_x, \bar{m} \rangle}$$

for crossing $\mathbf{u}^k[x] \in (\mathcal{P}^\ell)^{[n-1]}$, where $n_x \in \check{\Lambda}_{B, x}$ is primitive, annihilates $T_x \mathbf{u}^k[x]$, and is negative on tangent vectors pointing into the target chamber. Together with the change of strata morphisms of \mathfrak{X}_0 we obtain a scheme \mathfrak{X} over $\text{Spec } \mathbb{k}[t]/(t^{k+1})$ which reproduces the order k smoothing of \mathfrak{X}_0 constructed in [GS3].

Remark 5.13. Note that general position of Δ is not essential as long as we obtain the same virtual counts.

Proposition 5.14. *The conjecture is true if Δ contains no positive strata (that is monodromy polytopes of dimension $\max(1, \dim B - 1)$), for example for $\dim B = 2$.*

Proof. (Sketch) It is sufficient to show the following two claims:

Claim 1: Over R^k , the counting monomials arise via scattering. This can be proved by first decomposing the $\exp f_{\sigma, x}$ into products of binomials and then proceeding inductively by applying [GPS], Theorem 2.7 to each joint. Alternatively, one can adapt their proof directly:

Consider the thickening

$$\iota_\tau : \mathbb{k}[t_\tau]/t_\tau^{k+1} \hookrightarrow \frac{\mathbb{k}[u_{\tau i} \mid 1 \leq i \leq k,]}{(u_{\tau i}^2 \mid 1 \leq i \leq k)}, \quad t_\tau \mapsto \sum_{i=1}^k u_{\tau i}.$$

inducing a thickening $\bigotimes_{\tau} \iota_{\tau} : R^k \rightarrow \tilde{R}^k$ of the scattering parameter ring. Then consider virtual tropical disks with respect to the 2^k -fold branched covering of Δ whose branches τ_s^J over τ are labeled by $J \subset \{1, \dots, k\}$. We say such a disk special, if it has the following additional properties: The weight of a leaf is $|J|$ if it emanates from τ_s^J , and $u_{\tau J} u_{\tau J'} \neq 0$ whenever there are leaf vertices in τ_s^J and $\tau_s^{J'}$, where $u_{\tau J} := \prod_{i \in J} u_{\tau i}$. We can now attach the following function to the root tangent vector $-\bar{m}_{\tilde{h}}$ of such disks:

$$(5.5) \quad f_{\tilde{h}} := 1 + \text{length}(m_{\tilde{h}}) \text{vmult}'(\tilde{h}) \cdot z^{\bar{m}_{\tilde{h}}} t^{\text{ord } \tilde{h}} \prod_{\tau_s^J \cap \tilde{h}(\tilde{\Gamma}_{\text{leaf}}^{[0]}) \neq \emptyset} |J|! u_{\tau J}.$$

where vmult' equals vmult without the combinatorial factor.

Now the terms appearing in the thickening of the exponential of ?? are precisely the $f_{\tilde{h}}$ of those special disks \tilde{h} that contain only one edge. The others indeed arise from scattering, meaning the following: Whenever the root vertices of two special disks \tilde{h}, \tilde{h}' map to the same point p with transverse root leaves, there at most two ways to extend both disks beyond p locally: Either glue them to a single tropical disk, which is possible only if $f_{\tilde{h}} f_{\tilde{h}'} \neq 0$, or enlarge the root leaves such that p stays a point of intersection, which is always possible.

The functions attached to the two old and the three new roots then define a consistent scattering diagram, that is the counterclockwise product of the automorphisms

$$z^{\bar{m}} \mapsto z^{\bar{m}} f_{\tilde{h}}^{|\bar{m} \wedge \bar{m}_{\tilde{h}}|}$$

equals one. This is the content of Lemma 1.9 of [GPS], to which we refer for details. Now the proof of Theorem 2.17 shows that the sum $\sum_{\tilde{h}} \log f_{\tilde{h}}$ over all special disks with k -intersection type \mathbf{w} , root tangent vector $-\bar{m}$ and t -order ℓ equals indeed the thickening of the corresponding monomial in (5.3).

Claim 2: The counting functions (5.3) can be lifted. We must show that the scattering diagrams at each joint $j_V(\tilde{h})$ produce liftable monomials:

In case of a codimension zero joint this follows from the observation that each incoming non constant ray monomial has t -order greater than zero, hence working modulo (t^ℓ) implies working up to a finite k -order. In case of codimension one joints, by assumption there is only one non constant monomial of zero t -order present in each scattering diagram, namely that given by the log structure. In this case, we can apply [Gr3]. Finally, there are no codimension two joints by assumption. From both claims it follows that the gluing functions of both constructions must indeed coincide, as by our assumption on Δ both rely on scattering only, and scattering is unique up to equivalence. \square

5.4. Virtual counts of tropical Maslov index two disks. Assume now B fulfills the condition of subsection 5.1 such that $(B, \mathcal{P}, \varphi)$ fulfills Conjecture 5.12, for example $\dim B = 2$.

Proposition 5.15. *The coefficient a_{β} of the last monomial $z^{m_{\beta}}$ of a generic broken line β is the virtual number of tropical Maslov index two disks with root tangent vector m_{β} which*

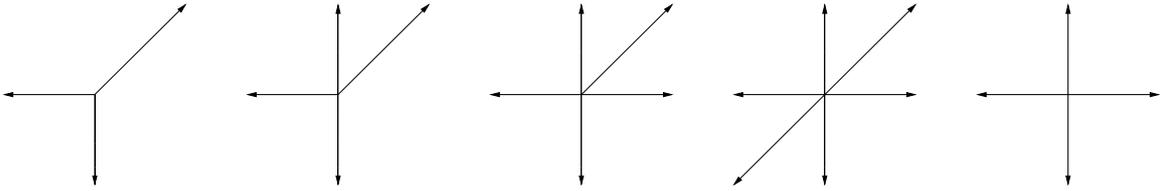


FIGURE 6.1. Fans of the five toric del Pezzo surfaces

complete β as in Lemma 5.4 and whose t -order equals the total change in the t -order of the exponents along β .

Proof. Let $az^m, a'z^{m'}$ be the functions attached to the edges adjacent to a fixed break point $\beta(t)$ of β . Let h be a virtual Maslov index zero disk bounded by $\beta(t)$ and with root tangent vector the required difference $\bar{m} - \bar{m}'$. Define the completed multiplicity of h as the virtual multiplicity of h times that of the break point. By definition, a'/a is a coefficient in the exponential of $a_{\mathfrak{b}} := |\bar{m} \wedge \bar{m}'| \log f_{\mathfrak{b}}$ for the function $f_{\mathfrak{b}}$ belonging to the wall or slab with tangent vector $\bar{m} - \bar{m}'$. By formula (5.3), $a_{\mathfrak{b}}$ equals the virtual number of completing virtual Maslov index zero disks with root tangent leaf $\bar{m} - \bar{m}'$ and t -order ℓ , completed by the break point multiplicity. Hence the required coefficient in $\exp a_{\mathfrak{b}}$ is given by counting *disconnected* virtual tropical disks of total t -order ℓ and total root tangent leaf \bar{m} with completed multiplicity. \square

6. TORIC DEGENERATIONS OF DEL PEZZO SURFACES

In this section we will compare superpotentials for different toric degenerations of del Pezzo surfaces, using broken lines and tropical Maslov index two disks. Recall that apart from $\mathbb{P}^1 \times \mathbb{P}^1$ all other del Pezzo surfaces dP_k can be obtained by blowing up \mathbb{P}^2 in $0 \leq k \leq 8$ points. Note that dP_k for $k \geq 5$ is not unique up to isomorphism but has a $2(k-4)$ -dimensional moduli space. For the anti-canonical bundle to be ample the blown-up points need to be in sufficiently general position. This means that no three points are collinear, no six points lie on a conic and no eight points lie on an irreducible cubic which has a double point at one of the points. However, rather than ampleness of $-K_X$ the existence of certain toric degenerations is central to our approach. For example, our point of view naturally includes the case $k = 9$.

6.1. Toric del Pezzo surfaces. Up to lattice isomorphism there are exactly five toric del Pezzo surfaces $X(\Sigma)$ whose fans Σ are depicted in Figure 6.1, namely \mathbb{P}^2 blown up torically in at most three points and $\mathbb{P}^1 \times \mathbb{P}^1$. To construct distinguished superpotentials for these surfaces we consider the following class of toric degenerations. Recall the notions of irreducibility (Definition 2.3) and simplicity ([GS1], §1.5) of toric degenerations.

Definition 6.1. A *distinguished toric degeneration of del Pezzo surfaces* is an irreducible, simple toric degeneration $(\mathfrak{X} \rightarrow T, \mathfrak{D})$ with \mathfrak{D} relatively ample over T and with generic fiber $\mathfrak{D}_\eta \subset \mathfrak{X}_\eta$ an anti-canonical divisor in a Gorenstein surface.

