On Functional Calculus Estimates

Felix L. Schwenninger

Promotiecommissie:

Voorzitter en secretaris:

prof. dr. P.M.G. Apers	Universiteit Twente

Promotor:

prof. dr. H.J. Zwart	Universiteit Twente

Referent:

prof. dr. M. Haase	Christian-Albrechts-Universität zu Kiel

Overige leden:

prof. dr. J.R. Partington	University of Leeds
prof. dr. A.C. Ran	Vrije Universiteit Amsterdam
prof. dr. A.A. Stoorvogel	Universiteit Twente
prof. dr. A.E. Veldman	Universiteit Twente
prof. dr. L. Weis	Karlsruher Institut für Technologie

UNIVERSITEIT TWENTE.

Universiteit Twente
Faculteit Elektrotechniek, Wiskunde en Informatica
Vakgroep Hybrid Systems

Dutch Institute for Systems and Control (DISC)

This research was financially supported by the Netherlands Organisation for Scientific Research (NWO) Project number 613.001.004.

CTIT Ph.D. Thesis Series No. 15-374
Centre for Telematics and Information Technology
P.O. Box 217, 7500 AE, Enschede, The Netherlands.

ISBN: 978-90-365-3962-3
ISSN: 1381-3617 (CTIT Ph.D. Thesis Series No. 15-374)
DOI: 10.3990/1.9789036539623

On Functional Calculus Estimates

PROEFSCHRIFT

ter verkrijging van
de graad van doctor aan de Universiteit Twente,
op gezag van de rector magnificus,
prof.dr. H. Brinksma,
volgens besluit van het College voor Promoties
in het openbaar te verdedigen
op vrijdag 25 september 2015 om 16.45 uur

door

Felix Leopold Schwenninger

geboren op 29 september 1987
te Wenen, Oostenrijk

Dit proefschrift is goedgekeurd door de promotor

prof. dr. H.J. Zwart

Preface

This digital version of the thesis only includes a shortened acknowledgements section.

First and foremost I want to express my deepest gratitude to my supervisor *Hans Zwart*. I was in the luxurious situation to be able to talk to him nearly whenever I wanted. His door was not only open for any type of mathematics, but also for a lot of other interesting discussions. We share the passion for trying mathematical puzzles on our own (which is sometimes not very efficient when reading papers) and I am really thankful for the freedom he has left me in doing research.
Hans, I am not the first to say this, but I can only agree: Your guidance is the best a PhD student can hope for. *Van harte bedankt voor de mogelijkheid om samen met jou onderzoek te kunnen doen.*

My PhD project has been funded by the Netherlands Organisation for Scientific Research (NWO), grant no. 613.001.004. This gave me the opportunity to make a passion to my profession – and at the same time to get to know a new country.

There are several people with whom I had the pleasure to discuss my research with. In particular, I would like to thank *Markus Haase* and *Jan Rozendaal* for fruitful and very interesting conversations on functional calculus.

I am also very grateful to the other members of my graduation committee, *Anton Stoorvogel, Jonathan Partington, André Ran, Arthur Veldman and Lutz Weis* for doing me the honor to serve on the committee and for reading the final version of my thesis.

In fall 2014, I visited the Department of Mathematics at Virginia Tech, USA. I am very grateful to *Joseph Ball* for hosting me and also giving me the opportunity to meet several people from slightly different mathematical fields. *Chris Beattie* and *Serkan Gugercin* introduced me to *Model Order Reduction* while having coffee at *Bollo's*. I really enjoyed these discussions (and highly appreciated the good coffee). *Chris* and *Serkan*, I am very grateful to both of you for spending so much time with me. Moreover, I would like to thank *Mark Embree* and *Eitan Tadmor* for very interesting discussions around the *Kreiss Matrix Theorem*. In particular, I am indebted to *Eitan Tadmor* for sharing an afternoon of his time with me, and providing me with the

unpublished proof of a 30-year-old result in [**Tad86**]. During my time in Blacksburg, parts of Chapter 4 of this thesis were written.

I would like to thank *Adam Bobrowski* for introducing Hans and me to the problem whether, for a cosine family C, $\sup_{t\geqslant 0}\|C(t)-I\| < 2$ implies $C(t) = I$. This was the starting point of the work presented in Chapter 6. Furthermore, I am grateful to *Wolfgang Arendt* for pointing out his $0-3/2$ law in [**Are12**, Theorem 1.1 in Three Line Proofs] to Hans and me. A warm word of thanks goes to *Wojtek Chojnacki* for providing me with his preprints of [**Cho15a, Cho15b**]. I am grateful for *Jean Esterle*, for discussions on the *zero-two* laws and for drawing my attention to his preprints of [**Est15a, Est15b, Est15c**] which were the basis for the results in Section 6.4.

In the past years, I was privileged to participate in several workshops where I learned a lot of mathematics, but also got in touch with many interesting people from all over the world. In particular, I would like to thank all the organizers of the yearly *Internet Seminar on Semigroups*, founded by *Rainer Nagel* and his group in Tübingen. Without these seminars, I would not have come in contact with Hans and his PhD project. Furthermore, I am very grateful for the *Elgersburg Schools*, organized by *Achim Ilchmann, Timo Reis* and *Fabian Wirth*, as well as the *Dutch Institute for Systems and Control (DISC)*, both giving me the opportunity to learn about the many facets of systems theory. I would also like to thank *Birgit Jacob* for nominating me as a junior fellow of the GAMM (International Association of Applied Mathematics and Mechanics).

Enschede, September 2015

Contents

CHAPTER 1

Introduction

1.1. A motivating example

Consider the linear differential equation

$$\begin{cases} \dot{x}(t) = Ax(t), \quad t > 0, \\ x(0) = x_0, \end{cases} \tag{1.1}$$

with $A \in \mathbb{C}^{N \times N}$ being a matrix of dimension N and $x(t), x_0 \in \mathbb{C}^N$. Clearly, the solution is given by the matrix exponential

$$e^{tA} = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n,$$

through $x(t) = e^{tA} x_0$. By using Jordan's normal form for A, it follows that the boundedness of the solution in t and the stability w.r.t. the initial condition can be 'read off' from the eigenvalues of A. For example, it can be shown that

$$\sup_{t \geqslant 0} \|x(t)\| < \infty, \quad \text{for all initial conditions } x_0,$$

if and only if the spectrum of A, $\sigma(A)$, is a subset of the closed left half-plane $\overline{\mathbb{C}_-}$ and for $\lambda \in \sigma(A) \cap i\mathbb{R}$ the geometric and algebraic multiplicities coincide.

However, for a large dimension N it can be difficult to actually compute e^{tA} and the spectrum $\sigma(A)$ correctly. Therefore, one may drop the idea of deriving the solution exactly, and instead make use of a numerical method to determine an approximation for x.
Let us fix a uniform stepsize $h > 0$ and let x_n denote an approximation to x at point nh, $n \in \mathbb{N}$. One of the most simple numerical methods is derived when replacing (1.1) by

$$\begin{cases} \dfrac{x_{n+1} - x_n}{h} = A\left(\dfrac{x_{n+1} + x_n}{2}\right), \quad n > 0, \\ x_0 = x(0), \end{cases} \tag{1.2}$$

which is known as the *Crank–Nicolson scheme*[1] **[CN47]**.

[1]For this special equation with A being a matrix, the method can also be seen as the *implicit midpoint rule* or the *trapezoidal rule*.

By rearranging terms, (1.2) yields

$$x_{n+1} = T^n x_0 := \left(\left(I + \tfrac{h}{2}A\right)\left(I - \tfrac{h}{2}A\right)^{-1} \right)^n x_0, \qquad n > 0, \tag{1.3}$$

where I denotes the identity matrix and we assume that $I - \frac{h}{2}A$ is invertible. Hence, we have an explicit formula for the numerical solution x_n.

Naturally, one might ask whether x_n is a 'good approximation' for x. There are various aspects of what a 'good approximation' means. We are particularly interested in the following questions concerning the asymptotic behavior: *Is the numerical solution* x_n *bounded in* n *for any initial condition* x_0*, if we know that for all initial conditions the exact solution is bounded in* t*?* and if so: *How do these bounds depend on each other?* Since T and e^{tA} are linear, these questions are equivalent to the following:

(Q1) Is T *power-bounded*, i.e., $\sup_{n\in\mathbb{N}} \|T^n\| < \infty$, if $\sup_{t\geqslant 0} \|e^{tA}\| < \infty$?

(Q2) How does $\mathrm{Pb}(T) = \sup_{n\in\mathbb{N}} \|T^n\|$ depend on $M_A = \sup_{t\geqslant 0} \|e^{tA}\|$?

In both these questions the matrix norms are the induced norms. As our situation is finite-dimensional, question (Q1) has an affirmative answer. To see this, we observe that by (1.3), T can be written as $T = \tau\left(\frac{h}{2}A\right)$ for the function

$$\tau : z \mapsto \frac{1+z}{1-z},$$

which is called the *Cayley transform*. Here, the definition

$$\tau\left(\tfrac{h}{2}A\right) = \left(I + \tfrac{h}{2}A\right)\left(I - \tfrac{h}{2}A\right)^{-1}$$

is formally 'clear' as τ is rational and we assumed that $I - \frac{h}{2}A$ is invertible. Let us remark that in general 'inserting an operator in a scalar function' is less obvious and in fact, crucial (see Section 1.2).
Since the Cayley transform maps the closed left half-plane $\overline{\mathbb{C}_-}$ onto the closed unit disc $\overline{\mathbb{D}}$, one can show that the spectral conditions on A for bounded e^{tA} (see above) translate into corresponding conditions for T. This fully answers (Q1).

As for (Q2), relating the bounds $\mathrm{Pb}(T)$ and M_A is not as simple. This question can actually be traced back to Kreiss **[Kre62]** who gave a first estimate. Finally Spijker **[Spi91]** proved that

$$\mathrm{Pb}(T) \leqslant e \cdot (N+1) \cdot M_A, \tag{1.4}$$

where e is the Euler constant and N the dimension of the space. For a discussion about the sharpness of this estimate we refer to Chapter 4.

We remark that studying $\sup_{t\geqslant 0} \|e^{tA}\|$ and $\sup_{n\in\mathbb{N}} \|T^n\|$ is also crucial for *stability* in terms of the evolution of errors in the initial conditions.

Let us now leave the finite-dimensional setting, and ask about corresponding results for *infinite-dimensional spaces*, where matrices get replaced by, possibly unbounded, operators A. In other words, (1.1) becomes a p.d.e. This comes with some difficulties. First of all the solution theory of (1.1) is not clear a priori. To obtain existence and uniqueness of solutions of (1.1), we assume that A generates a C_0-*semigroup of operators*, which we also denote by e^{tA}, on a Banach space X. Such a semigroup can be seen as the infinite dimensional analog of the matrix exponential.
As well as for matrices, boundedness of the solutions x, independent of the initial conditions, can be characterized by $\sup_{t\geqslant 0}\|e^{tA}\| < \infty$. However, the characterization in terms of the eigenvalues of A does not hold any more. If $\sup_{t\geqslant 0}\|e^{tA}\| < \infty$, it can be shown that $T = (I - \frac{h}{2}A)(I - \frac{h}{2}A)^{-1}$ is a bounded operator. Hence, we can pose question (Q1) again.
It is well-known that in general the answer to (Q1) is 'no' (which is not surprising in the view of estimate (1.4) which depends on the dimension N). However, under certain additional assumptions the answer is 'yes'. For instance, if the semigroup is *analytic*, or if the space X is a Hilbert space and $M_A = 1$. In such cases, we can study (Q2) and search for the optimal bound of $\mathrm{Pb}(T)$.

In this general infinite dimensional setting, Question (Q1) has also become known as *Cayley Transform problem* or the question of *Stability of the Crank-Nicolson scheme*. Despite the negative answer for general Banach spaces, it remains a notoriously open problem for bounded semigroups on Hilbert spaces. In this thesis (Chapter 5) it is shown that the latter is equivalent to the same question for *exponentially stable* semigroups on Hilbert spaces. We recall that e^{tA} is exponentially stable if there exist constants $M_A \geqslant 1, \omega > 0$ such that $\|e^{tA}\| \leqslant M_A e^{-\omega t}$ for all $t \geqslant 0$.

Taking a general viewpoint once more, we can understand $T = \tau\left(\frac{h}{2}A\right)$ as $f(A)$, i.e., a scalar function f 'applied' to A. For example, different functions f could describe different numerical schemes. Like for τ, the definition of $f(A)$ is 'straight-forward' if f is a rational function bounded on $\mathbb{C}_-$. Besides, we have already seen examples of $f(A)$ for a more complicated f, when we (formally) defined e^{tA}. In the view of A-*stability* for numerical schemes, it is natural to consider functions f which are bounded and analytic on $\mathbb{C}_-$, with $\sup_{z\in\mathbb{C}_-}|f(z)| \leqslant 1$. Let us assume that for such f we are able to define $f(A)$ in a certain way and that there exists a constant $K > 0$ (independent of f) such that

$$\|f(A)\| \leqslant K \sup_{z\in\mathbb{C}_-}|f(z)|. \tag{1.5}$$

Then, in particular, it follows that $\|T^n\| = \|\tau(\frac{h}{2}A)^n\| \leqslant K \cdot \sup_{z\in\mathbb{C}_-}|\tau(z)^n| = K$. Therefore, (Q1) has an affirmative answer whenever an estimate of the form (1.5) holds.
Although in general we cannot expect such estimates to hold, this property provides another class of examples with a positive answer to (Q1).

This discussion leads us to the general study of 'making sense of $f(A)$', which is known under the term *functional calculus*. In this thesis, estimates of the form (1.5) will be called *functional calculus estimates*. These notions are the subject of the following section.

1.2. Functional calculus estimates

Functional calculus is, loosely speaking, the procedure of defining a new operator $f(A)$ as the 'evaluation' of an operator A in a function f.
Probably, the simplest example of inserting an operator into a function is the square A^2 of a square matrix A with $f(z) = z^2$. Clearly, this definition can be extended to general powers A^n and polynomials $p(A)$ of a matrix. Other examples, which we have already seen in Section 1.1, are matrices

$$(I - A)^{-1}, \tau(A), e^{tA}.$$

However, the work of von Neumann **[vN96]** and Stone **[Sto90]** for self-adjoint operators on Hilbert spaces more than 80 years ago is actually considered as the beginnings of the theory of functional calculus. The word *calcul fonctionnel* is a little bit older and can be traced back to Frèchet **[Fré06]**, see also **[Haa05]** and **[Haa06a]** for more detailed historical remarks.
Since then, many different types of functional calculi[2] have been studied, all of them sharing some basic intuition what a *functional calculus* should be. In this section, we give a more precise explanation of how this notion *can* be generally defined. The structure of our presentation in this introduction shares the spirit of defining functional calculus as

.. a purely algebraic concept
and regard continuity properties as being accidental.[3]

Somewhat in contrast to the above viewpoint, we will not abandon *continuity properties* from this section, but instead divide the presentation into 'Algebra' and 'Analysis'. In short, the former deals with the definition of a *functional calculus* and the latter contains *functional calculus estimates*.

Algebra. In the following we try to make the intuitive understanding of a functional calculus rigorous. However, as a precise definition strongly depends on the various situations, this attempt can only succeed partially. We distinguish three steps.

[2]The plural of the latin word *calculus* is *calculi*. Therefore, although the word *calculuses* might be a possible plural form in the English language, we will, as in the majority of the literature, use the latin version.
[3]citation from M. HAASE, *The Functional Calculus for Sectorial Operators*, volume 169 of *Operator Theory: Advances and Applications*. Birkhäuser Verlag, Basel, 2006, page 125.

First of all, *functional calculus aims for defining objects* of the form

$$f(A), \tag{1.6}$$

for functions f and some operator A. At this point let us make the following assumptions. Throughout this thesis, A will always be a *linear* operator from a linear space $D(A) \subset X$ to X and f is assumed to be *scalar-valued*, i.e., a mapping from (a subset of) the complex numbers $\mathbb{C}$ to $\mathbb{C}$.[4]

However, the term *functional calculus* does not only concern the definition of $f(A)$. It also covers some properties of the mapping $f \mapsto f(A)$, which intuitively should be satisfied. This brings us to an important point in the comprehension of functional calculus. The operator A is fixed, while the function f is variable in a certain class.
Hence, as a **second step** towards a definition, we see *functional calculus as a mapping*

$$\Phi : F \to \text{operators on } X, \quad f \mapsto f(A), \tag{1.7}$$

for a class[5] of functions F and a given operator A. Therefore, 'an F-calculus' always refers to a fixed operator A and a class of functions F' in the following. This also means that Φ depends on A.

Without making more assumptions on Φ, F and A, which can depend on specific situations, it seems impossible to get a 'more rigorous' definition of a functional calculus. Typically, the class of functions has some algebraic structure, mostly a *group* or an *algebra* (over $\mathbb{C}$), which we want to preserve by the mapping Φ. In the above example of matrix polynomials, F is the algebra of polynomials $\mathbb{C}[z]$ and $\Phi : p \mapsto p(A)$ is linear and multiplicative, i.e., a homomorphism from $\mathbb{C}[z]$ to the algebra of square matrices on the space X.
Therefore, the ultimate goal is to obtain a *homomorphism* from the functions F to the operators on X. If we assume that F is an algebra of functions with operations $+, \cdot$ and that Op with $+_{\text{Op}}, \circ$ is an algebra of operators, then we symbolize such a homomorphism by

$$\Phi : (F, +, \cdot) \to (\text{Op}, +_{\text{Op}}, \circ), \quad f \mapsto f(A). \tag{1.8}$$

This represents the **third step** of the approach to define a *functional calculus*.
If we further assume that F is a vector space, a functional calculus is a mapping $f \mapsto f(A)$ such that

- $f \mapsto f(A)$ is linear,
- for $f = f_1 \cdot f_2 \in F$ we have $f(A) = f_1(A) \circ f_2(A)$,
- if there exist unity elements $1 \in F$ and $I \in \text{Op}$, then $\Phi(1) = I$.

[4] In the view of applications, these requirements are already quite strict. We observe that, for instance, there exists powerful theory for operator-valued functional calculus.
[5] The word *class* is not really precise. In general, this is nothing more than a set. However, we use the word *class* to indicate that this set may have some algebraic or topological structure.

Some examples. An example we have already seen in Section 1.1 is given by the matrix exponentials e^{tA} of a square matrix A. In fact, it can be shown that $(z \mapsto e^{tz}) \mapsto e^{tA}$ is a group homomorphism mapping $F = \{f_t(z) = e^{tz} : t \in \mathbb{R}\}$ (equipped with pointwise multiplication) to the square matrices e^{tA}, $t \geqslant 0$.
Furthermore, it is well-known that self-adjoint (or unitary operators) on a Hilbert space have a functional calculus with $\mathcal{A}$ being the set of continuous functions from $\mathbb{R}$ (or the torus $\mathbb{T}$ respectively) to $\mathbb{C}$, (von Neumann [**vN96**]). Further examples are the *Hille-Phillips calculus* [**HP57**] and the *Riesz-Dunford calculus* [**DS88**], which will both be discussed in this thesis, see Chapters 2 and 3.

Extending the homomorphism. Sometimes a homomorphism is not possible for the chosen pair of functions F and the operator A. In this case, one can try to weaken the homomorphism property by considering a subalgebra $\mathcal{E}$ of F first, on which a homomorphism Φ is possible, and extend Φ to F in an algebraic way.
In this thesis the homomorphisms Φ will mostly map to (the algebra of) bounded operators on some Banach space X. The mentioned extension of Φ will then typically map to *unbounded* operators. As the domains of the operators $f(A)$ may differ then, the above-listed properties have to be seen formally, and, in general, need to be made rigorous. Next, we introduce such an extension argument for a particular class of functions. See e.g. [**Haa06a**, Chapter 1] and the references therein.

Holomorphic calculus - some background. The following is a brief overview on the construction of the *holomorphic functional calculus*, which was abstractly done by Haase [**Haa05, Haa06a**][6]. Let Ω be an open set in the complex plane, and F be an algebra (including the 1-function) of holomorphic functions on Ω, equipped with pointwise multiplication. Further, let $\mathcal{E}$ be a subalgebra of F and let $\mathcal{B}(X)$ denote the algebra of bounded operators on some Banach space X (with unity I). Assume that Φ is an algebra homomorphism from $\mathcal{E}$ to $\mathcal{B}(X)$. Following Haase, we call Φ *primary calculus* and the tuple $(\mathcal{E}, F, \Phi)$ *abstract functional calculus*.
To extend the primary calculus to a larger set of functions in F we use a *regularization argument*, which can be sketched as follows. The set of *regularizers* is defined as

$$\text{Reg} = \{e \in \mathcal{E} : \Phi(e) \text{ is injective}\},$$

and the functions $f \in F$ which are *regularizable* by elements in Reg are denoted by

$$\mathcal{M}_{reg} = \{f \in F : \exists e \in \text{Reg with } (ef) \in \mathcal{E}\}.$$

If Reg is not empty, then for any $f \in \mathcal{M}_{reg}$, we can define

$$\Phi_{ext}(f) = [\Phi(e)]^{-1}\, \Phi(ef),$$

[6]The construction for sectorial operators already appeared in [**McI86**]. See also [**deL95**] for the first more general approach.

which can be shown to be independent of the choice of $e \in \text{Reg}$. By construction, Φ_{ext} is a mapping from $\mathcal{M}_{reg}$ to the closed (not necessarily bounded) operators[7] on X, which extends Φ. Sometimes we will identify Φ with its extension Φ_{ext}. Therefore, if $\mathcal{M}_{reg} = F$, i.e., every element in F is regularizable, then Φ can be seen as a mapping from F to the closed operators on X.

We remark that at this moment the operator A for which we want to define the calculus is not present. This is 'hidden' in the definition of Φ, see also [**Haa06a**, Chapter 1.3]. Since we see Φ as a mapping $f \mapsto f(A)$, we 'define' $f(A) := \Phi_{ext}(f)$ for $f \in \mathcal{M}_{reg}$. Furthermore, the above extension procedure works for any commutative unital algebra F, not necessarily holomorphic functions.

Analysis. So far, we have not considered any topological properties of a functional calculus, which, by (1.7) and (1.8), we have defined as a mapping/homomorphism $\Phi : F \to \text{Op}$. However, in the examples we can see that typically the function algebra and the operator algebra have a norm. In this case we can for instance study whether Φ is continuous. Let us from now on assume that the algebras F and Op are normed. By linearity of Φ, it follows that $\Phi : F \to \text{Op}$ is continuous if and only if there exists a constant $c > 0$ such that

$$\|\Phi(f)\|_{\text{Op}} \leqslant c\|f\|_F, \quad \forall f \in F. \tag{1.9}$$

In this case, the functional calculus (defined by Φ) is called *bounded*.

We want to see inequality (1.9) as an example for more general *functional calculus estimates* of the form

$$\| [\mathcal{K}(f)] (A)\|_{\text{Op}} \leqslant \|\mathcal{M}(f)\|_F, \quad \forall f \in F. \tag{1.10}$$

for mappings $\mathcal{K}, \mathcal{M} : F_0 \subset F \to F$ and a set F_0. Obviously, for $F_0 = F$, $\mathcal{K}(f) = f$ and $\mathcal{M}(f) = cf$ we arrive at the case above.
We admit that this definition may sound like abstract nonsense and, due to its generality, it does not seem to add deeper understanding in unifying concepts for functional calculus. In fact, there are even 'other' estimates for functional calculi which do not fit into this definition[8]. We rather see it as a notion to cover several estimates we can consider for functional calculi in this thesis.

We finish this section with the following comment about the *boundedness* of functional calculi derived by the extension procedure above. Let $\Phi = \Phi_{ext}$ be an extension of a homomorphism as derived in 1.2, and assume (for simplicity) that $\mathcal{M}_{\text{reg}} = F$. Thus, Φ maps F to the closed operators on a Banach space X. We say that the functional calculus Φ is *bounded*, if (1.9) holds with $\text{Op} = \mathcal{B}(X)$, which is the algebra of

[7]Note that the closed operators on a Banach space X, with the usual product $AB = A \circ B$, do not form an algebra.
[8]We refer to [**CEP15b**]. where *Lower estimates near the origin for functional calculus* are considered.

bounded operators on X (equipped with the induced operator norm). However, we remark that this definition implicitly requires that $\Phi(f)$ is in $\mathcal{B}(X)$ for every $f \in F$. Likewise, we say that a functional calculus is *unbounded*, if there exists an $f \in F$ such that $f(A)$ is not bounded.

1.3. Functional calculus estimates for cosine families

It is easy to see that if the (maximal) distance between a cosine function $\cos(t\sqrt{-a})$, with $a \leqslant 0$[9], and the constant function 1 is less than 2, i.e, if

$$\sup_{t\in\mathbb{R}} |\cos(t\sqrt{-a}) - 1| < 2, \tag{1.11}$$

then $\cos(t\sqrt{-a}) = 1$ for all $t \in \mathbb{R}$, or equivalently, $a = 0$. In other words, 1 is an isolated point in the set $\{\cos(t\sqrt{-a}) : a \geqslant 0\}$ equipped with the metric induced by the supremum norm. We also observe that the implication fails if the number 2 in (1.11) is replaced by any larger number. For a square matrix or, more general, for a bounded operator A on a Banach space X, we can define

$$\mathrm{Cos}(t) = \sum_{n=0}^{\infty} \frac{(-1)^n t^{2n}}{(2n)!}(-A)^n, \quad t \in \mathbb{R}. \tag{1.12}$$

It is easily seen that the norm of the sum can be majorized by $\cosh(t\sqrt{\|A\|})$. Hence, Cos defines an X-valued function on $\mathbb{R}$. It can be shown that the function Cos 'behaves as one would expect from the scalar cosine'. For instance, $\mathrm{Cos}(0) = I$ (where I denotes the identity), and *d'Alembert's identity* holds, i.e.,

$$\mathrm{Cos}(s+t) + \mathrm{Cos}(s-t) = 2\mathrm{Cos}(s)\mathrm{Cos}(t), \quad \forall s,t \in \mathbb{R}. \tag{1.13}$$

Furthermore, $\frac{d^2}{dt^2}C(t) = AC(t)$. Hence, Cos can be seen as the natural analog to the scalar cosine function for matrices or bounded operators. Coming back to the initial implication about the distance between $\cos(t\sqrt{-a})$ and 1, we can study a similar question for Cos. It can be shown that the corresponding implication,

$$\textit{if } \sup_{t\in\mathbb{R}} \|\mathrm{Cos}(t) - I\| < 2, \textit{ then } \mathrm{Cos}(t) = I \textit{ for all } t \in \mathbb{R}, \tag{1.14}$$

holds as well.

Looking at (1.12), we can view Cos in terms of a functional calculus for fixed A. Namely, by interpreting $\mathrm{Cos}(t)$ as $f_t(A)$ for $f_t(z) = \cos(t\sqrt{-z})$, $z \leqslant 0$, and using the power series of the cosine to define a mapping $f \mapsto f(A)$ for $f \in F_1 = \{f_t : t \in \mathbb{R}\}$.

Obviously, F_1 is not closed under (pointwise) summation. However, we can extend the mapping $f \mapsto f(A)$ to the linear combinations F of F_1. Clearly, $g_t(z) = f_t(z) - 1$ defines a function in F.

---

[9]Of course, by setting $b = \sqrt{-a}$, we could write $\cos(tb)$ instead of $\cos(t\sqrt{-a})$ here. The reason for our notation is that a equals the second derivative of $\cos\sqrt{-a}t$ at 0. This notation will be useful later on.

From this viewpoint the premise in (1.14) becomes

$$\sup_{t\in\mathbb{R}} \|g_t(A)\| < 2,$$

and can thus be seen as a *functional calculus estimate* of the form (1.10) for $F_0 = \{g_t : t \geqslant 0\}$, $\mathcal{K}(f) = f$ and $\mathcal{M}(f) \equiv 2$.

Our goal is to study the implication in (1.14) for general operator-valued *cosine families* (or *cosine functions*) C.

A *cosine family* $t \mapsto C(t)$ is defined as a function from $\mathbb{R}$ to the algebra of bounded linear operators on X such that $C(0)$ equals the identity I and *d'Alembert's identity*, $C(s+t) + C(s-t) = 2C(s)C(t)$ for $s, t \in \mathbb{R}$, is satisfied, cf. (1.13).

Furthermore, if the trajectories $c_x : t \mapsto C(t)x$ are continuous for all $x \in X$, then the cosine family is called *strongly continuous*. For such C one can define its *generator* A as, roughly speaking, the operator mapping x to the second derivative of c_x at 0, $\frac{\partial^2}{\partial t^2} c_x\big|_{t=0}$. As this derivative need not exist for every x, A can be unbounded. For a class of examples for cosine families with unbounded generator, see, e.g., **[BE04]**.

It can be shown that $\frac{d^2}{dt^2} C(t)x = AC(t)x$ for x in the domain of A. Hence, strongly continuous cosine families occur naturally in the solution of abstract second order differential equations of the form

$$\begin{cases} \ddot{x}(t) = Ax(t), \quad t > 0, \\ \dot{x}(0) = x_1, \\ x(0) = x_0, \end{cases} \tag{1.15}$$

where $x_0, x_1 \in X$.

In this thesis, we show that for strongly continuous cosine families C,

$$\sup_{t\in\mathbb{R}} \|C(t) - I\| < 2 \textit{ implies that } C(t) = I \textit{ for all } t \in \mathbb{R}, \tag{1.16}$$

which confirms the intuition we got from the special cases above. This implication had been open so far. In prior work, Bobrowski and Chojnacki **[BCG15]** derived a weaker form of (1.16), where the number 2 was replaced by $\frac{1}{2}$.

Furthermore, we also prove *scaled versions* of the form

$$\sup_{t\in\mathbb{R}} \|C(t) - \cos(t)I\| < 1 \implies C(t) = \cos(t)I.$$

Another question within this scope is whether the weaker condition

$$\limsup_{t\to 0^+} \|C(t) - I\| < 2, \tag{1.17}$$

implies that $\limsup_{t\to 0^+} \|C(t) - I\| = 0$. In other words, does (1.17) cause $C(\cdot)$ to be continuous at 0 in the operator norm? Whereas for the examples cos and Cos above the answer is clearly 'yes', the situation becomes much more difficult for general cosine families with unbounded generator A. So far, only partial results have been

known; for special classes of Banach spaces (more precisely, UMD spaces), such as Hilbert spaces, an affirmative answer was recently given by Fackler **[Fac13]**, using that a cosine family can be represented by a strongly continuous group in that case. On the other hand, Arendt **[Are12]** proved with a beautifully simple argument that for general cosine families, the implication holds if the number 2 in (1.17) is replaced by $\frac{3}{2}$.
We will show that (1.17) indeed implies that $\limsup_{t\to 0^+} \|C(t) - I\| = 0$ for strongly continuous cosine families and relate this result to (1.16).

Such assertions, originating from corresponding questions in semigroup theory, have become known as *Zero-two-laws*. Very recently, our results have been generalized by Chojnacki **[Cho15a]** and Esterle **[Est15b]** independently, dropping the strong continuity assumption on C. In turn, using techniques from Esterle, we show in this thesis that the $\sup_{t\in\mathbb{R}}$ can be weakened to $\limsup_{t\to\infty}$ in the assumption of (1.16).