If the general fiber of a toric degeneration as in the definition is smooth then it is a dP_k for some k , together with a smooth anti-canonical divisor. The point of this definition is both the irreducibility of the anti-canonical divisor and the fact that this divisor extends to a polarization on the central fiber. Starting from a reflexive polytope there is a canonical construction of the intersection complex of a distinguished toric degeneration of the associated polarized toric variety as follows.

Construction 6.2. Let Ξ be a reflexive polytope and $v_0 \in \Xi$ the unique interior integral vertex. Define the polyhedral decomposition $\check{\mathcal{P}}$ of $\check{B} = \Xi$ with maximal cells the convex hulls of the facets of Ξ and of v_0 . The affine chart at v_0 is the one defined by the affine structure of Ξ . At any other vertex define the affine structure by the unique chart compatible with the affine structure of the adjacent maximal cells and making $\partial\check{B}$ totally geodesic. This works because by reflexivity for any vertex v the integral tangent vectors of any adjacent facet together with $v - v_0$ generate the full lattice. Reflexivity also implies that $(\check{B}, \check{\mathcal{P}})$ has simple singularities in lowest codimension. Moreover, $(\check{B}, \check{\mathcal{P}})$ has a natural polarization of minimal degree by defining $\varphi(v_0) = 0$ and $\varphi(v) = 1$ for any other vertex v . By running [GS3] we thus obtain a distinguished, anti-canonically polarized toric degeneration with generic fibre isomorphic to the toric Fano variety $X(\Xi)$ defined by Ξ , together with an irreducible anti-canonical divisor.

Note that the discrete Legendre transform $(B, \mathcal{P}, \varphi)$ has a unique bounded cell σ_0 , isomorphic to the dual polytope of Ξ . Up to the addition of a global affine function the dual polarizing function $\check{\varphi}$ is the unique piecewise affine function changing slope by one along the unbounded facets.⁴

Example 6.3. Specializing to del Pezzo surfaces we start from the momentum polytopes of the five non-singular toric Fano surfaces. The result of the construction is depicted in Figure 6.2, which shows a chart in the complement of the dotted segments. Note that the discrete Legendre transform $(B, \mathcal{P}, \varphi)$, also depicted in Figure 6.2, indeed has parallel outgoing rays.

Conversely, in dimension 2 we have the following uniqueness result.

Theorem 6.4. *If $(\pi : \check{\mathcal{X}} \rightarrow T, \check{\mathcal{D}})$ is a distinguished toric degeneration of del Pezzo surfaces with non-singular generic fibre, then the associated intersection complex $(\check{B}, \check{\mathcal{P}})$ is isomorphic to one listed in Figure 6.2.*

Proof. Let $(\pi : \check{\mathcal{X}} \rightarrow T, \check{\mathcal{D}})$ be the given toric degeneration. Thus the generic fibre $\check{\mathcal{X}}_\eta$ is isomorphic to a del Pezzo surface dP_k over η for some $0 \leq k \leq 3$, or to $\mathbb{P}^1 \times \mathbb{P}^1$.

First we determine the number of integral points of \check{B} . Let \mathcal{L} be the polarizing line bundle on $\check{\mathcal{X}}$. By assumption

$$(6.1) \quad h^0(\check{\mathcal{X}}_\eta, \mathcal{L}|_{\check{\mathcal{X}}_\eta}) = h^0(dP_k, -K_{dP_k}) = \begin{cases} 10 - k, & \check{\mathcal{X}}_\eta \simeq dP_k \\ 9, & \check{\mathcal{X}}_\eta \simeq \mathbb{P}^1 \times \mathbb{P}^1. \end{cases}$$

⁴In the present case $\check{\varphi}$ is single-valued.

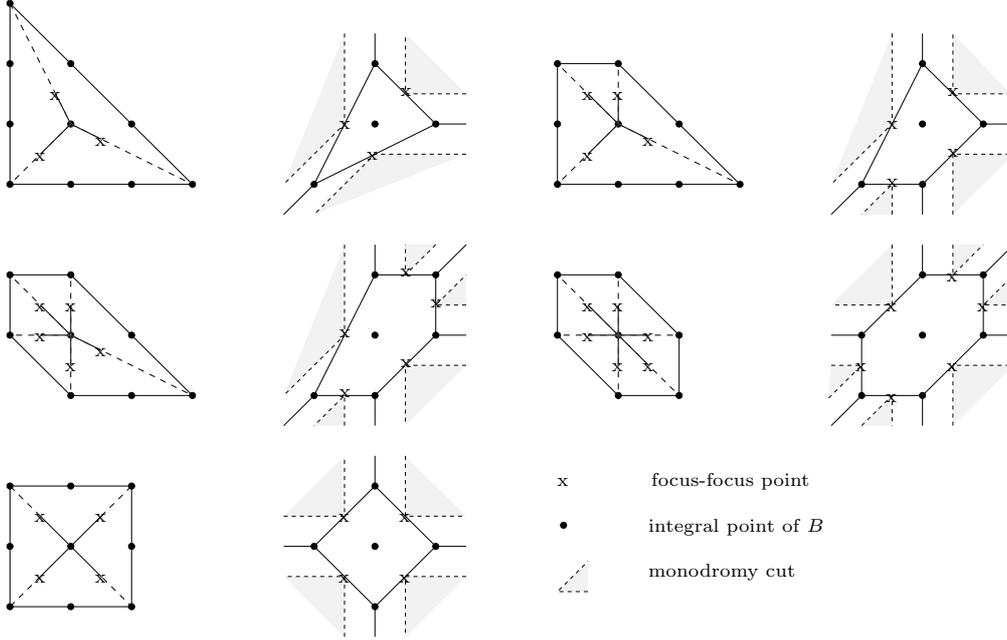


FIGURE 6.2. The intersection complexes $(\check{B}, \check{\mathcal{P}})$ of the five distinguished toric degenerations of toric del Pezzo surfaces and their Legendre duals (B, \mathcal{P}) .

Let $t \in \mathcal{O}_{T,0}$ be a uniformizing parameter and $\check{X}_n := \text{Spec}(\mathbb{k}[t]/(t^{n+1})) \times_T \check{\mathcal{X}}$ the n -th order neighbourhood of $\check{X}_0 := \pi^{-1}(0)$ in $\check{\mathcal{X}}$. Denote by $\mathcal{L}_n = \mathcal{L}|_{\check{X}_n}$.

Then for any n there is an exact sequence of sheaves on \check{X}_0 ,

$$0 \longrightarrow \mathcal{O}_{\check{X}_0} \longrightarrow \mathcal{L}_{n+1} \longrightarrow \mathcal{L}_n \longrightarrow 0.$$

By the analogue of [GS2], Theorem 4.2, for Calabi-Yau pairs, we know

$$h^1(\check{X}_0, \mathcal{O}_{\check{X}_0}) = h^1(\check{\mathcal{X}}_\eta, \mathcal{O}_{\check{\mathcal{X}}_\eta}) = 0.$$

Thus the long exact sequence on cohomology induces a surjection $H^0(\check{X}_0, \mathcal{L}_{n+1}) \twoheadrightarrow H^0(\check{X}_0, \mathcal{L}_n)$ for each n . By the theorem on formal functions and cohomology and base change ([Ha], Theorem 11.1 and Theorem 12.11) we thus conclude that $\pi_*\mathcal{L}$ is locally free, with fibre over 0 isomorphic to $H^0(\check{X}_0, \mathcal{L}_0)$. In view of (6.1) we thus conclude

$$h^0(\check{X}_0, \mathcal{L}_0) = \begin{cases} 10 - k, & \check{\mathcal{X}}_\eta \simeq dP_k \\ 9, & \check{\mathcal{X}}_\eta \simeq \mathbb{P}^1 \times \mathbb{P}^1. \end{cases}$$

Now on a toric variety the dimension of the space of sections of a polarizing line bundle equals the number of integral points of the momentum polytope. Since \check{X}_0 is a union of toric varieties each integral point $x \in \check{B}$ provides a monomial section of \mathcal{L}_0 on any

irreducible component $\check{X}_\sigma \subset \check{X}_0$ with $\sigma \in \mathcal{P}$ containing x . These provide a basis of sections of $H^0(\check{X}_0, \mathcal{L}_0)$.⁵ Hence \check{B} has $10 - k$ integral points.

An analogous argument shows that the number of integral points of $\partial\check{B}$ equals

$$h^0(\check{D}_0, \mathcal{L}_0) = h^0(\check{D}_\eta, \mathcal{L}_\eta),$$

which by Riemann-Roch equals $K_{dP_k}^2 = 9 - k$ or $K_{\mathbb{P}^1 \times \mathbb{P}^1}^2 = 8$. In either case we thus have a unique integral interior point $v_0 \in \check{B}$. In particular, \check{B} has the topology of a disk, and each singular point of the affine structure lies on an edge connecting v_0 to an integral point of $\partial\check{B}$.

Pushing the singular points p_1, \dots, p_l into $\partial\check{B}$, thereby trading them for corners, we arrive at an l -gon with a unique interior integral point, hence a reflexive polygon. Moreover, since the generic fibre of $\check{\mathfrak{X}} \rightarrow \check{\mathfrak{D}}$ is non-singular, at each vertex integral generators of the tangent spaces of the adjacent edges form a lattice basis. Thus by the classification of reflexive lattice polygons, up to adding some edges connecting v_0 to $\partial\check{B}$ the only configurations possible are the ones shown in Figure 6.2. \square

Remark 6.5. 1) The five types can be distinguished by $\dim H^0(\check{\mathfrak{X}}_\eta, \mathcal{L}_\eta)$, except for $\mathbb{P}^1 \times \mathbb{P}^1$ and dP_1 . Alternatively, by Proposition 6.11 one could use $H^1(\check{\mathfrak{X}}_\eta, \Omega_{\check{\mathfrak{X}}_\eta}^1)$.

2) For each $(\check{B}, \check{\mathcal{P}})$ there is a discrete set of choices of $\check{\varphi}$, which determines the local toric models of $\check{\mathfrak{X}} \rightarrow \check{\mathfrak{D}}$. This reflects the fact that the base of (log smooth) deformations of the central fibre \check{X}_0^\dagger as a space over the standard log point \mathbb{k}^\dagger is higher dimensional. In fact, let r be the number of vertices on $\partial\check{B}$. Then taking a representative of $\check{\varphi}$ that vanishes on one maximal cell, $\check{\varphi}$ is defined by the value at $r - 2$ vertices on $\partial\check{B}$. Convexity then defines a submonoid $Q \subset \mathbb{N}^{r-2}$ with the property that $\text{Hom}(Q, \mathbb{N})$ is isomorphic to the space of (not necessarily strictly) convex, piecewise affine functions on $(\check{B}, \check{\mathcal{P}})$ modulo global affine functions. Running the construction of [GS3] with parameters then produces a log smooth deformation with the given central fibre (\check{X}, \check{D}) over the completion at the origin of $\text{Spec } \mathbb{k}[Q]$. For the minimal polyhedral decompositions of Figure 6.2 with $r = l$ we have $\text{rk } Q = l - 2$, which by Remark 6.12,2 below agrees with the dimension of the space $H^1(\check{X}_0, \Theta_{\check{X}_0^\dagger/\mathbb{k}^\dagger})$ of infinitesimal log smooth deformations of $X_0^\dagger/\mathbb{k}^\dagger$. One can show that in this case the constructed deformation is in fact semi-universal. \square

The technical tool to compute superpotentials in the toric del Pezzo and in related examples in finitely many steps is the following lemma, suggested to us by Mark Gross. It greatly reduces the number of broken lines to be considered in situations fulfilling (2.1) and with a finite structure on the bounded cells.

⁵This also follows by the description of $(\check{X}_0, \mathcal{L}_0)$ by a homogeneous coordinate ring in [GS1], Definition 2.4.

Lemma 6.6. *Let \mathcal{S} be a structure for a non-compact, polarized tropical manifold $(B, \mathcal{P}, \varphi)$ that is consistent to all orders. We assume that there is a subdivision \mathcal{P}' of \mathcal{P} with vertices disjoint from Δ and with the following properties.*

- (1) *Each $\sigma \in \mathcal{P}'$ is affine isomorphic to $\rho \times \mathbb{R}_{\geq 0}$ for some bounded face $\rho \subset \sigma$.*
- (2) *$B \setminus \text{Int}(|\mathcal{P}'|)$ is compact and locally convex at the vertices (this makes sense in an affine chart).*
- (3) *If m is an exponent of a monomial of a wall (or slab) intersecting some $\sigma \in \mathcal{P}'$, $\sigma = \rho + \mathbb{R}_{\geq 0}\bar{m}_\sigma$, then $-\bar{m} \in \Lambda_\rho + \mathbb{R}_{>0} \cdot \bar{m}_\sigma$ (or $-\bar{m} \in \Lambda_\rho + \mathbb{R}_{\geq 0} \cdot \bar{m}_\sigma$ for slabs).*

Then the first break point t_1 of a broken line β with $\text{im}(\beta) \not\subset |\mathcal{P}'|$ can only happen after leaving $\text{Int}|\mathcal{P}'|$, that is,

$$t_1 \geq \inf \{t \in (-\infty, 0] \mid \beta(t) \notin |\mathcal{P}'|\}.$$

Proof. Assume $\beta(t_1) \in \sigma \setminus \rho$ for some $\sigma = \rho + \mathbb{R}_{\geq 0}\bar{m}_\sigma \in \mathcal{P}'$. Then $\beta|_{(-\infty, t_1]}$ is an affine map with derivative $-\bar{m}_\sigma$, and $\beta(t_1)$ lies on a wall. By the assumption on exponents of walls on σ , the result of nontrivial scattering at time t_1 only leads to exponents m_2 with $-\bar{m}_2 \in \Lambda_\rho + \mathbb{R}_{\geq 0}\bar{m}_\sigma$, the outward pointing half-space. In particular, the next break point can not lie on ρ . Going by induction one sees that any further break point in σ preserves the condition that β' does not point inward. Moreover, by the convexity assumption, this condition is also preserved when moving to a neighbouring cell in \mathcal{P}' . Thus $\text{im}(\beta) \subset |\mathcal{P}'|$. \square

Proposition 6.7. *Let (B, \mathcal{P}) be the dual intersection complex of a distinguished toric del Pezzo degeneration and let $\sigma_0 \subset B$ be the bounded cell. Then there is neighbourhood U of the interior vertex $v_0 \in \sigma_0$ such that for any $x \in U$ there is a canonical bijection between broken lines with endpoint x and rays of Σ .*

Proof. We can embed Σ in the tangent space at v_0 by extending the unbounded edges to v_0 in the chart shown in Figure 6.1. Note that for non-minimal $\check{\varphi}$ the size of the bounded polytope $\sigma_0 \subset B$ changes, but this does not affect the argument. Each ray of Σ can then be interpreted as the image of a unique broken line. Because each such broken lines has a positive distance from the shaded regions they can be moved with small perturbations of x . Conversely, by inspection of the five cases, the result of non-trivial scattering at $\partial\sigma_0$ leads to a broken line not entering $\text{Int}(\sigma_0)$. \square

Corollary 6.8. *Let $(\mathfrak{X} \rightarrow \text{Spec } \mathbb{k}[[t]], W)$ be the Landau-Ginzburg model mirror to a distinguished toric del Pezzo degeneration. Then there is an open subset $U \simeq \text{Spec } \mathbb{k}[[t]][x, y] \subset \mathfrak{X}$ such that $W|_U$ equals the usual Hori-Vafa monomial sum times t .* \square

Remark 6.9. 1) For other than anti-canonical polarizations the terms in the superpotential receive different powers of t , just as in the Hori-Vafa proposal.

2) Analogous arguments should work for smooth toric Fano varieties of any dimension.

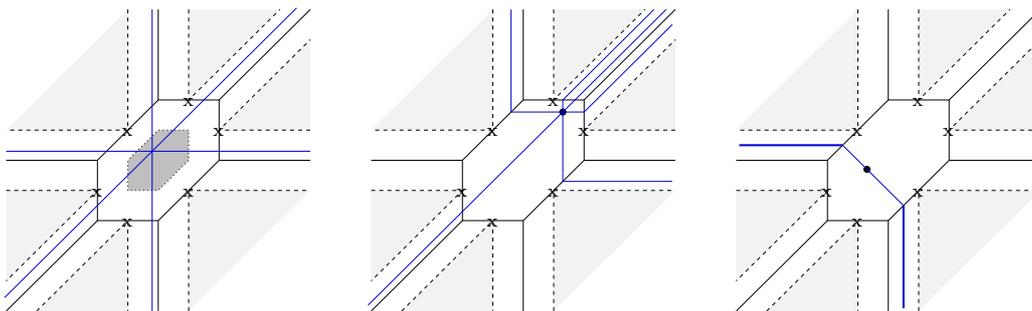


FIGURE 6.3. Tropical disks in the mirror of the distinguished base of an dP_3 indicating the invariance under change of root vertex.

Example 6.10. Let us study the distinguished toric degeneration of dP_3 with the minimal polarization $\check{\varphi}$ explicitly. In Figure 6.3 the first two pictures show all Maslov index two disks, respectively broken lines (using disk completion), for different choices of root vertex. First of all this illustrates the invariance under the change of root vertex proved in Lemma 4.7. The root tangent vectors are $(1, 0)$, $(1, 1)$, $(0, 1)$, $(-1, 0)$, $(-1, -1)$ and $(0, -1)$ giving the potential

$$W_{dP_3}^1(\sigma) = \left(x + y + xy + \frac{1}{x} + \frac{1}{y} + \frac{1}{xy}\right) \cdot t,$$

which for $t \neq 0$ has six critical points. The picture on the right shows tropical disks with weight two unbounded leaves, which therefore do not contribute to the superpotential. Moving the root vertex within the shaded open hexagon yields the same result, that is, none of the six broken lines has a break point.

An analogous picture arises for the other four distinguished del Pezzo degenerations.