1.4. Notation and some mathematical background

Let us describe the notation used throughout this thesis. We will use standard notation $\mathbb{C}$, $\mathbb{Z}$, $\mathbb{N}$ for the complex, integer and positive integer numbers, respectively. Further, let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For a set $\Omega \subset \mathbb{C}$, let $\overline{\Omega}$ be its closure and $\partial\Omega$ its boundary in $\mathbb{C}$. By $\mathbb{D}$ we denote the open unit disc in $\mathbb{C}$ and by $\mathbb{T} = \partial\mathbb{D}$ the unit circle. For $\mathbb{K} \in \{\mathbb{C}, \mathbb{R}\}$, we set $\mathbb{K}_- = \{z \in \mathbb{K} : \operatorname{Re} z < 0\}$, $\mathbb{K}_+ = \{z \in \mathbb{K} : \operatorname{Re} z > 0\}$. For $\theta \in (0, 2\pi)$, we define the open sector $\Sigma_\theta = \{z \in \mathbb{C} : z \neq 0, |\arg z| < \theta\}$, and set $\Sigma_0 = (0, \infty)$. The 'mirrored' sector $\mathbb{C} \setminus \overline{\Sigma_{\pi-\theta}}$ is denoted by $\reflectbox{$\Sigma$}_\theta$. By $B_r(z_0)$ we denote the open ball in $\mathbb{C}$ with radius r and centre z_0.
When we write $E(z) \lesssim F(z)$, where E, F are expressions depending on the variable z, we mean that there exists a universal constant K not depending on z such that $E(z) \leqslant KF(z)$ for all z. By $E(z) \sim F(z)$, we mean that $F(z) \lesssim E(z)$ and $E(z) \lesssim F(z)$. For example, $\|f(A)\| \lesssim \|f\|_\infty$ means that there exists a constant C, which may depend on A but not on f, such that $\|f(A)\| \leqslant C\|f\|_\infty$.

Functions. For an interval $I \subset \mathbb{R}$, a Banach space X and $p \in [1, \infty]$, $L^p(I, X)$ is the usual vector-valued L^p-space, where integrals are understood in the Bochner sense. For an open set $\Omega \subset \mathbb{C}$, let $H(\Omega)$ denote the (complex-valued) holomorphic functions on Ω. Let $H^\infty(\Omega)$ be the Banach algebra of bounded holomorphic functions on Ω equipped with the supremum norm $\|\cdot\|_{\infty,\Omega}$. Typical choices will be $\Omega \in \{\mathbb{C}_-, \mathbb{C}_+, \mathbb{D}\}$. For such Ω, the norm of an element $f \in H^\infty(\Omega)$ is attained at the boundary, see the Appendix for details and its vector-valued analog. Moreover, the theory of Hardy spaces allows for an isometric embedding of $H^\infty(\Omega)$ into $L^\infty(\partial\Omega)$ via the limit function at the boundary. We will use this identification without stating it explicitly. We refer to **[Gar07, Dur70, Nik02a]** for details about Hardy spaces (on the disc as well as on half-planes).

Operators and spaces. The operator theory we are dealing with will mostly be on Banach spaces, which in most of the cases will be denoted by X (with norm $\|\cdot\| = \|\cdot\|_X$). Sometimes, we will restrict ourselves to Hilbert spaces. Operators between Banach spaces are always understood to be linear (and single-valued), but not necessarily bounded. We say that A is an *operator on* X, if A maps from the Banach space X to X. $D(A)$ will denote the domain and $R(A)$ the range of A. For closed A, we write $\rho(A)$ for the resolvent set of A and $\sigma(A)$ for the spectrum. If $\lambda \in \rho(A)$, the resolvent $(\lambda I - A)^{-1} = (\lambda - A)^{-1}$ will be abbreviated by $R(\lambda, A)$. For Banach spaces X, Y, let $\mathcal{B}(X, Y)$ (or $\mathcal{B}(X)$ if $Y = X$) denote the Banach algebra of bounded operators from X to Y. For a Banach space X, we denote by X' its (continuous) dual and the duality brackets are denoted by

$$\langle y, x\rangle_{X',X} = \langle x, y\rangle_{X,X'} = y(x), \quad x \in X, y \in X'.$$

For an operator B in $\mathcal{B}(X, Y)$, B' denotes the adjoint, which then lies in $\mathcal{B}(Y', X')$. The Hilbert space adjoint will sometimes be denoted by B^*.

A main framework of this thesis is the theory of operator semigroups. Let us recall the most important facts. For a Banach space X, a function $T : [0, \infty) \to \mathcal{B}(X)$ is called a *semigroup* of operators if the following properties hold.

(i) $T(0) = I$,

(ii) $T(s + t) = T(s)T(t)$ for all $s, t \geqslant 0$.

T is said to be *strongly continuous* if the trajectories $T(\cdot)x$ are continuous for every $x \in X$. Strongly continuous semigroups are also called C_0-*semigroups*. One can show that for every C_0-semigroup T, there exist constants $M \geqslant 1$ and $\omega \in \mathbb{R}$ such that

$$\|T(t)\| \leqslant Me^{t\omega}, \quad \forall t \geqslant 0. \tag{1.18}$$

If a negative ω can be chosen, then T is called *exponentially stable*. For a C_0-semigroup the *generator* A is the operator defined by

$$Ax = \lim_{h \to 0^+} \frac{1}{h}(T(h)x - x) \text{ on } D(A) = \{x \in X : x \text{ such that the limit exists}\}.$$

The Hille-Yosida theorem gives a characterization of semigroup generators A.

THEOREM 1.1 (Hille-Yosida). *Let A be an operator on Banach space and let $M \geqslant 1$ and $\omega \in \mathbb{R}$. Then, A is the generator of a C_0-semigroup T satisfying* (1.18) *if and only if A is closed, densely defined, and for any λ with $\operatorname{Re}\lambda > \omega$, it holds that $\lambda \in \rho(A)$ and*

$$\|R(\lambda, A)^n\| \leqslant \frac{M}{(\operatorname{Re}\lambda - \omega)^n}, \quad \forall n \in \mathbb{N}, .$$

If A is the generator of a C_0-semigroup $T = (T(t))_{t \geqslant 0}$, we will also use the 'notation' $e^{tA} = T(t)$ (which is sometimes even justified by the considered functional calculus).

The Banach space $D(A)$ equipped with the graph norm of A will be referred to by $(X_1, \|.\|_1)$. Unless stated otherwise, all the semigroups we are considering in this thesis are strongly continuous.
A C_0-semigroup $T : [0,\infty) \to \mathcal{B}(X)$ is called *analytic* if for some $\theta \in (0, \frac{\pi}{2}]$ T can be extended to a sector $\Sigma_\theta \cup \{0\} \subset \mathbb{C}$ such that $T(s+t) = T(s)T(t)$ holds for all $s, t \in \Sigma_\theta$, T is analytic on Σ_θ and

$$\lim_{z\to 0, z\in\Sigma_{\theta'}} T(z)x = x, \quad \forall x \in X, \theta' \in (0,\theta).$$

We say that T is a *bounded analytic semigroup* if $T : \Sigma_\theta \to \mathcal{B}(X)$ is an analytic semigroup and $\sup_{z\in\Sigma_\theta} \|T(z)\| < \infty$. We have the following characterization. For a linear operator A on X, $T = e^{tA}$ is a bounded analytic semigroup if and only if A is densely defined, and there exists a $\delta \in [0, \frac{\pi}{2})$ such that

$$\rho(A) \supset \Xi_\delta \text{ and } \sup\{\|zR(z,A)\| : z \in \Xi_\delta\} < \infty.$$

This equivalence shows the relation to *sectorial operators*, which we will introduce in Chapter 3. Another characterization for T being an analytic semigroup is that

$$T(t)X \subset D(A)\ \forall t > 0, \text{ and } \sup_{t\geqslant 0} \|tAT(t)\| < \infty.$$

For an extensive introduction to semigroups we refer to the book by Engel and Nagel, **[EN00]**, see also **[Gol85, HP57, Paz83]**.

Furthermore, a nomenclature of the most important notions which will be defined in each chapter, can be found at end of the thesis.

1.5. Outline of the thesis and main contributions

In this section we collect some short overviews of the chapters in the two parts of this thesis. This also summarizes the main contributions. To make the chapters more self-contained, versions of the following summaries also appear as abstracts at the beginning of each chapter. Moreover, we remark that the style of this thesis allows for a study *à la carte*, which means that the chapters can be read independently. As Chapters 2 to 5 have the H^∞-functional calculus as common theme, they are merged into Part I. Part II only consists of Chapter 6 which deals with *zero-two laws* for cosine families.

Chapter 2. We show that, given a generator of an exponentially stable semigroup on a Banach space, a *weakly admissible* operator $g(A)$ can be defined for any bounded, analytic function g on the left half-plane $\mathbb{C}_-$. This yields an (unbounded) functional calculus. The construction uses a Toeplitz operator and is motivated by system theory. In Hilbert spaces, we even obtain admissibility. Furthermore, it is investigated when a bounded calculus can be guaranteed. For this we introduce the new notion of exact observability by direction. As an application of the approach we show an estimate characterizing the boundedness of the calculus for analytic semigroups on Hilbert spaces. Finally, it is shown that the calculus coincides with the (classical) holomorphic functional calculus derived by an algebraic extension procedure of the Hille-Phillips calculus. Thus, the approach can be seen as an alternate route for introducing the classical H^∞-calculus for strongly continuous semigroups.

Chapter 3. We investigate the boundedness of the H^∞-calculus by estimating the bound $b(\varepsilon)$ of the mapping $H^\infty \to \mathcal{B}(X)$: $f \mapsto f(A)T(\varepsilon)$ for ε near zero. Here, $-A$ generates the analytic semigroup T on a Banach space X and H^∞ is the space of bounded analytic functions on a domain strictly containing the spectrum of A.
In the view of (1.10), this estimate can be seen as a functional calculus estimate for $F_0 = F = H^\infty$, $\mathcal{K}(f) = (z \mapsto e^{-\varepsilon z} f(z))$ and $\mathcal{M}(f) = f$.
We show that $b(\varepsilon) = \mathcal{O}(|\log \varepsilon|)$ in general, whereas $b(\varepsilon) = \mathcal{O}(1)$ for bounded calculi. This generalizes a result by Vitse and complements work by Haase and Rozendaal for non-analytic semigroups. We discuss the sharpness of our bounds and show that single square function estimates yield $b(\varepsilon) = \mathcal{O}(\sqrt{|\log \varepsilon|})$.

Chapter 4. We prove H^∞-functional calculus estimates for Tadmor-Ritt operators T. These generalize and improve results by Vitse and are in conformity with the best known power-bounds for Tadmor-Ritt operators in terms of the constant dependence. In particular, we show estimates of the form $\|p(T)\| \leqslant c(m, n, T) \cdot \|p\|_{\infty,\mathbb{D}}$ for polynomials $p(z) = \sum_{j=m}^{n} a_j z^j$.
With $F_0 = F = \mathbb{C}[z]$, the algebra of polynomials, equipped with the supremum norm, $\mathcal{K}(\sum_i a_i z^i) = \sum_{j=m}^{n} a_j z^j$ and $\mathcal{M}(f) = f$, this estimate can be seen in terms of (1.10). We furthermore show the effect of having discrete square function estimates on these estimates.

Chapter 5. In the previous chapters we have seen some analogy between functional calculus results in *continuous* and *discrete time*. This chapter deals with the transformation from the continuous to the discrete setting via the Cayley transform. This leads to the prominent *Inverse Generator problem* and the *Cayley Transform problem* for C_0-semigroups. We show the equivalence of these two problems and the fact that we can even reduce these problems to the case where the semigroup is exponentially stable. Furthermore, we give an overview on existing results in the literature and state some open questions.

Chapter 6. We show that for $(C(t))_{t\geqslant 0}$ being a strongly continuous cosine family on a Banach space, the estimate $\limsup_{t\to 0^+} \|C(t) - I\| < 2$ implies that $C(t)$ converges to I in the operator norm. This implication has become known as the *zero-two law*. We further prove that the stronger assumption of $\sup_{t\geqslant 0} \|C(t) - I\| < 2$ yields that $C(t) = I$ for all $t \geqslant 0$. For discrete cosine families the assumption $\sup_{n\in\mathbb{N}} \|C(n) - I\| \leqslant r < \frac{3}{2}$ yields that $C(n) = I$ for all $n \in \mathbb{N}$. For $r \geqslant \frac{3}{2}$ this assertion does no longer hold. More general and using different techniques, we show that, for $(C(t))_{t\in\mathbb{R}}$ being a cosine family on a unital Banach algebra, the estimate $\limsup_{t\to\infty^+}\|C(t)-I\| < 2$ implies that $C(t) = I$ for all $t \in \mathbb{R}$. We also state the corresponding result for discrete cosine families and for semigroups. In the last part we consider scaled versions of above laws. We show that from the estimate $\sup_{t\geqslant 0}\|C(t)-\cos(at)I\| < 1$ we can conclude that $C(t)$ equals $\cos(at)I$. Here $(C(t))_{t\geqslant 0}$ is again a strongly continuous cosine family on a Banach space.

Part I

H^∞-calculus for semigroups

CHAPTER 2

Weakly admissible H^∞-calculus

Abstract. We show that, given a generator of an exponentially stable semigroup on a Banach space, a *weakly admissible* operator $g(A)$ can be defined for any bounded, analytic function g on the left half-plane. This yields an (unbounded) functional calculus. The construction uses a Toeplitz operator and is motivated by system theory. In Hilbert spaces, we even obtain admissibility. Furthermore, it is investigated when a bounded calculus can be guaranteed. For this we introduce the new notion of exact observability by direction. As an application of the approach we show an estimate characterizing the boundedness of the calculus for analytic semigroups on Hilbert spaces.
Finally, it is shown that the calculus coincides with the (classical) holomorphic functional calculus derived by an algebraic extension procedure of the Hille-Phillips calculus. Thus, the approach can be seen as an alternate route for introducing the classical H^∞-calculus for strongly continuous semigroups. [1]

2.1. Introduction

As we have seen in the Chapter 1, in various fields of mathematics (e.g., numerical analysis, operator theory), we encounter the task of 'evaluating' a function f where the argument is the operator A. Simple examples are polynomials, or rational functions, such as $(\alpha I - A)^{-1}$ with $\alpha \in \mathbb{C}$.
Functional calculus is the notion that covers the assignment $f \mapsto f(A)$ for a fixed (possibly unbounded) operator A on a Banach space X and functions F. If F has some algebraic structure (e.g., F is an algebra), one ultimately strives for a functional calculus such that the mapping $f \mapsto f(A)$ is a homomorphism from F to the bounded operators on X. As this is sometimes not possible, we aim for a mapping $f \mapsto f(A)$ which extends a homomorphism on a subalgebra of F, see Section 1.2 for more details.

[1]Parts of this chapter are adapted from the articles
F.L. SCHWENNINGER, H. ZWART, *Weakly admissible $\mathcal{H}^-_\infty$-calculus on reflexive Banach spaces, Indag. Math. 23, p. 796-815, 2012.*
F.L. SCHWENNINGER, H. ZWART, *Functional calculus for C_0-semigroups using infinite-dimensional systems theory*, Semigroups meet Complex Analysis, Harmonic Analysis and Mathematical Physics. *Eds. Arendt, Chill and Tomilov, vol. 250 of Op. Theory: Adv. Appl., Birkhäuser, to appear 2015.*

In this chapter, our goal is to construct a functional calculus for functions in $H^\infty(\mathbb{C}_-)$, i.e., functions which are bounded and analytic on the left half-plane of $\mathbb{C}$. For the operator A, we take a generator of an exponentially stable C_0-semigroup. The interest for this class lies e.g., in numerical analysis (as we have seen in Section 1.1) and system theory. In addition to the above-mentioned properties of a calculus, we want the mapping $f \mapsto f(A)$ to be consistent with the common definition of rational functions.