Morally speaking the last example shows that in toric situations ray generators of the fan are sufficient to compute the superpotential, but really they should be seen as special cases of tropical disks or broken lines.

6.2. Non-toric del Pezzo surfaces. In this section we consider del Pezzo surfaces dP_k for $k \geq 4$, referred to as higher del Pezzo surfaces. Let us first determine the topology of B and the number of singular points of the affine structure.

Proposition 6.11. *Let (B, \mathcal{P}) be the dual intersection complex of an irreducible, simple toric degeneration $(\pi : \check{\mathfrak{X}} \rightarrow T, \check{\mathfrak{D}})$ of two-dimensional log Calabi-Yau pairs. In particular, the generic fibre $\check{\mathfrak{X}}_\eta$ is a proper surface with $\check{\mathfrak{D}}$ a smooth anti-canonical divisor.*

Then B is homeomorphic to \mathbb{R}^2 , and the affine structure has $l = \dim H^1(\check{\mathfrak{X}}_\eta, \Omega_{\check{\mathfrak{X}}_\eta}^1) + 2$ singular points.

Proof. Since the relative logarithmic dualizing sheaf $\omega_{\check{\mathfrak{X}}/\check{\mathfrak{D}}}(-\log \check{\mathfrak{D}})$ is trivial, the generalization of [GS1], Theorem 2.39, to the case of log Calabi-Yau pairs shows that B is orientable. By the classification of surfaces with effective anti-canonical divisor we know

$H^i(\check{\mathcal{X}}_\eta, \mathcal{O}_{\check{\mathcal{X}}_\eta}) = 0$, $i = 1, 2$. As in the proof of Theorem 6.4 this implies $H^1(\check{X}_0, \mathcal{O}_{\check{X}_0}) = 0$. Thus by the log Calabi-Yau analogue of [GS1], Proposition 2.37,

$$H^1(B, \mathbb{k}) = H^1(\check{X}_0, \mathcal{O}_{\check{X}_0}) = 0.$$

In particular, \check{B} has the topology of \mathbb{R}^2 .

As for the number of singular points the generalization of [GS2], Theorem 3.21 and Theorem 4.2, shows that $\dim H^1(\check{\mathcal{X}}_\eta, \Omega_{\check{\mathcal{X}}_\eta}^1)$ is related to an affine Hodge group:

$$\dim H^1(\check{\mathcal{X}}_\eta, \Omega_{\check{\mathcal{X}}_\eta}^1) = \dim H^1(B, i_* \Lambda^* \otimes_{\mathbb{Z}} \mathbb{R}).$$

To compute $H^1(B, i_* \Lambda^* \otimes_{\mathbb{Z}} \mathbb{R})$ we choose the following Čech cover of B . Since B is homeomorphic to \mathbb{R}^2 there is an open disk $U_0 \subset B$ with all singular points contained in ∂U_0 . Order the singular points $p_1, \dots, p_l \in B$ by following the circle ∂U_0 . Then there exist open sets $U_1, \dots, U_l \subset B$ with the following properties. (1) $U_i \cap \{p_1, \dots, p_l\} = \{p_i\}$, (2) $B \setminus U_0 \subset \bigcup_{i=1}^l U_i$, (3) $U_i \cap U_0$, $U_i \cap U_{i+1}$ and $U_l \cap U_1$ are contractible and disjoint from $\{p_1, \dots, p_l\}$, (4) For pairwise disjoint $i, j, k \geq 1$ we have $U_i \cap U_j \cap U_k = \emptyset$. Then $\mathcal{U} := \{U_0, U_1, \dots, U_l\}$ is a Leray covering of B for $i_* \Lambda_{\mathbb{R}}^* := i_* \Lambda^* \otimes_{\mathbb{Z}} \mathbb{R}$ (cf. [GS1], Lemma 5.5). The terms in the Čech complex are

$$\begin{aligned} C^0(\mathcal{U}, i_* \Lambda_{\mathbb{R}}^*) &= \mathbb{R}^2 \times \prod_{i=1}^l \mathbb{R}^1 \\ C^1(\mathcal{U}, i_* \Lambda_{\mathbb{R}}^*) &= \prod_{i=1}^l \mathbb{R}^2 \times \prod_{i=1}^l \mathbb{R}^2 \\ C^2(\mathcal{U}, i_* \Lambda_{\mathbb{R}}^*) &= \prod_{i=1}^l \mathbb{R}^2. \end{aligned}$$

It is easy to see that the Čech differential $C^0(\mathcal{U}, i_* \Lambda_{\mathbb{R}}^*) \rightarrow C^1(\mathcal{U}, i_* \Lambda_{\mathbb{R}}^*)$ is injective while $C^1(\mathcal{U}, i_* \Lambda_{\mathbb{R}}^*) \rightarrow C^2(\mathcal{U}, i_* \Lambda_{\mathbb{R}}^*)$ is surjective. Hence

$$\dim H^1(B, i_* \Lambda_{\mathbb{R}}^*) = 4l - 2l - (l + 2) = l - 2$$

determines the number l of focus-focus points as claimed. \square

Remark 6.12. 1) From the analysis in Proposition 6.4 and Proposition 6.11 it is clear that for del Pezzo surfaces of degree at least four the anti-canonical polarization is too small to extend over a toric degeneration. The associated tropical manifold would simply not have enough integral points to admit the required number of singular points.

2) Essentially the same argument also computes the dimension of the space of infinitesimal deformations:

$$h^1(\check{\mathcal{X}}_\eta, \Theta_{\check{\mathcal{X}}_\eta}(\log \check{\mathcal{D}}_\eta)) = h^1(\check{X}_0, \Theta_{\check{X}_0^\dagger/\mathbb{k}^\dagger}) = h^1(B, i_* \Lambda_{\mathbb{R}}) = l - 2. \quad \square$$

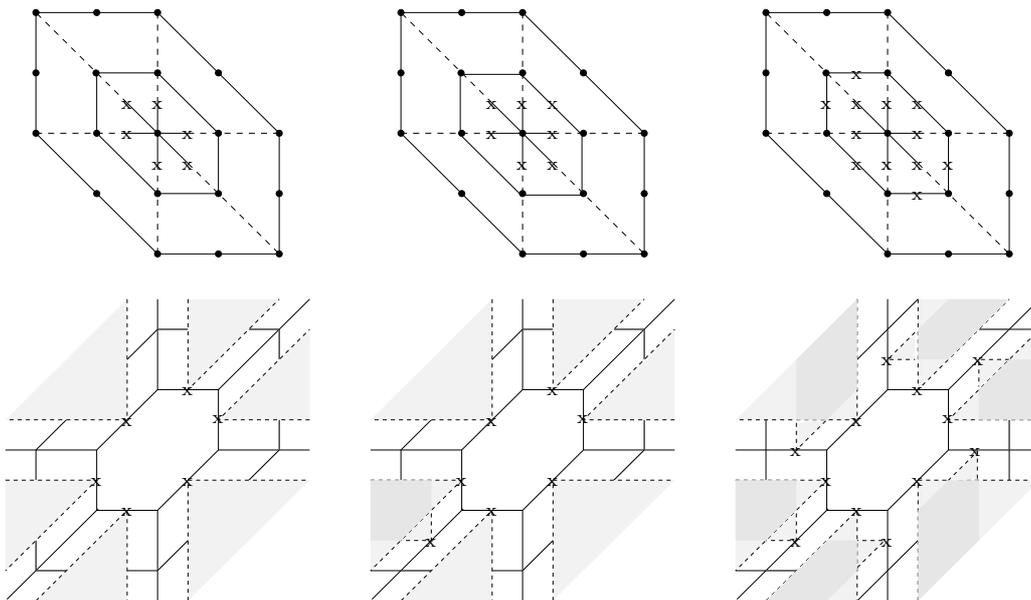


FIGURE 6.4. Straight boundary models for higher del Pezzo surfaces obtained by changing affine data for dP_3 and their Legendre duals.

It is easy to write down toric degenerations of non-toric del Pezzo surfaces, since they can be represented as hypersurfaces or complete intersections in weighted projective spaces, as for example done for dP_5 in [GS4, Example 4.2]. The most natural toric degenerations in this setup have central fibre the toric boundary divisor of the ambient space. But because this construction gives reducible D_η such toric degenerations are never distinguished. To obtain proper superpotentials we therefore need a different approach.

Construction 6.13. Start from the intersection complex $(\check{B}, \check{\mathcal{P}})$ of the distinguished dP_3 . The six focus-focus points in the interior of the bounded two cell make the boundary ρ straight, There is no space to introduce more singular points because all interior edges already contain a singular point. To get around this, polarize by $-2 \cdot K_{dP_3}$ and adapt \mathcal{P} in the obvious way, see Figure 6.4. This scales the affine manifold B by two, but keeps the singular points fixed. The new boundary now has 12 integral points and the union γ of edges neither intersecting the central vertex nor ∂B is a geodesic. We can then introduce new singular points as visualized in Figure 6.4. Moreover, let $\check{\varphi}$ be unchanged on the interior cells and change slope by one when passing to a cell intersecting ∂B . Plugging in up to five singular points, one can show that toric degenerations obtained from the tropical data are in fact degenerations of dP_k , $4 \leq k \leq 9$. \square

Unlike in the anti-canonically polarized case, the models constructed in this way are not unique. The geodesic γ is divided into six segments by \mathcal{P} , and the choice on which of these segments we place the singular points results in different models. We will see in Example 6.16 how this choice influences tropical curve counts. Although there are other ways to define distinguished models for higher del Pezzo surfaces, for example by choosing

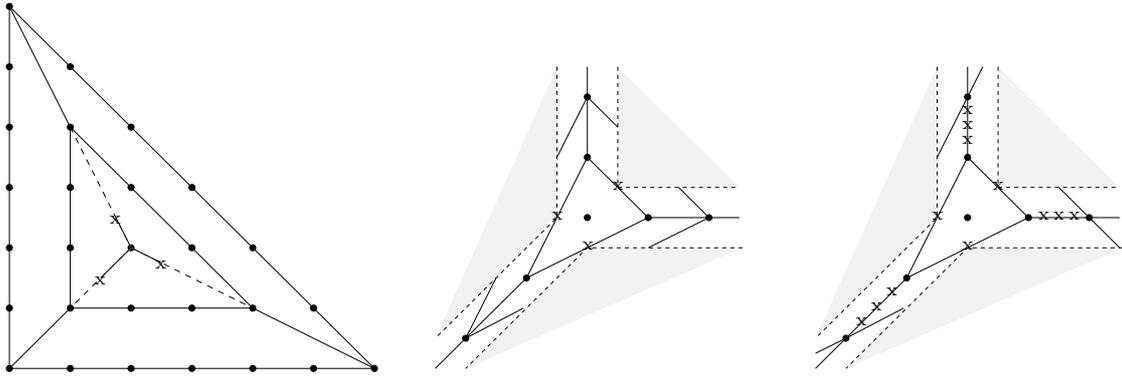


FIGURE 6.5. An alternative base for higher del Pezzo surfaces and their mirror.

another polarization, in this way we can extend the unique toric models most naturally, since all tropical disks and broken lines we studied before arise in these models without any change.

Remark 6.14. Note that introducing six new points, for instance as in the rightmost picture in Figure 6.4, corresponds to a blow up of \mathbb{P}^2 in nine points, which is not Fano anymore, but from our point of view still has a Landau-Ginzburg mirror. From a different point of view this has already been noted in [AKO], where the authors construct a compactification of the Hori-Vafa mirror as a symplectic Lefschetz fibrations as follows. Start with the standard potential $x + y + \frac{1}{xy}$ for \mathbb{P}^2 and compactify by a divisor at infinity consisting of nine rational curves. Then by a deformation argument it is possible to push k of those rational curves to the finite part and decompactify to obtain a potential for dP_k , including $k = 9$. We can reproduce this result from our point of view by starting with \mathbb{P}^2 rather than with dP_3 , as illustrated in Figure 6.5. Moving rational curves from infinity to the finite part is analogous to introducing new focus-focus points. In the present case one may put three focus-focus points on each unbounded ray of (B, \mathcal{P}) until the respective Legendre dual corner becomes straight. Figure 6.5 on the right shows nine such points (corresponding to the case $k = 9$ above), and any additional singular point would result in a concave boundary. This can be seen as an affine-geometrical explanation for why the compactification constructed by the authors in [AKO] has exactly nine irreducible components. Note that it is possible to introduce more singular points when passing to larger polarizations, but in this way we will not end up with degenerations of del Pezzo surfaces.

In order to determine the superpotential we depicted in Figure 6.4 an appropriate chart of the relevant (B, \mathcal{P}) . When two regions to be removed overlap we shaded them darker to indicate the non-trivial transformation there.

Example 6.15. Figure 6.6 shows the dual intersection complex (B, \mathcal{P}) of a toric degeneration of dP_4 from Construction 6.13. The additional focus-focus point changes the

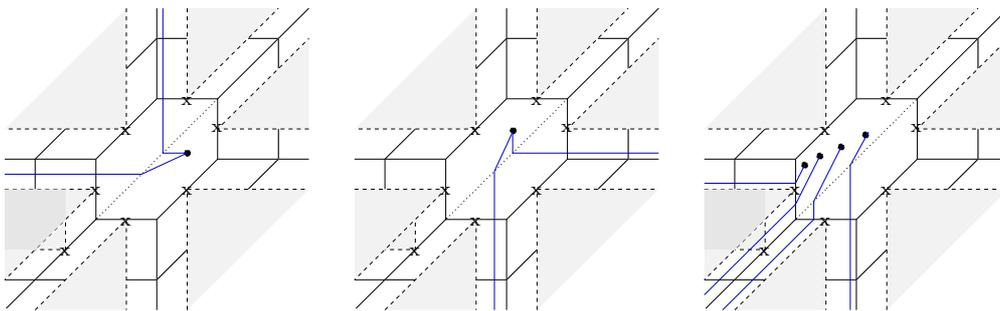


FIGURE 6.6. Mirror base to dP_4 showing new broken lines contributing to $W_{dP_4}^2$, indicating the wall crossing phenomenon and the invariance under change of endpoint within a chamber.

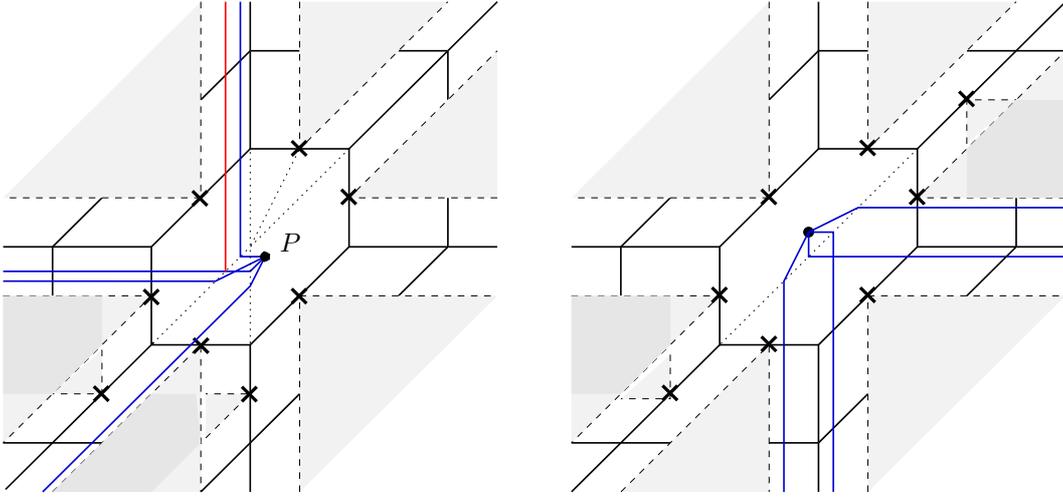
structure \mathcal{S} and allows broken lines to scatter with the wall in direction $(1,1)$ in the central cell σ_0 , which yields new root tangent directions. A broken line coming from infinity in direction $(\pm 1, 0)$ will produce root tangent vector $(-1 \pm 1, -1)$, whereas one with direction $(0, \pm 1)$ will take direction $(-1, -1 \pm 1)$. By construction, every broken line reaching the interior cell σ_0 will have t -order at least 2. Moreover, note that the scattering diagram locally in σ_0 is already consistent to all orders. A broken line can have at most one break point within σ_0 , in which case the t -order increases by one. So let us compute $W_{dP_4}^3$. We get two generically new root tangent directions, namely $(-1, -2)$ and $(-2, -1)$, and possibly more contributions from direction $(0, -1)$ and $(-1, 0)$. The two rightmost pictures in Figure 6.6 show all new broken lines for different choices of root vertex, apart from the six toric ones we have already encountered in Example 6.10. Depending on this choice, the superpotential to order three is therefore either given by

$$W_{dP_4}^3 = W_{dP_4}^3(\sigma_0) = \left(x + y + xy + \frac{1}{x} + \frac{1}{y} + \frac{1}{xy}\right) \cdot t^2 + \left(\frac{1}{x} + \frac{1}{x^2y}\right) \cdot t^3 \quad \text{or}$$

$$W_{dP_4}^3 = W_{dP_4}^3(\sigma_0) = \left(x + y + xy + \frac{1}{x} + \frac{1}{y} + \frac{1}{xy}\right) \cdot t^2 + \left(\frac{1}{y} + \frac{1}{xy^2}\right) \cdot t^3,$$

both of which have seven critical points, as expected. These superpotentials are not only related by interchanging x and y , for symmetry reasons, but also by wall crossing along the wall separating σ_0 into two chambers. Furthermore one can check that no matter on which unbounded ray we place the focus-focus point, we get a potential with seven critical points already to t -order three. In the leftmost picture we indicated the behaviour of a single broken line of root tangent direction $(-1, -2)$ under change of root vertex. If the root vertex changes a chamber by passing one of the dotted lines drawn, the broken line changes accordingly.