Let us consider the *Toeplitz operator* M_g *with symbol* $g \in H^\infty(\mathbb{C}_-)$ defined by

$$M_g : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+), f \mapsto M_g f = \mathfrak{L}^{-1}\Pi(g \cdot \mathfrak{L}(f)),$$

where $\mathfrak{L}$ is the Laplace transform and Π denotes the projection onto $\mathcal{H}^2$, the Hardy space on the right half-plane, see Section 2.1.2 for details. Since for fixed $a < 0$,

$$g(s) \cdot \mathfrak{L}(e^{at})(s) = \frac{g(s)}{s-a} = \frac{g(a)}{s-a} + \frac{g(s)-g(a)}{s-a},$$

where the last sum is an orthogonal decomposition in $\mathcal{H}^2$ and $\mathcal{H}^2_\perp$, we conclude that

$$M_g(e^{at}) = g(a)e^{at}. \tag{2.1}$$

In system theoretical words, ‘exponential input yields exponential output’. Obviously, $g \mapsto g(a)$ is a homomorphism from $H^\infty(\mathbb{C}_-)$ to $\mathbb{C}$. Our idea is to replace the exponential by the semigroup $e^{At} = T(t)$. In fact, we show that the formally defined function

$$y(t) = M_g(T(\cdot)x_0)(t)$$

can be seen as the *output* of the linear system

$$\begin{cases} \dot{x}(t) &= Ax(t), \quad x(0) = x_0 \\ y(t) &= Cx(t) \end{cases} \tag{2.2}$$

for some (unbounded) operator C. Thus, formally $y(t) = CT(t)x_0$. This means that C takes the role of $g(a)$ in (2.1). Hence, the task is to find C given the *output mapping* $x_0 \mapsto y(t)$. By G. Weiss, **[Wei89]**, this can be done uniquely, incorporating the notion of *admissibility*, see Lemma 2.3.

The work for (separable) Hilbert spaces by Zwart, **[Zwa12]**, serves as the main motivation. The aim of this chapter is to give a general approach for Banach spaces. The lack of the Hilbert space structure leads to a weak formulation which will be introduced in Section 2.2. In general, this yields a calculus of *weakly admissible* operators. Then, we turn to the task of giving sufficient conditions on A that guarantee bounded $g(A)$ for all $g \in H^\infty(\mathbb{C}_-)$, Section 2.3. In Section 2.3.2 a connection to the results for the ‘strong’ calculus from **[Zwa12]** is established and we see that the weak approach extends the Hilbert space case.
We use the approach to show that $\|g(A)T(\varepsilon)\|$ can be bounded by $|\log \varepsilon|$ for $\varepsilon \to 0$ if A is an analytic semigroup on a Hilbert space, Section 2.4.
In Section 2.5 it is shown that the derived *weakly admissible calculus* coincides with the classical approach to H^∞-calculus based on the abstract axiomatics of holomorphic

function calculi, see [**Haa06a**, Chapter 1]. This is mainly due to the fact that the calculi coincide on the *primary calculus*, which in this case is the *Hille-Phillips* calculus, i.e., the assignment of $f(A)$ for f being the Laplace transform of a Borel measure with bounded total variation, see Lemma 2.20 and Theorem 2.40.

2.1.1. Classical approaches to $H^\infty(\mathbb{C}_-)$-calculus. The class of bounded analytic functions has attracted much interest in functional calculus in the last decades. Early work was done by McIntosh, [**McI86**], or can be found for instance in [**CDMY96**]. There, the considered operators are *sectorial* and the main idea is to extend the *Riesz–Dunford*-calculus by the abstract *regularization* argument seen in Section 1.2. We refer to Chapter 3 for a brief introduction and, for an extensive overview, to the book by Haase [**Haa06a**].
For the generator A of an exponentially stable semigroup, $-A$ is sectorial of angle $\pi/2$. Hence, there exists a sectorial calculus for A for bounded, analytic functions on a larger sector (containing the left half plane). However, since the spectrum of A lies in a half-plane bounded away from the imaginary axis, the more appropriate notion (rather than sectorial operator) is the one of a *half-plane operator* which has been studied in [**BHM13**], [**Haa06b**] and [**Mub11**]. Moreover, as A generates a strongly continuous semigroup, the operator defined by

$$\Psi(\mu)x = \int_0^\infty T(t)x\,d\mu(t), \quad x \in X,$$

is bounded for any Borel measure μ with bounded variation. Denoting by $f_\mu = \mathfrak{L}(\mu)$ the Laplace transform of μ, it follows that $f_\mu \mapsto \Psi(\mu)$ is a homomorphism from a subalgebra of $H^\infty(\mathbb{C}_-)$ to $\mathcal{B}(X)$, see Section 2.5. Using this homomorphism as *primary calculus*, by means of the regularization argument seen in Section 1.2, there exists an extension of Ψ to $H^\infty(\mathbb{C}_-)$. A brief introduction will be given in Section 2.5.1.
In general, it is not clear whether an $H^\infty(\mathbb{C}_-)$-calculus is unique. At least if it is bounded and shares some continuity property, this can be guaranteed, see page 116 in [**Haa06a**]. However, if the calculi coincide on the *primary calculus*, and share some fundamental properties for *abstract functional calculus* as defined by Haase, then they also coincide, [**Haa05, Haa06a**]. See [**Haa06a**, Chapter 1] for a detailed discussion.

2.1.2. Admissibility and Toeplitz operators. For a Hilbert space Y, we introduce the vector-valued Hardy spaces $\mathcal{H}^2(\mathbb{C}_+, Y) = \mathcal{H}^2(Y)$ and $\mathcal{H}^2_\perp(\mathbb{C}_-, Y) = \mathcal{H}^2_\perp(Y)$ on the half-planes $\mathbb{C}_-, \mathbb{C}_+$. A function $f : \mathbb{C}_+ \to Y$ lies in $\mathcal{H}^2(Y)$ if f is holomorphic and

$$\|f\|^2_{\mathcal{H}^2(Y)} := \sup_{x>0} \int_{\mathbb{R}} \|f(x+iy)\|^2_Y dy < \infty.$$

With the norm $\|\cdot\|_{\mathcal{H}^2(Y)}$, $\mathcal{H}^2(Y)$ becomes a Banach space. Analogously, $g : \mathbb{C}_- \to Y$ lies in $\mathcal{H}^2_\perp(Y)$ if $g(-\,\cdot) \in \mathcal{H}^2(Y)$. An element f in $\mathcal{H}^2(Y)$ or $\mathcal{H}^2_\perp(Y)$ has a (non-tangential-limit) boundary function $f(i\cdot)$ (which exists a.e.) on the imaginary axis.

The function $f(i\cdot)$ lies in $L^2(i\mathbb{R}, Y)$ and $\|f\|_{\mathcal{H}^2(Y)} = \|f(i\cdot)\|_{L^2}$ or $\|f\|_{\mathcal{H}^2_\perp(Y)} = \|f(i\cdot)\|_{L^2}$, respectively. Hence, we can identify elements of the Hardy spaces with their boundary functions. We will often use this fact without stating it explicitly. Therefore, $\mathcal{H}^2(Y)$ and $\mathcal{H}^2_\perp(Y)$ are even Hilbert spaces equipped with the inner product

$$\langle f, g\rangle := \int_{\mathbb{R}} f(it)\overline{g(it)}dt.$$

Moreover, by the Paley Wiener theorem, the (two-sided) Laplace transform

$$\mathfrak{L}(f)(z) = \int_{\mathbb{R}} f(t)e^{-zt}dt, \quad z \in \mathbb{C},$$

is an isomorphism from $L^2(\mathbb{R}_+, Y)$ to $\mathcal{H}^2(Y)$ and from $L^2(\mathbb{R}_-, Y)$ to $\mathcal{H}^2_\perp(Y)$, respectively (here, we identify f with its 'zero-extension' on $\mathbb{R}$). This yields the orthogonal decomposition $\mathcal{H}^2(Y) \oplus \mathcal{H}^2_\perp(Y) = L^2(i\mathbb{R}, Y)$. Let $\Pi_Y : L^2(i\mathbb{R}, Y) \to \mathcal{H}^2(Y)$ denote the orthogonal projection onto $\mathcal{H}^2(Y)$ with kernel $\mathcal{H}^2_\perp(Y)$. For $Y = \mathbb{C}$ we write $\mathcal{H}^2 = \mathcal{H}^2(\mathbb{C})$, $\Pi = \Pi_{\mathbb{C}}$ and so on.

The Fourier transform, defined for $f \in L^1(\mathbb{R}, Y)$ by

$$(\mathcal{F}f)(s) = \int_{\mathbb{R}} e^{-its} f(t)\, dt,$$

extends to an isomorphism from $L^2(\mathbb{R}, Y)$ to $L^2(\mathbb{R}, Y)$ with $\|\mathcal{F}f\|_{L^2} = \sqrt{2\pi}\|f\|_{L^2}$ for all $f \in L^2(\mathbb{R}, Y)$. For $f \in L^2(\mathbb{R}_+, Y)$, we denote by f_{ext} the extension of f to $\mathbb{R}$ by $f|_{\mathbb{R}_-} = 0$. We will often use the following relation between the Fourier transform and the Laplace transform,

$$(\mathfrak{L}f)(i\cdot) = (\mathcal{F}f_{ext})(\cdot), \quad f \in L^2(\mathbb{R}_+, Y), \tag{2.3}$$

where $\mathfrak{L}f \in \mathcal{H}^2(Y)$ gets identified with its boundary function on $i\mathbb{R}$.
We refer to the book by Rosenblum and Rovnyak **[RR97]** for a detailed treatment of Hilbert-space valued Hardy spaces, see also **[CZ95]** and **[ABHN11]**.
In the following let $\sigma_\tau : L^2(\mathbb{R}_+, Y) \to L^2(\mathbb{R}_+, Y)$, $\tau \geqslant 0$, denote the left shift,

$$\sigma_\tau f = f(. + \tau). \tag{2.4}$$

DEFINITION 2.1. Let Y be a Banach space. A linear function $\mathfrak{O} : X \to L^2(\mathbb{R}_+, Y)$ is called an **output mapping** for the C_0-semigroup $T(\cdot)$ if

- $\mathfrak{O}$ is bounded, and
- *shift-invariant*, i.e., for all $\tau \geqslant 0$ and $x \in X$,

$$\sigma_\tau(\mathfrak{O}x) = \mathfrak{O}(T(\tau)x). \tag{2.5}$$

All output mappings that we are going to use correspond to the considered semigroup $T(\cdot)$ with generator A. In system theory this notion is often named *well-posed* *infinite-time output mapping*. Considering a system like (2.2), the intuitive candidate for an output mapping is an extension of the densely defined mapping $x_1 \mapsto \mathfrak{D}x_1 = CT(\cdot)x_1$, $x_1 \in D(A)$. Therefore, let us introduce operators C which indeed yield that this $\mathfrak{D}$ is an output mapping. Recall that X_1 denotes the domain $D(A)$ equipped with the graph norm.

DEFINITION 2.2. Let Y be a Banach space. An operator $C \in \mathcal{B}(X_1, Y)$ is called **admissible** for the semigroup $T(\cdot)$ if there exists a constant m_1 such that

$$\|CT(\cdot)x_1\|_{L^2(\mathbb{R}_+,X)} \leqslant m_1\|x_1\| \quad \forall x_1 \in D(A).$$

It is easy to see that if C is admissible, then the mapping $x \mapsto CT(\cdot)x$ can be extended uniquely to an output mapping.
Conversely, the following result due to G. Weiss [**Wei89**] states that every output mapping can be derived by an admissible operator C. This fact is fundamental for the construction of our functional calculus.

LEMMA 2.3 (G. Weiss). *Let Y be a Banach space and $\mathfrak{D} : X \to L^2(\mathbb{R}_+, Y)$ an output mapping for the semigroup $T(\cdot)$. Then there exists a unique $C \in \mathcal{B}(X_1, Y)$ such that*

$$\mathfrak{D}x_1 = CT(\cdot)x_1 \quad \forall x_1 \in D(A),$$

(where the equality holds in L^2-sense). This implies that C is admissible, see Def. 2.2.

In order to use the previous lemma, we will define an output mapping via a Toeplitz operator. Therefore, we need the following notions and results which can already be found in [**Zwa12**].

Before we introduce Toeplitz operators, we observe that for $g \in H^\infty(\mathbb{C}_-)$ and $h \in \mathcal{H}^2(H)$ for some Hilbert space H, we can multiply g and h by means of their boundary functions on $i\mathbb{R}$. Therefore, because (up to identification) $g \in L^\infty(i\mathbb{R})$ and $h \in L^2(i\mathbb{R}, H)$, we get that $gh \in L^2(i\mathbb{R}, H) = \mathcal{H}^2(H) \oplus \mathcal{H}^2_\perp(H)$. Thus, for $f \in L^2(\mathbb{R}_+, H)$, it follows that $g \cdot \mathfrak{L}f \in \mathcal{H}^2(H) \oplus \mathcal{H}^2_\perp(H)$.

DEFINITION 2.4. Let H be a Hilbert space. For a function $g \in H^\infty(\mathbb{C}_-)$, we define the **Toeplitz operator**

$$M_g : L^2(\mathbb{R}_+, H) \to L^2(\mathbb{R}_+, H), f \mapsto \mathfrak{L}^{-1}\Pi_H(g \cdot \mathfrak{L}f),$$

where $\mathfrak{L}^{-1}$ denotes the inverse Laplace transform and Π_H is the orthogonal projection onto $\mathcal{H}^2(H)$ (see above).

Let M_- denote the Borel measures supported in $(-\infty, 0]$ with bounded variation $\|\cdot\|_M$. Recall that the convolution of such a measure ν with a function f is given by

$$\nu * f = \int_{-\infty}^{0} f(t-s)d\nu(s)$$

LEMMA 2.5. *Let* H *be a Hilbert space,* $f \in L^2(\mathbb{R}_+, H)$ *and* $g_1, g_2 \in H^\infty(\mathbb{C}_-)$*. Then, the following properties hold:*

(i) $M_g \in \mathcal{B}(L^2(\mathbb{R}_+, H))$ *and* $\|M_g\| \leqslant \|g\|_\infty$.

(ii) $\sigma_\tau M_g = M_g \sigma_\tau$ *for all* $\tau \geqslant 0$.

(iii) $M_g B = B M_g$ *for all* $B \in \mathcal{B}(H)$*, i.e., for all* $f \in L^2(\mathbb{R}_+, H)$

$$M_g(Bf) = B(M_g f),$$

where $(Bf)(t) = B(f(t))$ *for all* $t \geqslant 0$.

(iv) $M_{g_1 \cdot g_2} = M_{g_1} M_{g_2}$.

(v) If g *is either*

(a) $\mathfrak{L}(\nu)$ *for* $\nu \in M_-$*, or*

(b) an element of $H^\infty(\mathbb{C}_-) \cap \mathcal{H}^2_\perp$,

then,

$$M_g f = (\mathfrak{L}^{-1}(g) * f_{ext})|_{\mathbb{R}_+}$$

where f_{ext} *denotes the extension of the function* f *to* $\mathbb{R}$ *by* $f_{ext}|_{\mathbb{R}_-} = 0$.

PROOF. For *(i)* to *(iv)*, see **[Zwa12]**. *(v)* follows by the following consequence of the convolution theorem and (2.3). Consider first case *(a)*. It holds that

$$\begin{aligned}(g \cdot \mathfrak{L}(f))(i\cdot) &= (\mathcal{F}(\nu) \cdot \mathcal{F}(f_{ext}))(\cdot) = \mathcal{F}(\nu * f_{ext})(\cdot) \\ &= \mathfrak{L}((\nu * f_{ext})\big|_{(0,\infty)})(i\cdot) + \mathfrak{L}((\nu * f_{ext})\big|_{(-\infty,0))})(i\cdot), \qquad (2.6)\end{aligned}$$

where the Fourier transform of ν is defined by $\mathcal{F}(\nu)(s) = \int_{\mathbb{R}} e^{-its} d\nu(t)$. Since $\nu * f_{ext} \in L^2(\mathbb{R}, H)$ by Young's inequality (for the vector-valued version, see e.g., [**Haa06a**, Appendix E.3]), Eq. (2.6) yields

$$M_g f = \mathfrak{L}^{-1} \Pi_{\mathcal{H}^2(H)}(g \cdot \mathfrak{L} f) = (\nu * f_{ext})\big|_{(0,\infty)}. \qquad (2.7)$$

This shows the assertion.