Given a toric degeneration, we have seen that the superpotential W^k is globally defined and independent under change of chambers and reference point. We define the full potential to be $W := \varinjlim W^k$. As scattering diagrams are almost never finite, the full potential is an infinite expression most of the time. However, if there exists an open set

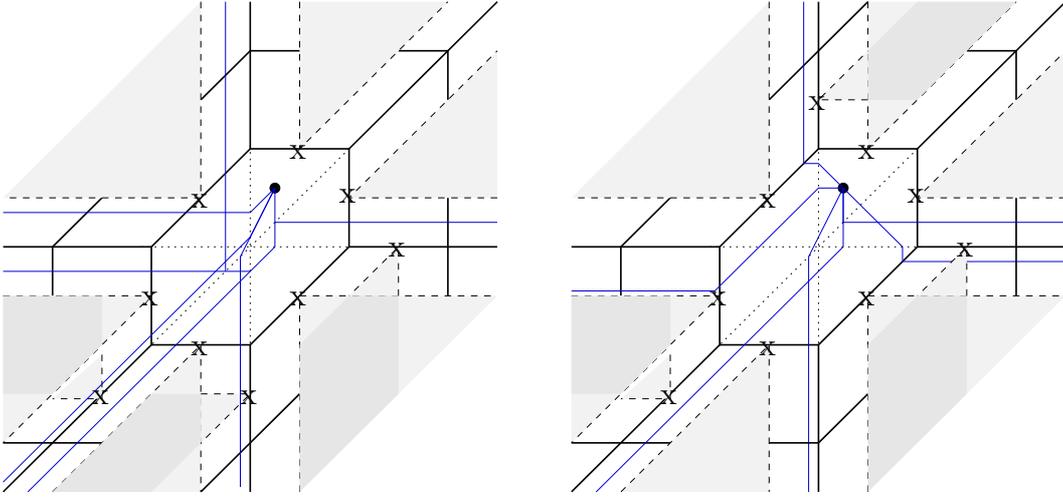
FIGURE 6.7. Mirror bases to dP_5 showing all new broken lines.

$U \subset B$ such that only finitely many walls and slabs intersect U then an open subset of the formal scheme $\lim_{\rightarrow} X_k$ has an algebraic torus \mathbb{G}_m^n over \mathbb{A}^1 as an algebraic model. If furthermore there exists a point $x \in U$ such that there are only finitely many broken lines with endpoint x then W can be represented by an algebraic function on this algebraic torus, hence a Laurent polynomial with coefficients in t . In such a situation we say that W is *manifestly algebraic*. In Example 6.15 we have seen a non-trivial manifestly algebraic superpotential.

Example 6.16. Attaching another singular point on the unbounded ray in direction $(0, -1)$ as in Figure 6.7 on the left we arrive at a degeneration of dP_5 . For a structure consistent to all orders there are three walls in the bounded maximal cell necessary, indicated by dotted lines in the figure. They are the extensions of the slabs with tangent directions $(1, 1)$ and $(0, 1)$ caused by additional singular points, and the result of scattering of these, the wall with tangent direction $(1, 2)$. Because $(1, 1)$ and $(0, 1)$ form a lattice basis the scattering procedure at the origin does not produce any additional walls. In any case, any broken line coming in from direction $(1, 1)$ and with endpoint P as indicated in Figure 6.7 can not interact with any of the scattering products. Tracing any possible broken lines starting from $t = -\infty$ one arrives at only five broken lines with endpoint P , with only one, drawn in red, having more than one breakpoint. We therefore obtain the following superpotential:

$$W_{dP_5}(\sigma_0) = \left(x + y + xy + \frac{1}{x} + \frac{1}{y} + \frac{1}{xy}\right) \cdot t^2 + \left(\frac{1}{y} + \frac{1}{xy} + \frac{1}{x^2y} + \frac{1}{xy^2}\right) \cdot t^3 + \frac{1}{xy}t^4.$$

Example 6.17. We study another model of the mirror to dP_5 , which differs from the last one only in the position of the second focus-focus point. Instead of placing it on the ray with generator $(0, -1)$ we move it to the ray generated by $(1, 1)$, as shown in Figure 6.7

FIGURE 6.8. Two mirror bases to dP_6 .

on the right. This more particular choice yields the superpotential

$$W_{dP_5}(\sigma) = W_{dP_5}^3(\sigma) = \left(x + y + xy + \frac{1}{x} + \frac{1}{y} + \frac{1}{xy}\right) \cdot t^2 + \left(\frac{1}{y} + x + x^2y + \frac{1}{xy^2}\right) \cdot t^3.$$

It is an interesting question to understand in detail the effect of particular choices of singular points and the corresponding degenerations.

Example 6.18. As a last example, we study broken lines in the mirror of a distinguished model of dP_6 , depicted in the middle in Figure 6.8. This time we obtain the superpotential

$$W_{dP_6}^5(\sigma) = \left(x + y + xy + \frac{1}{x} + \frac{1}{y} + \frac{1}{xy}\right) \cdot t^2 + \left(2 \cdot \frac{1}{xy^2} + \frac{1}{xy} + 2 \cdot \frac{1}{y}\right) \cdot t^3 + \frac{1}{y} \cdot t^4 + \frac{1}{y} \cdot t^5,$$

with nine critical points. Again, this potential comes from a special choice of positions of critical points and root vertex among many others. However, this choice does not appear to be manifestly algebraic, as the three walls meeting at the origin scatter infinitely often. In contrast, the rightmost picture shows the mirror base of a dP_6 -degeneration with manifestly algebraic superpotential

$$W_{dP_6}(\sigma) = W_{dP_6}^3(\sigma) = \left(x + y + xy + \frac{1}{x} + \frac{1}{y} + \frac{1}{xy}\right) \cdot t^2 + \left(\frac{y}{x} + \frac{1}{x} + \frac{1}{xy^2} + 2 \cdot \frac{1}{y} + \frac{x}{y}\right) \cdot t^3.$$

These examples illustrate that if we leave the realm of toric geometry, Landau-Ginzburg potentials for del Pezzo surfaces can, at least locally, still be described by Laurent polynomials, as in the toric setting.

7. SMOOTH NON-FANO AND SINGULAR FANO TORIC SURFACES

Having studied smooth Fano surfaces, we now show that our approach also works if we admit Gorenstein singularities or drop the Fano condition.

7.1. Singular Fano toric surfaces. A toric variety is Fano with at most Gorenstein singularities if and only if its anti-canonical polytope is reflexive. There are 16 reflexive polytopes in dimension two, among which five give rise to smooth varieties, studied in the last section. The remaining eleven polytopes are characterized by the property that the dual polytope has at least one integral boundary point that is not a corner. Call the variety associated to such a polytope a *singular toric del Pezzo surface*. Let $\Xi \subset M$ be a two-dimensional reflexive polytope and denote by Σ the fan constructed by cones over faces of Ξ . To study the Landau-Ginzburg model of the Fano variety $X := X(\Xi) := X(\Sigma)$, we relax Definition 6.1 to allow for singular del Pezzo surfaces as well. By polarizing X by a smooth divisor D in the class $-K_X$, we obtain an affine manifold $(B, \mathcal{P}, \varphi)$ with straight boundary, as before. The uniqueness of these models is immediate, as the proof of Theorem 6.4 applies without any change, that is

Corollary 7.1. *The intersection complex $(\check{B}, \check{\mathcal{P}})$ of a distinguished toric degeneration of del Pezzo surfaces is unique. \square*

Theorem 7.2. *Let $(\mathfrak{X} \rightarrow \mathrm{Spec}(k[[t]]), D)$ be a distinguished del Pezzo degeneration, where $X := X(\Xi)$ is singular and Ξ is a reflexive polytope. Denote by $\Xi(0)$ the set of vertices of Ξ . Then there is a choice of root vertex $x \in \sigma$ such that the superpotential on the bounded cell $\sigma_0 \subset \check{B}$ is given by*

$$W(\sigma_0) = \left(\sum_{m \in \partial \Xi_0} z^m \right) \cdot t + \left(\sum_{m \in (\partial \Xi \cap M) \setminus \Xi_0} z^m \right) \cdot t^2.$$

Proof. Let $p \in (\Xi \cap M) \setminus \Xi(0)$ be an integral point. Since Ξ is reflexive, it follows that p is primitive and lies on a one-face ω with vertices p_+ and p_- . The face ω spans a vector space with primitive integral points v_+ and v_- , respectively. Choose a root vertex $x \in \sigma$ and consider a ray in direction p based at x . As this ray ρ hits the wall ω , without loss of generality the focus-focus point attached to ω will scatter it by a (multiple of) v_+ in direction p_+ . By definition of \check{B} , p_+ is a generator of one of the unbounded rays of the structure. Therefore ρ defines a broken line and it is easy to see that every broken line has to be of this form. This proves the theorem. In Figure 7.1 we depicted the dual bases \check{B} for all eleven degenerations with a choice of root vertex and all broken lines. \square

7.2. Hirzebruch surfaces. As another application we will study superpotentials for Hirzebruch surfaces \mathbb{F}_m . We fix the fan Σ in $N \cong \mathbb{Z}^2$ of \mathbb{F}_m to be the fan generated by the four primitive vectors $\rho_0 = (0, 1)$, $\rho_1 = (-1, 0)$, $\rho_2 = (0, -1)$ and $\rho_3 = (1, m)$. Since \mathbb{F}_m is only Fano in the cases $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{F}_1 = dP_1$, for $m \geq 2$ the normal fan of the anti-canonical polytope will not be the fan of a Hirzebruch surface. Denote by D_{ρ_i} the torus-invariant divisor associated to ρ_i . Instead of the anti-canonical divisor, we consider a smooth divisor D in the class $m \cdot D_{\rho_0} + m \cdot D_{\rho_1} + m \cdot D_{\rho_2} + D_{\rho_3}$. Then a toric degeneration of the pair (\mathbb{F}_m, D) defines a tropical affine manifold B with straight boundary. B is obtained from the Newton polytope Ξ_D of D by trading corners for focus-focus singularities,

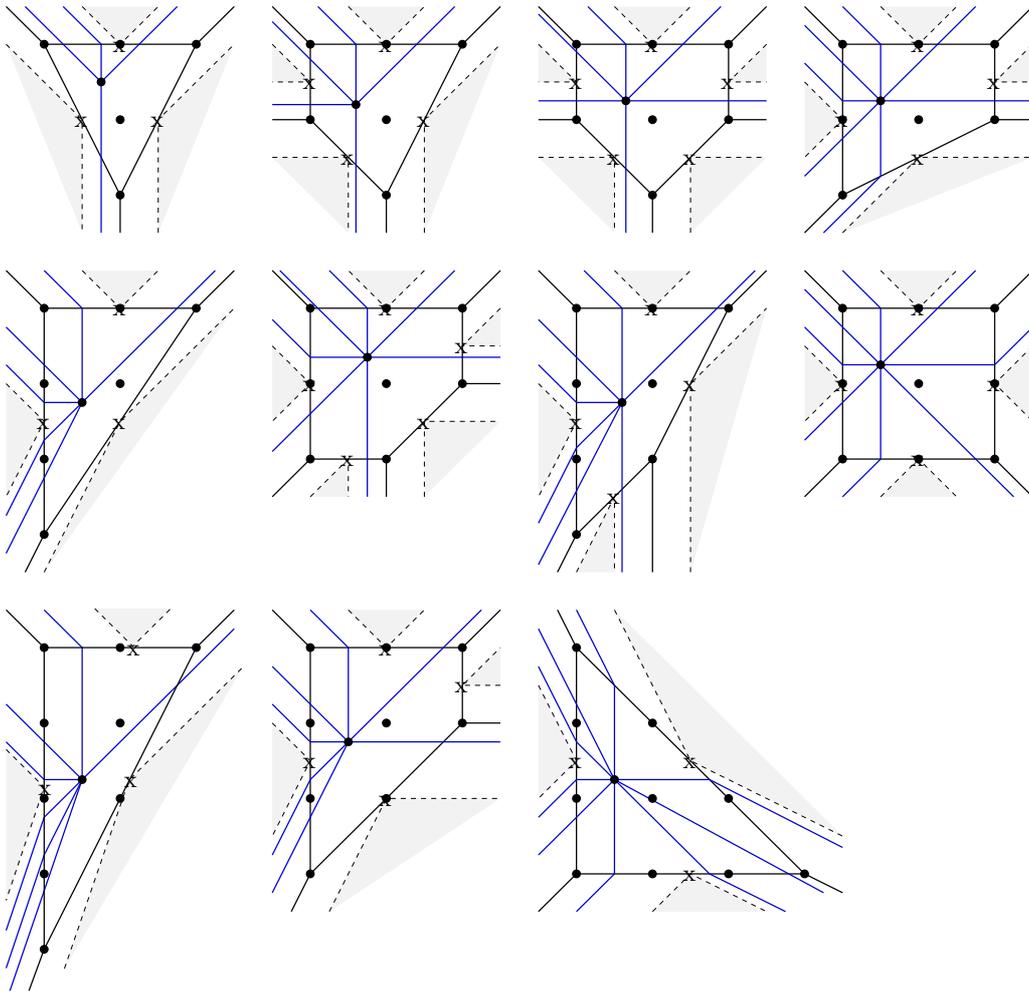


FIGURE 7.1. The broken lines contributing to the proper superpotential of the eleven singular toric del Pezzo surfaces.

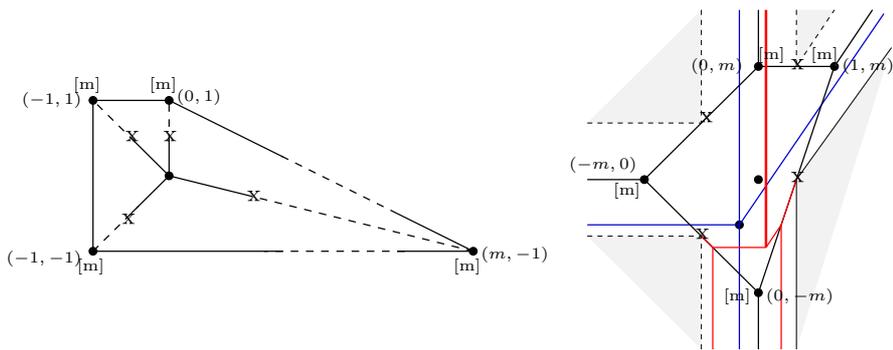


FIGURE 7.2. A straight boundary model for the Hirzebruch surface \mathbb{F}_m with unique tropical $(-m)$ -curve.

as depicted in Figure 7.2 on the left. The polyhedral decomposition \mathcal{P} is the one induced by joining the interior point with the corners and $\varphi(v) = m$ for every $v \in \partial B$, denoted by square brackets in the picture. From this description it follows that the Legendre dual $(\check{B}, \check{\mathcal{P}}, \check{\varphi})$ has four unbounded parallel rays in direction of the ρ_i 's seen from the bounded cell σ_0 and $\check{\varphi}(v) = m$ for all integral points on the boundary of σ_0 . For a choice of root vertex $x \in \sigma_0$ we get the superpotential

$$W_{(\mathbb{F}_m, D)}^1(\sigma) = \left(y + \frac{1}{x} + \frac{1}{y} + xy^m\right) \cdot t.$$

8. THREE-DIMENSIONAL EXAMPLES

So far we restricted ourselves to $\dim B = 2$ to avoid problems arising from unbounded joints and discriminant loci. Nevertheless it is instructive to consider higher dimensional examples as well.

Example 8.1. Starting from \mathbb{P}^3 with its anti-canonical polarization, we obtain a model with a distinguished base by trading corners *and edges* for focus-focus singularities, similar to what we did in two dimensions. More precisely, subdivide the anti-canonical polytope Ξ by introducing six two-faces spanned by the origin and two distinct corners of Ξ . Then choose the discriminant locus Δ to be defined by the first barycentric subdivision of these six affine triangles, as shown in Figure 8.1. This makes the boundary $\partial\Xi$ totally geodesic. By setting $\varphi(v) = 1$ for every corner v of Ξ we arrive at the tropical affine manifold $(B, \mathcal{P}, \varphi)$ as depicted. The discrete Legendre transform $(\check{B}, \check{\mathcal{P}}, \check{\varphi})$, also drawn in Figure 8.1, has four parallel unbounded rays and a discriminant locus $\check{\Delta}$ with six unbounded rays. If these rays were bounded, $\check{\Delta}$ would be homeomorphic to Δ . Call the bounded three-cell σ_0 . Every bounded two-face is subdivided into three chambers by $\check{\Delta}$ and at every vertex of \check{B} three such chambers meet. We claim that

$$W_{\mathbb{P}^3}^1(\sigma_0) = \left(x + y + z + \frac{1}{xyz}\right) \cdot t.$$

To see this we choose a root vertex $x \in \sigma_0$ and consider only broken lines β that pass through one of the three chambers adjacent to $(0, 0, 1)$, that is broken lines that come from infinity in direction $(0, 0, -1)$. The other cases work analogously. Denote the root tangent vector of β by m . Then either $m = (0, 0, 1)$ and β is a straight line or it is broken at the boundary of σ_0 such that the result of scattering of m is $(0, 0, 1)$. By symmetry we may assume that β hits the chamber \mathbf{u} on the face spanned by $(0, 0, 1)$, $(0, 1, 0)$ and $(-1, -1, -1)$. The monodromy invariant plane for \mathbf{u} is spanned by the vectors $(0, -1, 1)$ and $(1, 1, 2)$. Therefore m has to be either $(0, 1, 0)$ or $(-1, -1, -1)$. Thus, the set of root tangent vectors of broken lines ending in x is contained in the set of vertices of \check{B} . Moreover, it is easy to see that indeed every vertex occurs exactly once, by following a ray starting at x in direction m and breaking at a wall in $\partial\sigma_0$ if necessary.