As for *(b)*, it follows that $\mathfrak{L}^{-1}(g) \in L^2(\mathbb{R}_-)$ since $g \in \mathcal{H}^2_\perp$. Since g lies also in $H^\infty(\mathbb{C}_-)$, we have that $g \cdot \mathfrak{L}(f) \in L^2(i\mathbb{R})$. Then, the proof follows analogously as for *(a)*. □

2.2. $H^\infty(\mathbb{C}_-)$-calculus on Banach spaces

Unless stated otherwise, the following convention holds for the rest of the chapter. Let X be a Banach space and let $T = T(\cdot)$ be an exponentially stable C_0-semigroup on X with generator A, see Section 1.4 for a short overview on semigroups. Furthermore, g will always denote a function in $H^\infty(\mathbb{C}_-)$.

2.2.1. General weak approach.

DEFINITION 2.6. Let Z be a Banach space. A bilinear map $B : X \times Z \to L^2(\mathbb{R}_+)$ is called a **weakly admissible output form** for $T(\cdot)$ if the following holds.

- B is bounded, i.e., there exists $b > 0$ such that
$$\|B(x,z)\|_{L^2(\mathbb{R}_+)} \leqslant b\|x\|\|z\|_Z \qquad \forall x \in X, z \in Z, \tag{2.8}$$
- and $B(.,z)$ is *shift-invariant*, i.e.,
$$\sigma_\tau B(x,z) = B(T(\tau)x, z) \qquad \forall \tau > 0, x \in X, z \in Z. \tag{2.9}$$

Clearly, if B is a *weakly admissible output form*, then $B(\cdot,z) : X \to L^2(\mathbb{R}_+)$ is an *output mapping* for all $z \in Z$, cf. Definition 2.1. An example for such B is given by $B(x,z) = \langle z, T(\cdot)x\rangle_{X',X}$ with $Z = X'$. This choice fulfills the assumptions of Definition 2.6 because $T(\cdot)$ is exponentially stable.

DEFINITION 2.7. Let $B : X \times Z \to L^2(\mathbb{R}_+)$ be a weakly admissible output form, $g \in H^\infty(\mathbb{C}_-)$ and $y \in Z$. Define
$$\mathfrak{D}^B_{g,y} : X \to L^2(\mathbb{R}_+), x \mapsto M_g(B(x,y)), \tag{2.10}$$
where M_g denotes the Toeplitz operator on $L^2(\mathbb{R}_+)$ with symbol g (see Definition 2.4 with $H = \mathbb{C}$).

LEMMA 2.8. *Let* $B : X \times Z \to L^2(\mathbb{R}_+)$ *be a weakly admissible output form,* $g \in H^\infty(\mathbb{C}_-)$ *and* $y \in Z$. *Then,*
$$\mathfrak{D}^B_{g,y} : X \to L^2(\mathbb{R}_+), x \mapsto M_g(B(x,y))$$
is an output mapping for $T(\cdot)$ *and there exists a* b *only depending on* B *such that*
$$\|\mathfrak{D}^B_{g,y}x\|_{L^2(\mathbb{R}_+)} \leqslant b\|g\|_\infty\|y\|_Z\|x\|. \tag{2.11}$$
Furthermore, there exists a unique operator $L^B_{g,y} \in \mathcal{B}(X_1, \mathbb{C})$ *such that*
$$\mathfrak{D}^B_{g,y}x_1 = L^B_{g,y}T(\cdot)x_1, \quad x_1 \in D(A), \tag{2.12}$$
and for $x_0 \in X$, $x_1 \in D(A)$, *and* $s \in \mathbb{C}_+$, *the following two identities hold.*
$$\mathfrak{L}[\mathfrak{D}^B_{g,y}x_0](s) = L^B_{g,y}(sI - A)^{-1}x_0, \tag{2.13}$$
$$\begin{aligned} L^B_{g,y}x_1 &= \int_0^\infty [\mathfrak{D}^B_{g,y}(sI - A)x_1](t)e^{-st}\,dt \\ &= \mathfrak{L}[\mathfrak{D}^B_{g,y}(sI - A)x_1](s). \end{aligned} \tag{2.14}$$

PROOF. By Lemma 2.5 *(ii)*, and (2.9), for $x \in X$,
$$\sigma_\tau\mathfrak{D}^B_{g,y}x = \sigma_\tau M_g(B(x,y))$$

$$
\begin{aligned}
&= M_g(\sigma_\tau(B(x,y)))\\
&= M_g(B(T(\tau)x,y))\\
&= \mathfrak{D}^B_{g,y}T(\tau)x.
\end{aligned}
$$

Thus $\mathfrak{D}^B_{g,y}$ is shift-invariant. By Lemma 2.5 *(i)*, M_g is bounded on $L^2(\mathbb{R}_+, H)$ with bound less than $\|g\|_\infty$. Since $B(\cdot, y) : X \to L^2(\mathbb{R}_+)$ is bounded by (2.8), it follows that $\mathfrak{D}^B_{g,y}$ is bounded and that (2.11) holds. Thus, $\mathfrak{D}^B_{g,y}$ is an output mapping.
Now that we know that $\mathfrak{D}^B_{g,y}$ is an output mapping, Lemma 2.3 yields the existence of an operator $L^B_{g,y} \in \mathcal{B}(X_1, \mathbb{C})$ such that (2.12) holds. Taking the Laplace transform of (2.12), which exists for $s \in \mathbb{C}_+$ since $\mathfrak{D}^B_{g,y} \in L^2(\mathbb{R}_+)$, and, using that the integrals exist in X_1, we deduce

$$\mathfrak{L}[\mathfrak{D}^B_{g,y}x_1](s) = L^B_{g,y}(sI - A)^{-1}x_1$$

for $x_1 \in D(A)$. Since $D(A)$ is dense and by boundedness of the operators $x_1 \mapsto \mathfrak{L}[\mathfrak{D}^B_{g,y}x_1](s)$, $L^B_{g,y}(sI - A)^{-1}$, (2.13) follows. Taking $x_0 = (sI - A)x_1$ yields (2.14). □

Using the lemma above, we can deduce properties of the mapping $y \mapsto L^B_{g,y}x$.

LEMMA 2.9. *Under the assumptions of Lemma 2.8, the following assertions hold.*

(i) There exists $b_2 > 0$ *such that*

$$|L^B_{g,y}x_1| \leqslant b_2\|g\|_\infty\|y\|_Z\|x_1\|_1 \qquad x_1 \in D(A), y \in Z. \tag{2.15}$$

(ii) For fixed $x_1 \in D(A)$ *the mapping*

$$L^B_{g,\cdot}x_1 : Z \to \mathbb{C}, \qquad y \mapsto L^B_{g,y}x_1$$

is linear and bounded, hence, $L^B_{g,\cdot}x_1 \in Z'$*, i.e., there exists a unique element* f_{x_1} *in* Z' *such that*

$$L^B_{g,y}x_1 = \langle y, f_{x_1}\rangle_{Z,Z'} \qquad \forall y \in Z. \tag{2.16}$$

PROOF. For (i), fix an $s \in \mathbb{C}_+$. Note that by Cauchy-Schwarz and (2.11),

$$
\begin{aligned}
\left|\int_0^\infty [\mathfrak{D}^B_{g,y}(sI - A)x_1](t)e^{-st}\,dt\right| &\leqslant (\operatorname{Re} s)^{-\frac{1}{2}}\|\mathfrak{D}^B_{g,y}(sI - A)x_1\|_{L^2(\mathbb{R}_+)}\\
&\leqslant (\operatorname{Re} s)^{-\frac{1}{2}} b\|g\|_\infty\|y\|_Z\|(sI - A)x_1\|.
\end{aligned}
$$

By (2.13), the left-hand side equals $|L^B_{g,y}x_1|$ and we obtain (2.15) because $(sI - A) \in \mathcal{B}(X_1, X)$.
Having (i), for (ii), it remains to show the linearity of $L_{g,\cdot}x_1$ for fixed $x_1 \in D(A)$. By the linearity of $B(x_0, \cdot)$ and M_g it is clear that $\mathfrak{D}_{g,\cdot}x_0$ is linear, for fixed $x_0 \in X$. Hence, using (2.14) again for some fixed $s \in \mathbb{C}_+$, we have for $y, z \in Z$ and $\lambda \in \mathbb{C}$

$$L_{g,y+\lambda z}x_1 = \mathfrak{L}[\mathfrak{D}^B_{g,y+\lambda z}(sI - A)x_1](s)$$

$$= \mathfrak{L}[\mathfrak{D}^B_{g,y}(sI-A)x_1](s) + \lambda\mathfrak{L}[\mathfrak{D}^B_{g,z}(sI-A)x_1](s)$$
$$= L_{g,y}x_1 + \lambda L_{g,z}x_1.$$

□

Motivated by the previous lemma, we can consider the map

$$g^B(A): D(A) \to Z', \qquad x_1 \mapsto f_{x_1} = \left(y \mapsto L^B_{g,y}x_1\right). \tag{2.17}$$

The mapping $g^B(A)$ is linear since $L^B_{g,y}x_1$ is linear in x_1 and by (2.15) it is bounded, i.e., $g^B(A) \in \mathcal{B}(X_1, Z')$. Moreover, by Lemma 2.9 (ii) for all $x_1 \in D(A)$, $y \in Z$,

$$\langle y, g^B(A)x_1\rangle_{Z,Z'} = L^B_{g,y}(A)x_1. \tag{2.18}$$

Now, we are able to state the main result of the general weak approach.

THEOREM 2.10. *Let A be the generator of an exponentially stable C_0-semigroup T and $B: X \times Z \to L^2(\mathbb{R}_+)$ be a weakly admissible output form for T (see Def. 2.6). For $g \in H^\infty(\mathbb{C}_-)$, let $\mathfrak{D}^B_{g,y}x = M_g(B(x,y))$, see (2.10).*

Then there exists a unique operator $g^B(A) \in \mathcal{B}(X_1, Z')$ such that

$$\mathfrak{D}^B_{g,y}x_1 = \langle y, g^B(A)T(\cdot)x_1\rangle_{Z,Z'} \tag{2.19}$$

for all $y \in Z$ and $x_1 \in D(A)$.

Furthermore, the following assertions hold.

(i) There exists a constant $\alpha > 0$ such that for all $x \in X$, $s \in \mathbb{C}_+$,

$$\left\|g^B(A)(sI-A)^{-1}x\right\|_{Z'} \leqslant \frac{\alpha}{\sqrt{\mathrm{Re}(s)}}\|g\|_\infty\|x\|. \tag{2.20}$$

(ii) If $Z = X'$ and for all $t \geqslant 0, x \in X, y \in X'$,

$$B(T(t)x, y) = B(x, T(t)'y), \tag{2.21}$$

then, for $t \geqslant 0, x_1 \in D(A), y \in X'$,

$$\langle y, g^B(A)T(t)x_1\rangle_{X',X''} = \langle T(t)'y, g^B(A)x_1\rangle_{X',X''}. \tag{2.22}$$

(iii) If $Z = X'$ and $B(x,y) = \langle y, T(\cdot)x\rangle$, then $g^B(A) \in \mathcal{B}(X_1, X)$ and

$$\langle y, g^B(A)T(\cdot)x_1\rangle_{X',X} = M_g(\langle y, T(\cdot)x_1\rangle_{X',X}), \tag{2.23}$$

for all $x_1 \in D(A)$.

PROOF. Let $g^B(A) \in \mathcal{B}(X_1, Z)$ be defined by (2.17) (see considerations above Theorem 2.10). Then (2.19) follows by the following chain of identities. For $t \geqslant 0$,

$$\langle y, g^B(A)T(t)x\rangle_{Z,Z'} \overset{(2.18)}{=} L^B_{g,y}T(t)x_1 \overset{(2.12)}{=} \left(\mathfrak{D}^B_{g,y}x_1\right)(t),$$

where the last equality holds for a.e. $t \geqslant 0$. The uniqueness of $g^B(A)$ follows from (2.19) and the fact that $T(t)x_1 \to x_1$ in X_1 for $t \to 0^+$.

Item (i): This is a consequence of (2.13). In fact, by Cauchy-Schwarz,

$$\begin{aligned}\left|\langle y, g^B(A)(sI-A)^{-1}x\rangle_{Z,Z'}\right| &\overset{(2.18)}{=} L^B_{g,y}(sI-A)^{-1}x \\ &\overset{(2.13)}{=} \left|\mathfrak{L}[\mathfrak{D}^B_{g,y}x](s)\right| \\ &\overset{\text{C.S.}}{\leqslant} \frac{1}{\sqrt{\mathrm{Re}(2s)}}\left\|\mathfrak{D}^B_{g,y}x\right\|_{L^2(\mathbb{R}_+,X)} \\ &\overset{(2.11)}{\leqslant} \frac{\alpha}{\sqrt{\mathrm{Re}(s)}}\|g\|_\infty\|x\|\|y\|_Z.\end{aligned}$$

In the last step we used the boundedness of the output mapping.
Item (ii): We use (2.14). Let $t>0$, $s\in\mathbb{C}_+$, $y\in X'$ and $x_1\in D(A)$. Then,

$$\begin{aligned}\langle y, g^B(A)T(t)x_1\rangle_{X',X''} &\overset{(2.18)}{=} L^B_{g,y}T(t)x_1 \\ &\overset{(2.14)}{=} \mathfrak{L}[\mathfrak{D}^B_{g,y}(sI-A)T(t)x_1](s) \\ &\overset{(2.10)}{=} \mathfrak{L}[M_gB((sI-A)T(t)x_1,y)](s). \qquad (2.24)\end{aligned}$$

By exploiting the additional assumption on B, (2.21), we further deduce

$$\begin{aligned}\mathfrak{L}[M_gB((sI-A)T(t)x_1,y)](s) &= \mathfrak{L}[M_gB((sI-A)x_1,T'(t)y)](s) \\ &= \mathfrak{L}[\mathfrak{D}_{g,T'(t)y}(sI-A)x_1](s) \\ &= L_{g,T'(t)y}x_1 \\ &= \langle T(t)'y, g^B(A)x_1\rangle_{X',X''}.\end{aligned}$$

Together with (2.24), this gives (2.22).
Item (iii): Since X is isometrically embedded into X'', we have to show that $g^B(A)$ indeed maps into X. Then, $g^B(A)\in\mathcal{B}(X_1,X)$ and (2.23) follows from (2.19). By (2.18) and (2.13), for $s=1$, $x_1\in D(A)$,

$$\begin{aligned}\langle y, g^B(A)x_1\rangle_{X',X''} = L^B_{g,y}x_1 &= \int_0^\infty e^{-t}[\mathfrak{D}^B_{g,y}(I-A)x_1](t)dt \\ &= (h*f_{ext})(0^+), \qquad (2.25)\end{aligned}$$

where $h(t)=e^t\mathbb{1}_{\mathbb{R}_-}(t)$ and $f=\mathfrak{D}^B_{g,y}(I-A)x_1$. Since $h\in L^1(\mathbb{R}_-)$, we get by Lemma 2.5 *(v)* (with assumption *(a)*) that

$$\begin{aligned}(h*f_{ext})|_{\mathbb{R}_+} &= M_{\mathfrak{L}(h)}f \\ &= M_{\mathfrak{L}(h)}\mathfrak{D}^B_{g,y}(I-A)x_1 \\ &= M_{\mathfrak{L}(h)}M_gB((I-A)x_1,y) \\ &= M_{\mathfrak{L}(h)g}B((I-A)x_1,y), \qquad (2.26)\end{aligned}$$

where we used the definition of $\mathfrak{D}^B_{g,y}$ and Lemma 2.5 *(iv)*. It is easy to see that $(\mathfrak{L}(h)g)(z)=\frac{g(z)}{1-z}$. Hence, $\mathfrak{L}(h)g\in H^\infty(\mathbb{C}_-)\cap\mathcal{H}^2_\perp$ and by Lemma 2.5 *(v)* (with assumption *(b)*), we conclude in (2.26) that

$$(h * f_{ext})|_{\mathbb{R}_+} = \mathfrak{L}^{-1}(\mathfrak{L}(h)\cdot g) * \langle y, T(\cdot)(I-A)x_1\rangle_{X',X}|_{\mathbb{R}_+}$$
$$= \langle y, \mathfrak{L}^{-1}(\mathfrak{L}(h)\cdot g) * T(\cdot)(I-A)x_1\rangle_{X',X}|_{\mathbb{R}_+},$$

where the last equality follows since the vector-valued convolution exists. With (2.25) this yields, by continuity of the dual brackets,

$$\langle y, g^B(A)x_1\rangle_{X',X''} = \langle y, \left(\mathfrak{L}^{-1}(\mathfrak{L}(h)\cdot g) * T(\cdot)(I-A)x_1\right)(0^+)\rangle_{X',X}.$$

Thus, we conclude that $g^B(A)x_1 \in X$ for $x_1 \in D(A)$. □

Theorem 2.10 and estimate (2.11) motivate the introduction of the following notion.

DEFINITION 2.11. Let Y be a Banach space. An operator $C \in \mathcal{B}(X_1, Y)$ is called **weakly admissible** if there exists $m > 0$ such that for all $x \in D(A)$ and $y' \in Y'$

- $\langle y', CT(.)x\rangle \in L^2(\mathbb{R}_+)$ and
- $\|\langle y', CT(.)x\rangle\|_{L^2(\mathbb{R}_+)} \leqslant m\|y'\|_{Y'}\|x\|$.

REMARK 2.12.

- From this definition we get immediately that if $C \in \mathcal{B}(X_1, Y)$ is weakly admissible, then $\tilde{B}(x, y) = \langle y, CT(.)x\rangle_{Y',Y}$ defined on $D(A) \times Y'$ can be uniquely extended to a bilinear mapping B on $X \times Y'$. This B fulfills the assumptions in Definition 2.6 ($Z = Y'$) and because of this, $\mathfrak{D}^C_{g,y}, L^C_{g,y}, g^C(A)$ will denote $\mathfrak{D}^B_{g,y}, L^B_{g,y}, g^B(A)$ respectively. Note that this B does not satisfy (2.21) in general even if $Y = X'$.
- From Theorem 2.10 and (2.11), it follows that $g^B(A)$ is weakly admissible.

REMARK 2.13. The notion of weak admissibility and its connection to admissibility are well-studied. Obviously, if $C \in \mathcal{B}(X_1, Y)$ is admissible, then C is also weakly admissible. In [**Wei91**], G. Weiss conjectured that both notions are equivalent[2]. Although the *Weiss conjecture* is known to be true under certain assumption on the semigroup and the space Y, see e.g., [**Wei91, JP01**], it fails in general. In [**ZJS03**], Zwart, Jacob and Staffans gave a first counterexample, even for bounded A and C. Subsequently, Jacob, Partington and Pott [**JPP02**] showed that the conjecture also fails in general for contraction semigroups. We refer to [**JPP02**] for an overview about the various results. In [**LM03**], Le Merdy gave a characterization of the problem for analytic semigroups and in [**Zwa05**], Zwart presented sufficient conditions on C implying admissibility.
Over the years, the Weiss conjecture has remained an active field of study. We refer to [**LM14a, Wyn10**] for general *α-admissibility* and discrete-time analogs. Recently, a stochastic version of the conjecture was studied in [**AHvN13**].

[2]Actually, he conjectured that the condition $\sup_{\lambda\in\mathbb{C}_+}\sqrt{\operatorname{Re}\lambda}\|CR(\lambda, A)\| < \infty$ implies that C is admissible. It is easy to see that a weakly admissible operator C satisfies this condition

2.2.2. The calculus. In the following we will set $Z = X'$.

DEFINITION 2.14. Let $g(A)$ denote the operator $g^B(A)$ from Theorem 2.10 for the weakly admissible output form $B(x,y) = \langle y, T(\cdot)x\rangle_{X',X}$.
Consistently, we will write $\mathfrak{D}_{g,y}$ for $\mathfrak{D}^B_{g,y}$ (see (2.10)) when this specific B is meant.

We are going to need the following lemmata several times. Recall that we say that an operator $B : D(B) \subset X \to X$ commutes with an operator $P \in \mathcal{B}(X)$, if

$$D(B) \subset D(BP) \text{ and } PBx = BPx \quad \forall x \in D(B).$$

We denote this by $PB \subset BP$.

LEMMA 2.15. *The operator $g(A)$ is a bounded operator from X_1 to X which commutes with the semigroup, i.e.,*

$$g(A)T(t) = T(t)g(A) \tag{2.27}$$

on $D(A)$ for all $t > 0$. Therefore, for $\lambda \in \rho(A)$

$$g(A)R(\lambda, A)x_1 = R(\lambda, A)g(A)x_1 \qquad \forall x_1 \in D(A). \tag{2.28}$$

In particular, $g(A)D(A^2) \subset D(A)$.

PROOF. The first assertions follow all directly from Theorem 2.10. (2.28) follows from the identity (2.27) applied to an element $x_1 \in D(A)$ by taking the Laplace transform. $\square$

LEMMA 2.16. *Let Y be a Banach space and $C \in \mathcal{B}(X_1, Y)$.*

(i) If C is weakly admissible, then

$$Cg(A)x_2 = g^C(A)x_2, \qquad x_2 \in D(A^2), \tag{2.29}$$

where $g^C(A)$ is the operator from Theorem 2.10 with $B(x,y) = \langle y, CT(\cdot)x\rangle_{Y',Y}$ (see Remark 2.12). Hence, $Cg(A)$ can be extended uniquely to a weakly admissible operator.

(ii) If $Y = X$ and if C commutes with $T(\cdot)$ on $D(A)$, then,

$$Cg(A)x_2 = g(A)Cx_2, \quad x_2 \in D(A^2). \tag{2.30}$$

If C is even in $\mathcal{B}(X)$, then (2.30) *holds for $x \in D(A)$.*

PROOF. Let $x_2 \in D(A^2)$ and $y \in Y'$. Then $Ax_2 \in D(A)$. Using (2.28) and that $CA^{-1} \in \mathcal{B}(X, Y)$, we obtain

$$\begin{aligned}\langle y, Cg(A)T(t)x_2\rangle_{Y',Y} &= \langle y, CA^{-1}g(A)T(t)Ax_2\rangle_{Y',Y} \\ &= \langle (CA^{-1})'y, g(A)T(t)Ax_2\rangle_{X',X}\end{aligned}$$

$$\overset{(2.23)}{=} \big(M_g(\langle (CA^{-1})'y, T(\cdot)Ax_2\rangle_{X',X})\big)(t)$$
$$= \big(M_g(\langle y, CT(\cdot)x_2\rangle_{Y',Y})\big)(t), \tag{2.31}$$

for a.e. $t \geqslant 0$.

Item (i). Since C is weakly admissible, we get by Remark 2.12 that $M_g(\langle y, CT(\cdot)x_2\rangle = \mathfrak{D}^C_{g,y}$ which further equals $\langle y, g^C(A)T(\cdot)x_2\rangle_{Y',Y''}$ by (2.19). Hence,

$$\langle y, Cg(A)T(t)x_2\rangle_{Y',Y} = \langle y, g^C(A)T(t)x_2\rangle_{Y',Y''}.$$

The equality holds for all $t \geqslant 0$ point-wise since both the right and the left hand-side are continuous functions for $x \in D(A^2)$. Thus (2.29) follows.

Item (ii). If $C \in \mathcal{B}(X_1, X)$ commutes with $T(\cdot)$, we can apply this to (2.31), and derive $Cg(A)x = g(A)C$ on $D(A^2)$. If C is bounded, this even holds for $D(A)$. □

As pointed out in Remark 2.12, $g^C(A)$ will not commute with the semigroup in general. However, if $C \in \mathcal{B}(X_1, X)$ commutes with $T(\cdot)$, then

$$B(T(t)x, y) = \langle y, CT(\cdot)T(t)x\rangle_{X',X} = \langle T'(t)y, CT(\cdot)x\rangle_{X',X} = B(x, T'(t)y)$$

for all $t \geqslant 0$ and $x \in X$. Hence, by Theorem 2.10 *(ii)*, we conclude that $g^C(A)T(t) = T(t)g^C(A)$ for all $t \geqslant 0$ in this case.

It may happen that $g(A)$ is bounded in the norm of X. However, by construction it is only defined on $D(A)$. In this case, we would like to identify $g(A)$ with its bounded extension to X. Moreover, even when $g(A)$ is not bounded in X, we can extend it to a closed operator as we will see in the following.

DEFINITION 2.17. For a Banach space Y and $C \in \mathcal{B}(X_1, Y)$, the operator given by

$$C_\Lambda x = \lim_{\lambda\to\infty} \lambda CR(\lambda, A)x,$$
$$D(C_\Lambda) = \{x \in X : \text{the above limit exists}\}$$

is called the **Lambda extension** of C.

LEMMA 2.18 (Properties of the Lambda extension). *Let $C \in \mathcal{B}(X_1, Y)$. The following assertions hold for C_Λ defined in Definition 2.17.*

(i) $C \subset C_\Lambda$.

(ii) If C is bounded in X, then $C_\Lambda \in \mathcal{B}(X, Y)$.

(iii) If $Y = X$, $P \in \mathcal{B}(X)$ and $PCR(\lambda, A) = CR(\lambda, A)P$ for all $\lambda \in \rho(A)$, then $PC_\Lambda \subset C_\Lambda P$.

(iv) If $Y = X$ and C commutes with some (any) $R(\mu, A) = (\mu I - A)^{-1}$, then C_Λ is closed. Moreover, C_Λ equals the closure $\overline{C}$ of C.

PROOF. Recall the following property of a C_0-semigroup ([**EN00**, Lemma II.3.4])

$$\lim_{\lambda\to\infty} \lambda R(\lambda, A)x = x \qquad \forall x \in X. \tag{2.32}$$

Item (i). Let $x \in D(C) = D(A)$. Then (2.32) implies that $\lambda R(\lambda, A)x$ converges to x even in X_1. Since $C \in \mathcal{B}(X_1, Y)$, we concluded that $\lambda CR(\lambda, A) \to Cx$ as $\lambda \to \infty$. Thus, C_Λ is an extension of C.

Item (ii). If C is bounded in X, then there exists a unique extension $\overline{C} \in \mathcal{B}(X, Y)$, $C \subset \overline{C}$. By (2.32), it follows that $C_\Lambda = \overline{C}$.

Item (iii). Let $x \in D(C_\Lambda)$. If $P \in \mathcal{B}(X)$ commutes with $CR(\lambda, A)$, then

$$PC_\Lambda x = \lim_{\lambda\to\infty} \lambda PCR(\lambda, A)x = C_\Lambda Px. \tag{2.33}$$

Hence, C_Λ commutes with P.

Item (iv). We show that C_Λ is a closed operator first. Let $\{x_n\}$ be a sequence in $D(C_\Lambda)$ such that $x_n \to x$ and $C_\Lambda x_n \to z$ for $n \to \infty$. Since C commutes with the resolvent, we have by *(iii)* that for all $n \in \mathbb{N}$,

$$R(\mu, A)C_\Lambda x_n = C_\Lambda R(\mu, A)x_n = CR(\mu, A)x_n,$$

where the last equality holds since $R(\mu, A)x_n \in D(A)$. Since $CR(\mu, A) \in \mathcal{B}(X)$, we deduce for the limit $n \to \infty$

$$R(\mu, A)z = CR(\mu, A)x.$$

Multiply by μ and let $\mu \to \infty$. By (2.32) the limit exists and

$$z = \lim_{\mu\to\infty} \mu CR(\mu, A)x$$

holds. Thus, $x \in D(C_\Lambda)$ and $C_\Lambda x = z$.

Seeing the operators C and C_Λ as graphs in the space $X \times X$, it is clear that $\overline{C} \subset C_\Lambda$. Conversely, let $(x, y) \in C_\Lambda$, i.e., $x \in D(C_\Lambda), y = C_\Lambda x$. Define

$$x_n = nR(n, A)x \in D(C) = D(A).$$

Observe that for $n \to \infty$, $x_n \to x$ by (2.32) and $Cx_n \to y$ by the definition of C_Λ. Hence, $(x, y) \in \overline{C}$ and thus, $\overline{C} = C_\Lambda$. □

In the following let $g_\Lambda(A) = (g(A))_\Lambda$ denote the Lambda extension of $g(A)$. We make the convention that for (unbounded) operators F, G the domain of $F + G$ is $D(F) \cap D(G)$.

THEOREM 2.19. $g \mapsto g_\Lambda(A)$ *fulfills the properties of an (unbounded) functional calculus, i.e.,*

(i) $g \equiv 1 \Rightarrow \; g(A) = I$,

(ii) $(g_1 + g_2)_\Lambda(A) \supset g_{1,\Lambda}(A) + g_{2,\Lambda}(A)$,

(iii) $(g_1 g_2)_\Lambda(A) \supset g_{1,\Lambda}(A) g_{2,\Lambda}(A)$ *and*

$$D\big(g_{1,\Lambda}(A) g_{2,\Lambda}(A)\big) = D\big((g_1 g_2)_\Lambda(A)\big) \cap D\big(g_{2,\Lambda}(A)\big). \tag{2.34}$$

If $g_2(A)$ *is bounded, then equality holds in (ii) and (iii).*

PROOF. Obviously, for $g \equiv 1 \in H^\infty(\mathbb{C}_-)$, $\mathfrak{D}_{g,y} f = f$ and thus, $g(A) = I$. Since the Toeplitz operator M_g is linear in the symbol g, it follows that

$$(g_1 + g_2)(A) = g_1(A) + g_2(A)$$

defined on $D(A)$. For $x \in D(g_{1,\Lambda}(A) + g_{2,\Lambda}(A)) = D(g_{1,\Lambda}(A)) \cap D(g_{2,\Lambda}(A))$ it follows that

$$\lim_{\lambda\to\infty} \lambda(g_1(A) + g_2(A))R(\lambda, A)x \tag{2.35}$$

exists. Hence, x lies in the domain of $(g_1 + g_2)_\Lambda(A)$. If $g_2(A)$ is bounded, then $D(g_{2,\Lambda}(A)) = X$. Thus, the existence of (2.35) implies that $x \in D(g_{1,\Lambda}(A))$.