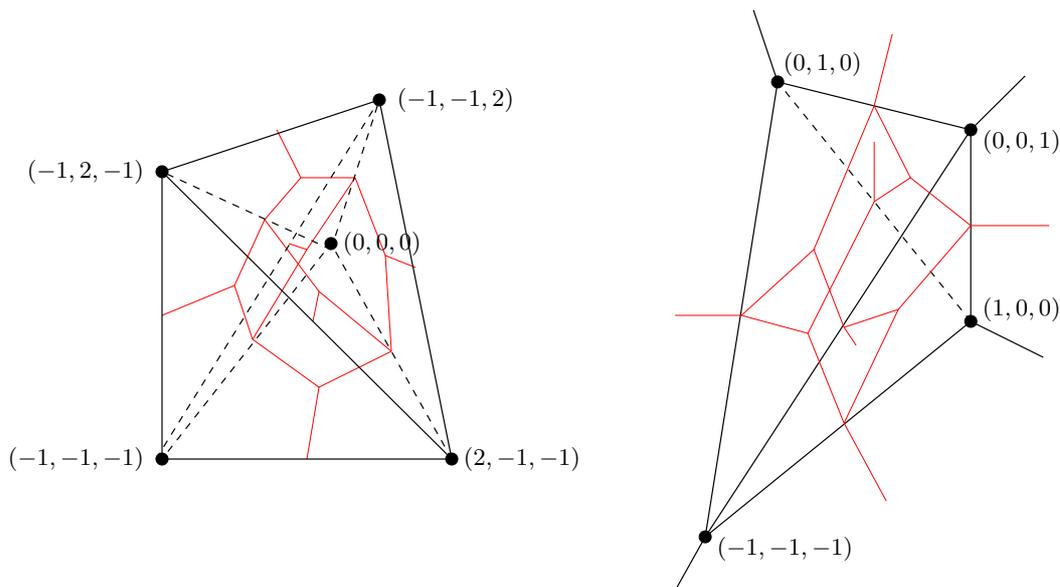


FIGURE 8.1. The distinguished model of \mathbb{P}^3 and its affine Legendre dual.

Example 8.2. Consider the reflexive polytope Ξ depicted in the upper left in Figure 8.2 and denote its polar polytope by $\check{\Xi}$. We will study cone and fan picture for this polytope and the behaviour of the superpotential under an MPCP subdivision of Ξ .

Fan picture: First view Ξ as the anti-canonical polarization on the toric variety $X := X(\Xi)$ associated to its normal fan Σ . In exactly the same way as before we arrive at a tropical affine manifold $(B, \mathcal{P}, \varphi)$ and its dual as shown in Figure 8.2, with homeomorphic discriminant loci, if we truncate unbounded edges. One can prove that for a choice of root vertex the full potential on the bounded cell $\sigma \subset \check{B}$ is given by

$$W_X(\sigma) = \left(xyz + \frac{1}{xyz} + \frac{x}{yz} + \frac{y}{xz} + \frac{z}{xy} \right) \cdot t$$

and the monomials occurring in this expression are in one-to-one correspondence with ray generators $\rho \in \Sigma$. Now choose a maximal partial crepant projective (MPCP) subdivision $\check{\Xi}$ and construct tropical degeneration data $(\check{B}, \check{\mathcal{P}}, \check{\varphi})$ with a distinguished boundary, as before. The discrete Legendre transform drawn in the lower right picture of Figure 8.2 is then essentially given by cutting off the three middle edges of \check{B} and adapting the structure accordingly. The boundary of the bounded cell $\tilde{\sigma}$ therefore has three two cells that are squares in the affine structure, with tangent planes perpendicular to $(1, -1, -1)$, $(-1, 1, -1)$ and $(-1, -1, 1)$, respectively. By this description one can check by hand that $W_{\Xi}(\sigma) = W_{\check{\Xi}}(\tilde{\sigma})$, so the superpotential does not change on this open part.

Cone picture: Next, consider the normal fan $\check{\Sigma}$ of $\check{\Xi}$, that is the fan generated by cones over faces of Ξ , and denote the resulting toric variety by \check{X} . Then proceed as before to

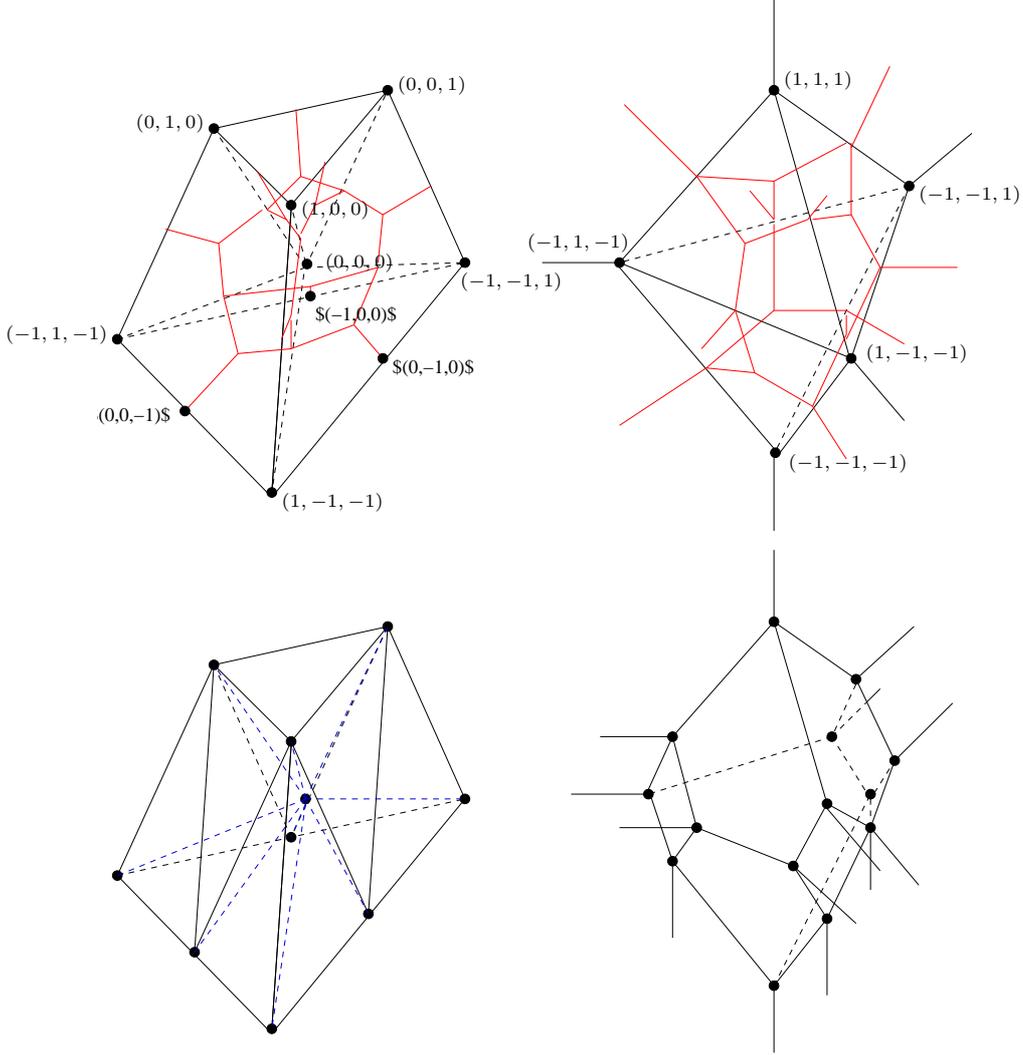


FIGURE 8.2. A reflexive polytope with MPCP subdivision and dual bases

obtain a straight boundary model from $\tilde{\Xi}$, which we denote $(B', \mathcal{P}', \varphi')$ for notational reasons. Along the same lines as in the proof of Theorem 7.2 we see that for a generic choice of root vertex in the unique bounded cell $\tilde{\sigma}'$ the superpotential for \tilde{X} is given by

$$W_{\tilde{X}}(\tilde{\sigma}') = \left(x + y + z + \frac{x}{yz} + \frac{y}{xz} + \frac{z}{xy}\right) \cdot t + \left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) \cdot t^2,$$

which is closely related to the MPCP resolution $\tilde{X} \rightarrow \tilde{X}$ induced by $\tilde{\Xi}$, as this is obtained by introducing the three new ray generators $(-1, 0, 0)$, $(0, -1, 0)$ and $(0, 0, -1)$.

8.1. Outlook. Most of the results presented in this section are expected to hold in higher dimensions. For example the uniqueness of distinguished toric models and the stabilization of the superpotential seem to hold in all dimensions. Furthermore, it would be interesting to compute Landau-Ginzburg models for non-toric Fano threefolds with our method.

In [Pr1, Pr2] so called *very weak Landau-Ginzburg potentials* are found. The terms and coefficients of these Laurent polynomials have to be chosen very carefully. As the potentials presented there do not come from a specific algorithm, but rather are written down in an ad hoc way, one would like to have an interpretation of the terms occurring. One might ask whether there are toric degenerations reproducing the potentials in [Pr1, Pr2] via tropical disk counting, as in the examples here.

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