Item (iii). We verify $(g_1 \cdot g_2)(A) = g_1(A) g_2(A)$ on $D(A^2)$ first. According to Lemma 2.16, it suffices to prove $g_1^C(A) = (g_1 \cdot g_2)(A)$ for $C = g_2(A)$. Let $y \in X'$ and $x \in D(A^2)$. Then,

$$\begin{aligned}
\langle y, (g_1 g_2)(A) T(t) x\rangle &= \big(M_{g_1 g_2}(\langle y, T(.)x\rangle)\big)(t) \\
&= \big(M_{g_1} M_{g_2}(\langle y, T(.)x\rangle)\big)(t) \\
&= \big(M_{g_1}(\langle y, g_2(A) T(.)x\rangle)\big)(t) \\
&= \langle y, g_1^C(A) T(t) x\rangle,
\end{aligned}$$

where we used (2.23) several times as well as the fact that $M_{g_1 g_2} = M_{g_1} M_{g_2}$ (see Lemma 2.5). Since $x \in D(A^2)$, the equality holds point-wise for $t \geqslant 0$. Thus,

$$(g_1 \cdot g_2)(A) x_2 = g_1(A) g_2(A) x_2 \qquad \forall x_2 \in D(A^2). \tag{2.36}$$

Now, let $x \in D(g_{1,\Lambda}(A) g_{2,\Lambda}(A))$. This means that

$$\lim_{\mu\to\infty} \mu g_2(A) R(\mu, A)x = g_{2,\Lambda}(A)x \quad \text{exists as well as}$$

$$\lim_{\lambda\to\infty} \lambda g_1(A) R(\lambda, A) g_{2,\Lambda}(A)x = g_{1,\Lambda}(A) g_{2,\Lambda}(A)x.$$

Since $g_1(A)R(\lambda, A) \in \mathcal{B}(X)$ and since $R(\lambda, A)$ commutes with $g_2(A)$ on $D(A)$, (2.28), we obtain that

$$g_{1,\Lambda}(A) g_{2,\Lambda}(A)x = \lim_{\lambda\to\infty} \lim_{\mu\to\infty} (\lambda\mu) g_1(A) g_2(A) R(\lambda, A) R(\mu, A)x$$

Clearly, $R(\lambda, A)R(\mu, A)x \in D(A^2)$. Thus, by (2.36),

$$g_{1,\wedge}(A)g_{2,\wedge}(A)x = \lim_{\lambda\to\infty}\lim_{\mu\to\infty}(\lambda\mu)(g_1g_2)(A)R(\lambda, A)R(\mu, A)x.$$

Using the resolvent identity, this can be written as

$$g_{1,\wedge}(A)g_{2,\wedge}(A)x = \lim_{\lambda\to\infty}\lim_{\mu\to\infty}\frac{\lambda\mu}{\mu-\lambda}(g_1g_2)(A)\Big[R(\lambda, A)x - R(\mu, A)x\Big]. \tag{2.37}$$

By (2.20), we have that $(g_1g_2)(A)R(\mu, A)x \to 0$ as $\mu \to \infty$. Therefore,

$$\lim_{\mu\to\infty}\frac{\lambda\mu}{\mu-\lambda}(g_1g_2)(A)R(\mu, A)x = 0.$$

Furthermore,

$$\lim_{\mu\to\infty}\frac{\lambda\mu}{\mu-\lambda}(g_1g_2)(A)R(\lambda, A)x = \lambda(g_1g_2)(A)R(\lambda, A)x.$$

Together, this yields the limit in (2.37),

$$g_{1,\wedge}(A)g_{2,\wedge}(A)x = \lim_{\lambda\to\infty}\lambda(g_1g_2)(A)R(\lambda, A)x$$

which means that $x \in D((g_1g_2)_\wedge(A))$ and $(g_1g_2)_\wedge(A)x = g_{1,\wedge}(A)g_{2,\wedge}(A)x$. This also shows the inclusion '$\subseteq$' in (2.34) since $x \in D(g_{2,\wedge}(A))$ by assumption. To show the other inclusion, we observe that for $x \in X$ and $\mu \in \rho(A)$

$$\begin{aligned}(g_1g_2)(A)R(\mu, A)x &= \lim_{\lambda\to\infty}\lambda(g_1g_2)(A)R(\lambda, A)R(\mu, A)x\\ &= \lim_{\lambda\to\infty}\lambda g_1(A)g_2(A)R(\lambda, A)R(\mu, A)x\\ &= \lim_{\lambda\to\infty}\lambda g_1(A)R(\lambda, A)g_2(A)R(\mu, A)x,\end{aligned}$$

where we used (2.36) and that $R(\lambda, A)R(\mu, A)x$, $R(\mu, A)x$ lie in $D(A^2)$ and $D(A)$, respectively. This gives that $g_2(A)R(\mu, A)x \in D(g_{1,\wedge}(A))$ and

$$(g_1g_2)(A)R(\mu, A)x = g_{1,\wedge}(A)g_2(A)R(\mu, A)x.$$

For $x \in D((g_1g_2)_\wedge(A)) \cap D(g_{2,\wedge}(A))$ this yields that the limit

$$\lim_{\mu\to\infty}\mu(g_1g_2)(A)R(\mu, A)x = \lim_{\mu\to\infty}g_{1,\wedge}(A)\mu g_2(A)R(\mu, A)x$$

exists. Since $\mu g_2(A)R(\mu, A)x \to g_{2,\wedge}(A)$ for $\mu \to \infty$ and the closedness of $g_{1,\wedge}(A)$ we deduce

$$g_{2,\wedge}(A)x \in D(g_{1,\wedge}(A)) \quad \text{and} \quad g_{1,\wedge}(A)g_{2,\wedge}(A)x = (g_1g_2)_\wedge(A)x.$$

This shows that $x \in D\big(g_{1,\wedge}(A)g_{2,\wedge}(A)\big)$. For bounded $g_2(A)$, (2.34) directly shows the equality. □

Since for any $g \in H^\infty(\mathbb{C}_-)$, $g(A)$ is weakly admissible, see Remark 2.12, we will call the mapping $g \mapsto g(A)$ the *weakly admissible calculus for* A.

Next, we see that our calculus is an extension of the *Hille-Phillips calculus*, see e.g., **[Haa06a, HP57]** and Section 2.5.1.

PROPOSITION 2.20. *If* $g \in H^\infty(\mathbb{C}_-)$ *is such that either*

(a) $g = \mathfrak{L}(\nu)$ *for* $\nu \in M_-$, *or*

(b) $g = \mathfrak{L}(\nu)$ *for* $\nu \in L^2(\mathbb{R}_-)$ *(* $\iff$ $g \in \mathcal{H}^2_\perp$*),*

then $g_\wedge(A) \in \mathcal{B}(X)$ *and for* $x \in X$,

$$g_\wedge(A)x = \int_0^\infty T(s)x\, d\nu(-s). \tag{2.38}$$

Here, we identify $d\nu(s)$ *with* $\nu(s)ds$ *in the case of* (b).

PROOF. Since $D(A)$ is dense and the operator $x \mapsto \int_0^\infty T(s)x d\nu(-s)$ is bounded, it suffices to show (2.38) for $x \in D(A)$ only. Let $y \in X'$, $x \in D(A)$. By equation (2.23) of Theorem 2.10,

$$\langle y, g(A)T(\cdot)x_1\rangle_{X',X} = M_g\langle y, T(\cdot)x_1\rangle_{X',X}.$$

Lemma 2.5*(v)*, with $f = \langle y, T(\cdot)x_1\rangle_{X',X}$, yields

$$M_g f = (\nu * f_{ext})\big|_{\mathbb{R}_+} = \langle y, \nu * T(\cdot)x_1\big|_{\mathbb{R}_+}\rangle_{X',X}.$$

Together, we conclude

$$\langle y, g(A)T(t)x_1\rangle_{X',X} = \langle y, \int_{-\infty}^0 T(t-s)x_1 d\nu(s)\rangle_{X',X},$$

which yields the assertion by the strong continuity of $T(\cdot)$. □

Note that $\mathfrak{L}(\nu) \in \mathcal{H}^2_\perp$ implies that $d\nu(s) = h(s)ds$ for some $h \in L^2(\mathbb{R}_-)$ in condition (b) in Proposition 2.20.

We collect some basic results of our calculus.

THEOREM 2.21. *The functional calculus has the following properties.*

(i) Define $\mathcal{H}^\mathcal{B} = \{g \in H^\infty(\mathbb{C}_-) : g_\wedge(A) \in \mathcal{B}(X)\}$. *Then,*

$$\Phi : \mathcal{H}^\mathcal{B} \to \mathcal{B}(X), \quad g \mapsto g_\wedge(A)$$

is an algebra homomorphism.

(ii) If $P \in \mathcal{B}(X)$ *commutes with* A, $PA \subset AP$, *then* P *commutes with* $g_\wedge(A)$ *for any* $g \in H^\infty(\mathbb{C}_-)$. *In particular,* $T(t)g_\wedge(A) \subset g_\wedge(A)T(t)$.

(iii) For $\mu \in \mathbb{C}_+$, $g(z) = \frac{1}{\mu - z}$ *we have* $g_\wedge(A) = R(\mu, A)$.

(iv) For $t \geqslant 0$, $g(z) = e^{tz}$ *we have* $g_\wedge(A) = T(t)$.

PROOF. *(i)* Let g_1, g_2 be in $\mathcal{H}^\mathcal{B}$. By Theorem 2.19 *(iii)*, $(g_1g_2)_\wedge(A)$ is an extension of $g_{1,\wedge}(A)g_{2,\wedge}(A)$. Since the latter is a bounded operator defined on X, also $(g_1g_2)_\wedge(A) \in \mathcal{B}(X)$. Thus, $(g_1g_2)_\wedge(A) \in \mathcal{H}^\mathcal{B}$. The rest is clear from Theorem 2.19.

(ii) Using the Laplace transform, it is easy to see that $PA \subset AP$ implies that $PT(t) = T(t)P$ for any $t \geqslant 0$. By Lemma 2.16*(ii)*, $Pg(A) \subset g(A)P$ and hence, $Pg_\Lambda(A) \subset g_\Lambda(A)P$, see Lemma 2.18.
(iii) and *(iv)* follow directly from Proposition 2.20 using that $\frac{1}{\mu - z} = \mathfrak{L}(e^{\mu \cdot}|_{\mathbb{R}_-})(z)$ and $e^{tz} = \mathfrak{L}(\delta_{-t})(z)$, where δ_{-t} denotes the Dirac measure at $-t$. □

We conclude the construction of the $H^\infty(\mathbb{C}_-)$-calculus by proving that the main identity of the construction, (2.23), can be extended for the Lambda extension.

PROPOSITION 2.22. *For* $g \in H^\infty(\mathbb{C}_-)$ *we have that for all* $x \in D(g_\Lambda(A))$, $y \in X'$,

$$\langle y, g_\Lambda(A)T(\cdot)x\rangle = [\mathfrak{D}_{g,y}x](\cdot) \overset{Def.}{=} M_g\langle y, T(\cdot)x\rangle_{X',X}. \tag{2.39}$$

PROOF. For $x \in D(A)$, (2.39) holds by Theorem 2.10 *(iii)*.
Let now $x \in D(g_\Lambda(A))$ and define $x_n := nR(n, A)x$. It follows that $x_n \in D(A)$ and that $x_n \to x$ for $n \to \infty$ (the latter holds by **[EN00**, Lemma II.3.4**]**). Hence, we have that (2.39) holds for x_n, $n \in \mathbb{N}$, and since $\mathfrak{D}_{g,y}$ is a bounded operator from X to L^2, we conclude that

$$\langle y, g_\Lambda(A)T(\cdot)x_n\rangle = [\mathfrak{D}_{g,y}x_n](\cdot) \overset{L^2}{\to} [\mathfrak{D}_{g,y}x](\cdot) \qquad \text{as } n \to \infty, \tag{2.40}$$

we have that $\mathfrak{D}_{g,y}x_n \to \mathfrak{D}_{g,y}x$ in $L^2(\mathbb{R}_+)$. Furthermore, by the definition of the Lambda extension, it follows that $g_\Lambda(A)x_n = g(A)x_n \to g_\Lambda(A)x$ as $n \to \infty$. Since $T(t)$ commutes with $g_\Lambda(A)$ by Theorem 2.21 *(ii)*, this implies that the left hand side of (2.40) goes to $\langle y, g_\Lambda(A)T(\cdot)x\rangle$ pointwise. This yields the assertion. □

2.2.3. Admissible H^∞-calculus on Hilbert spaces. In this section we compare our weakly admissible H^∞-calculus with the calculus for exponentially stable semigroups on separable Hilbert spaces derived in **[Zwa12]**. First, we remark that the assumption of separability is actually not needed in **[Zwa12]** (the proofs of the results remain completely the same)[3], therefore, we state the following for general Hilbert spaces. To be consistent with our notation of the duality brackets, the inner product of a Hilbert space is assumed to be linear in the second component.

For the ease of presentation, we will call the functional calculus derived in **[Zwa12]** the *strong calculus* and denote it by $g \mapsto g_s(A)$.

The strong calculus is constructed by choosing the output mapping

$$\mathfrak{D}_g : X \to L^2(\mathbb{R}_+, X) : \qquad x \mapsto M_g(T(\cdot)x).$$

Note that $M_g = \mathfrak{L}^{-1}\Pi_X(g \cdot \mathfrak{L})$ is now defined via the Laplace transform on $L^2(\mathbb{R}_+, X)$ and the projection Π_X, which is well-defined since X is a Hilbert space.

[3]The assumption of considering separable Hilbert spaces appears when studying general Toeplitz operators M_g with operator-valued symbol $g : \mathbb{C}_- \to \mathcal{B}(H)$, $g \in H^\infty(\mathbb{C}_-, \mathcal{B}(X))$, see **[Mik08, RR97]**. Whereas we only consider scalar-valued g.

Let $g_s(A) \in \mathcal{B}(X_1, X)$ be defined as the *admissible operator* from Lemma 2.3 such that

$$\mathfrak{D}_g x = M_g(T(\cdot)x) = g_s(A)T(\cdot)x, \tag{2.41}$$

for $x \in D(A)$.

Since $g_s(A)$ is admissible, it follows that $g_s(A)$ is weakly admissible. Of course, a weakly admissible need not be admissible in general, see Remark 2.13. However, we will see that the weak and strong calculus coincide when the considered space is a Hilbert space. To prove this, we make use of the following elementary result.

LEMMA 2.23. *Let* X *be a Hilbert space,* $f \in L^2(\mathbb{R}_+, X)$*,* $g \in \mathcal{H}^2(X)$ *and* $h \in L^2(i\mathbb{R}, X)$*. Then, for* $y \in X$

(i) $\langle y, \mathfrak{L}f\rangle = \mathfrak{L}\langle y, f\rangle$ *and* $\langle y, \mathfrak{L}^{-1}g\rangle = \mathfrak{L}^{-1}\langle y, g\rangle$*,*

(ii) $\langle y, \Pi_X h\rangle = \Pi\langle y, h\rangle$.

PROOF. The first assertion holds because $\mathfrak{L}f$ and $\mathfrak{L}^{-1}g$ exist strongly and by the continuity of the inner product. To see the second assertion, we use that $L^2(i\mathbb{R}, X) = \mathcal{H}^2(X) \oplus \mathcal{H}^2_\perp(X)$. Hence, we can find $h_1 \in \mathcal{H}^2(X)$ and $h_2 \in \mathcal{H}^2_\perp(X)$ such that $h = h_1 + h_2$. From the first part of this lemma we have that $\langle y, h_1\rangle \in \mathcal{H}^2$ and $\langle y, h_2\rangle \in \mathcal{H}^2_\perp$ which yields

$$\langle y, \Pi_X h\rangle = \langle y, h_1\rangle = \Pi\langle y, h_1\rangle = \Pi\langle y, h\rangle.$$

□

THEOREM 2.24. *Let* X *be a Hilbert space and let* A *generate an exponentially stable semigroup on* X*. Then, for* $g \in H^\infty(\mathbb{C}_-)$*,* $g(A) = g_s(A)$*, thus,* $g_\Lambda(A) = (g_s(A))_\Lambda$*.*

PROOF. It suffices to show that

$$\langle y, g(A)T(t)x\rangle = \langle y, g_s(A)T(t)x\rangle \tag{2.42}$$

for $t > 0, y \in X'$ and $x \in D(A)$. By Theorem 2.10 and its counterpart for the strong calculus (see (2.41)), we have that

$$\langle y, g(A)T(.)x\rangle = \mathfrak{D}_{g,y}x,$$
$$\langle y, g_s(A)T(.)x\rangle = \langle y, \mathfrak{D}_g x\rangle.$$

where $\mathfrak{D}_g x = M_g(T(.)x)$ with $M_g \in \mathcal{B}(L^2(\mathbb{R}_+, X))$. By the definition of M_g and Lemma 2.23 we see that

$$\begin{aligned}\langle y, \mathfrak{D}_g x\rangle &= \langle y, \mathfrak{L}^{-1}\Pi_X(g \cdot \mathfrak{L}[T(.)x])\rangle \\ &= \mathfrak{L}^{-1}\Pi(g \cdot \mathfrak{L}[\langle y, T(.)x\rangle]) \\ &= M_g(\langle y, T(.)x\rangle)\end{aligned}$$

$$= \mathfrak{D}_{g,y}x,$$

where this last M_g is an element in $\mathcal{B}(L^2(\mathbb{R}_+))$. Hence, the equality in (2.42) holds for almost every $t > 0$. Since both functions are continuous in t, it holds even point-wise and in particular for $t = 0$. □

REMARK 2.25. As a consequence of Theorem 2.24, it follows that the weakly admissible calculus of Section 2 is automatically admissible in the Hilbert space case.

2.3. Sufficient conditions for a bounded calculus

2.3.1. Exact Observability by Direction. In order to give a sufficient condition for a bounded functional calculus, we introduce a refined notion of *observability*.

DEFINITION 2.26. For an operator $C \in \mathcal{B}(X_1, Y)$, the pair (C, A) is called **exactly observable by direction** if there exist $m, K > 0$ such that for every $x \in D(A)$ there is a $y_x \in Y'$ with $\|y_x\|_{Y'} = 1$ such that

$$K\|x\| \leqslant \|\langle y_x, CT(\cdot)x\rangle_{Y',Y}\|_{L^2(\mathbb{R}_+)} \leqslant m\|x\|. \tag{2.43}$$

THEOREM 2.27. *If there exists an operator* $C \in \mathcal{B}(X_1, Y)$, *where* Y *is a Banach space, such that* (C, A) *is exactly observable by direction, then* $g \mapsto g_\wedge(A)$ *is a bounded* $H^\infty(\mathbb{C}_-)$*-calculus with*

$$\|g_\wedge(A)\| \leqslant \frac{m}{K}\|g\|_\infty, \tag{2.44}$$

where m, K *are the constants from* (2.43).

PROOF. Let $x \in D(A^2)$. Then, there exists a $y_x \in X'$ with norm 1 such that

$$\begin{aligned} K\|g(A)x\| &\leqslant \|\langle y_x, CT(\cdot)g(A)x\rangle_{Y',Y}\|_{L^2(\mathbb{R}_+)} \\ &= \|\langle y_x, Cg(A)T(\cdot)x\rangle_{Y',Y}\|_{L^2(\mathbb{R}_+)}, \end{aligned}$$

where we used that $g(A)$ commutes with the semigroup. Since $\langle y_x, Cg(A)T(\cdot)x\rangle = \langle (CA^{-1})'y_x, g(A)T(\cdot)Ax\rangle = M_g\langle y_x, CT(\cdot)x\rangle$,

$$\|\langle y_x, Cg(A)T(\cdot)x\rangle_{Y',Y}\|_{L^2(\mathbb{R}_+)} = \|M_g\langle y_x, CT(\cdot)x\rangle_{Y',Y}\|_{L^2(\mathbb{R}_+)}.$$

We can further estimate by the norm of the Toeplitz operator, Lemma 2.5 *(i)*, and by using the assumption. Hence,

$$\|M_g\langle y_x, CT(\cdot)x\rangle_{Y',Y}\|_{L^2(\mathbb{R}_+)} \leqslant m\|g\|_\infty\|x\|.$$

Altogether, we have for $x \in D(A^2)$

$$\|g(A)x\| \leqslant \frac{m}{K}\|g\|_\infty\|x\|, \tag{2.45}$$

which proves the assertion, since $D(A^2)$ is dense in X. □

REMARK 2.28. Of course, one should ask for examples of operators C such that (C, A) is exactly observable by direction. Since, in particular, the lower inequality in (2.43) is hard to check, this is a difficult question in general.
We refer to **[CDMY96]** and **[KW10]** for other 'weak' estimates implying a bounded calculus. In particular, it is an interesting question whether Theorem 2.27 and the result in [**KW10**, Theorem 5.2] can be related.

2.3.2. Exact Observability vs. Exact Observability by Direction. Let us consider the strong calculus $g \mapsto g_s(A)$ for Hilbert space operators A from Section 2.2.3. In **[Zwa12]**, the following notion is used to guarantee a bounded $H^\infty(\mathbb{C}_-)$-calculus.

DEFINITION 2.29. Let Y be a Hilbert space. For an operator $C \in \mathcal{B}(X_1, Y)$, the pair (C, A) is called **exactly observable** if there exist $m, K > 0$ satisfying

$$K\|x\| \leqslant \|CT(\cdot)x\|_{L^2(\mathbb{R}_+,Y)} \leqslant m\|x\| \tag{2.46}$$

for all $x \in D(A)$.

In the following we study the relation between *exact observability* and *exact observability by direction*.

REMARK 2.30.

(i) Since for $\|y\|_Y = 1$ and $x \in D(A)$

$$|\langle y, CT(t)x\rangle_Y| \leqslant \|CT(t)x\|,$$

there is a relation between *exact observability* (2.46), and *exact observability by direction* (2.43), as indicated in the following scheme.

$$\begin{array}{ccccc} K\|x\| & \leqslant & \|\langle y_x, CT(.)x\rangle\|_{L^2(\mathbb{R}_+,Y)} & \leqslant & m\|x\| \quad \text{(Ex. Obs. by dir.)} \\ & \Downarrow & & \Uparrow & \\ K\|x\| & \leqslant & \|CT(.)x\|_{L^2(\mathbb{R}_+,Y)} & \leqslant & m\|x\| \quad \text{(Ex. Obs.)} \end{array}$$

The implication arrows denote which inequalities follow from each other.

(ii) If the 'right-hand' estimate in the definition of exact observability by direction is assumed to hold, i.e., if

$$\exists m > 0\ \forall x \in D(A)\ \exists y_x, \|y_x\| = 1, \qquad \langle y_x, CT(.)x\rangle\|_{L^2(\mathbb{R}_+,Y)} \leqslant m\|x\|,$$

then, the pair (C, A) is not exactly observable by direction iff there exists a sequence $\{x_k\} \subset D(A)$ with $\|x_k\| = 1$, $k \in \mathbb{N}$ such that

$$\|\langle y, CT(t)x_k\rangle\|_{L^2(\mathbb{R}_+)} < \frac{1}{k}$$

for all $y \in Y$ with $\|y\|_Y = 1$.
We will use this characterization later.

(iii) Exact observability can be defined for general Banach spaces X, Y since the definition does not need the Hilbert space structure.

The following result is the Hilbert space counterpart of Theorem 2.27 for the strong calculus, see [**Zwa12**].

THEOREM 2.31. *If there exists an operator* $C \in \mathcal{B}(X_1, Y)$ *such that* (C, A) *is exactly observable, then* $g_s(A)$ *is bounded for all* $g \in H^\infty(\mathbb{C}_-)$.
Hence, the strong calculus, $g \mapsto (g_s(A))_\Lambda$ *(where* $(g_s(A))_\Lambda$ *denotes the Lambda extension of* $g(A)$*) is bounded.*

PROPOSITION 2.32. *For finite dimensional* Y *and weakly admissible* $C \in \mathcal{B}(X_1, Y)$, *exact observability and exact observability by direction of* (C, A) *are equivalent.*

PROOF. Since for finite dimensional Y the notions of admissibility and weak admissibility coincide, in the view of Remark 2.30 it remains to show that (2.46) implies (2.43). Assume that (C, A) is not exactly observable by direction. Hence, there exists a sequence x_n in $D(A)$ with $\|x_n\| = 1$ such that

$$\|\langle y, CT(.)x_n\rangle_Y\|_{L^2(\mathbb{R}_+)} < \frac{1}{n} \qquad \forall y \in Y', \|y\|_Y = 1, \tag{2.47}$$

for all $n \in \mathbb{N}$. Let $\{\phi_k : k = 1, .., N\}$ and $\{\phi'_k : k = 1, .., N\}$ be bases of Y and Y', respectively, such that $\|\phi_k\| = 1$ and $\langle \phi'_k, \phi_j\rangle = \delta_{kj}$ for $k, j = 1, .., N$. Then, for $t \geqslant 0$

$$CT(t)x_n = \sum_{k=1}^{N} \langle \phi'_k, CT(t)x_n\rangle \phi_k$$
$$\Rightarrow \|CT(t)x_n\|_Y^2 \leqslant N \sum_{k=1}^{N} |\langle \phi'_k, CT(t)x_n\rangle|^2.$$

Integrating and using (2.47), this yields $\|CT(.)x_n\|_{L^2(\mathbb{R}_+, Y)} \to 0$ for $n \to \infty$. This contradicts the exact observability of (C, A). □

Note that if (C, A) is exactly observable by direction, then C need not be weakly admissible. Therefore, this is additionally required in the proposition above. Finally, we give an example that, even given admissibility, in general exact observability does not imply observability by direction,

EXAMPLE 2.33. We consider a Hilbert space X with orthonormal basis $\{\phi_n\}_{n\in\mathbb{N}}$ and a set $\{\lambda_n, n \in \mathbb{N}\} \subset \mathbb{R}_-$. If $\sup\{\lambda_n\} < 0$, then the operators

$$T(t) \sum_{n=1}^{N} x_n \phi_n := \sum_{n=1}^{N} e^{\lambda_n t} x_n \phi_n, \qquad N \in \mathbb{N}, t \geqslant 0.$$

define an exponentially stable semigroup, see e.g., **[ZJS03]** or Section 3.3.1 in Chapter 3. It can be shown that the generator of T is given by

$$Ax = \sum_{n=1}^{\infty} \lambda_n x_n \phi_n,$$

with $D(A) = \left\{x \in X : \sum_{n=1}^{\infty} |\lambda_n x_n|^2 < \infty\right\}$. For C, we take the square root of $(-A)$, which is given by

$$C \sum_{n=1}^{N} x_n \phi_n = \sum_{n=1}^{N} \sqrt{-\lambda_n} x_n \phi_n,$$

and domain $D(C) = \left\{x \in X : \sum_{n=1}^{\infty} |\sqrt{-\lambda_n} x_n|^2 < \infty\right\}$.

Define $f_n(.) = \sqrt{-2\lambda_n} e^{\lambda_n \cdot}$ and choose $\lambda_n = -2^n$. By **[Nik02b]** Theorem D.4.2.2. (and the appropriate version for the left half-plane), it follows that f_n is a Riesz sequence in $L^2(\mathbb{R}_+)$, i.e., there exist constants $m, M > 0$ such that

$$m \sum_{n=1}^{N} |c_n|^2 \leqslant \left\| \sum_{n=1}^{N} c_n f_n \right\|^2_{L^2(\mathbb{R}_+)} \leqslant M \sum_{n=1}^{N} |c_n|^2, \tag{2.48}$$

for all finite sequences of complex numbers $(c_1, ..., c_N)$.

Let us apply these results to our situation. Define

$$x_N = \sum_{n=1}^{N} \frac{1}{\sqrt{N}} \phi_n.$$

Then, $\|x_N\| = 1$ and for all $y \in X$ with $\|y\|^2 = \sum_{n=1}^{\infty} |y_n|^2 = 1$ there holds

$$\begin{aligned}
\|\langle y, CT(.)x_N\rangle\|^2_{L^2(\mathbb{R}_+)} &= \| \sum_{n=1}^{N} \sqrt{-\lambda_n} e^{\lambda_n \cdot} x_{N,n} y_n \|^2_{L^2(\mathbb{R}_+)} \\
&= \frac{1}{2} \| \sum_{n=1}^{N} x_{N,n} y_n f_n \|^2_{L^2(\mathbb{R}_+)} \\
&\leqslant \frac{M}{2} \sum_{n=1}^{N} \frac{1}{N} |y_n|^2 \\
&\leqslant \frac{M}{2N},
\end{aligned}$$

where we used (2.48). Hence, (C, A) is not exactly observable by direction (see Remark 2.30).
However, by

$$\|CT(t)x\|^2_{L^2(\mathbb{R}_+,X)} = \frac{1}{2} \int_0^{\infty} \| \sum_{n=1}^{N} x_n f_n(t) \phi_n \|^2 dt$$

$$= \frac{1}{2}\int_0^\infty \sum_{n=1}^N |x_n|^2 |f_n(t)|^2 dt = \frac{1}{2}\|x\|^2,$$

we see that (C, A) is exactly observable and, therefore, by Theorem 2.31, we obtain a bounded functional calculus.

REMARK 2.34. Let us consider the situation of Example 2.33, but now with $\lambda_n = \lambda_0 < 0$ for all n. Then, for $x \in D(A)$,

$$\|\langle y, CT(.)x\rangle\|^2_{L^2(\mathbb{R}_+)} = \int_0^\infty |\langle y, \sqrt{-\lambda_0}e^{\lambda_0 t}x\rangle|^2 \, dt = \frac{1}{2}|\langle y, x\rangle|^2.$$

If we choose $y_x = \frac{x}{\|x\|}$, we get

$$\|\langle y_x, CT(.)x\rangle\|^2_{L^2(\mathbb{R}_+)} = \frac{1}{2}\|x\|^2,$$

hence, (C, A) is exactly observable by direction.

2.4. An application for analytic semigroups on Hilbert spaces

We are going to show how our approach can be applied to derive results about the boundedness of the functional calculus. Here, we will deal with the Hilbert space case only. In Chapter 3, the results will be proved for general Banach spaces using different techniques and in more generality. First we state the *admissible version* of Lemma 2.16. In Section 2.2.3, we have seen that the strong calculus $g \mapsto g_s(A)$ for Hilbert spaces coincides with our weakly admissible calculus. Therefore, in the following let $g(A) = g_s(A)$.

THEOREM 2.35 (Lemma 2.1 in **[Zwa12]**). *Let* X, Y *be Hilbert spaces and let* A *be the generator of an exponentially stable semigroup* T *on* X. *If* $C \in \mathcal{B}(X_1, Y)$ *is admissible, then*

$$(M_g(CT(\cdot)x_0))(t) = Cg(A)T(t)x_0, \qquad x_0 \in D(A^2).$$

Moreover, $Cg(A)$ *extends to an admissible output operator.*

For rest of the section we restrict to exponentially stable, analytic semigroups and and show that the norm of $g(A)T(\varepsilon)$ behaves like $|\log(\varepsilon)|$ for ε close to zero. Recall that for an analytic semigroup T, for any $x \in X$ and $t > 0$, $T(t)x$ lies in $D(A)$. Therefore, since $g(A) \in \mathcal{B}(X_1, X)$, $g(A)T(t) \in \mathcal{B}(X)$ for $t > 0$. We recall that for an exponentially stable, analytic semigroup $T(t)$, there exists a $M, \omega > 0$ such that

$$\|(-A)^{\frac{1}{2}}T(t)\| \leqslant \frac{M}{\sqrt{t}}e^{-\omega t}, \qquad t > 0, \tag{2.49}$$

see **[Paz83**, Theorem 2.6.13]. Here $(-A)^{\frac{1}{2}}$ is defined as fractional power of semigroup generators (by a Riesz-Dunford integral for the inverse $(-A)^{-\frac{1}{2}}$, see **[EN00]**). Using this inequality, we prove the following estimate.

THEOREM 2.36. *Let* A *generate an exponentially stable, analytic semigroup* T *on a Hilbert space* X. *Then there exists* $m, \varepsilon_0 > 0$ *such that for every* $g \in H^\infty(\mathbb{C}_-)$, $\varepsilon \in (0, \varepsilon_0)$

$$\|g(A)T(\varepsilon)\| \leqslant m\|g\|_\infty |\log(\varepsilon)|. \tag{2.50}$$

If we assume that $(-A^*)^{\frac{1}{2}}$ *or* $(-A)^{\frac{1}{2}}$ *is admissible, then*

$$\|g(A)T(\varepsilon)\| \leqslant m\|g\|_\infty \sqrt{|\log(\varepsilon)|} \quad \text{for} \quad \varepsilon \in (0, \varepsilon_0). \tag{2.51}$$

If both $(-A^*)^{\frac{1}{2}}$ *and* $(-A)^{\frac{1}{2}}$ *are admissible, then* $g(A)$ *is bounded.*

PROOF. For $y \in D(A^*)$, $x \in D(A^2)$ we have

$$\begin{aligned} \frac{1}{2}\langle y, g(A)T(2\varepsilon)x\rangle &= \int_0^\infty \langle y, (-A)T(2t)g(A)T(2\varepsilon)x\rangle dt \\ &= \int_0^\infty \langle (-A^*)^{\frac{1}{2}}T(\varepsilon)^*T(t)^*y, g(A)T(t)(-A)^{\frac{1}{2}}T(\varepsilon)x\rangle dt, \end{aligned}$$

where we used that $g(A)$ commutes with the semigroup and $(-A)^{\frac{1}{2}}$. Using Cauchy-Schwarz's inequality, we find

$$\begin{aligned} \frac{1}{2}|\langle y, g(A)T(2\varepsilon)x\rangle| &\leqslant \|(-A^*)^{\frac{1}{2}}T(\varepsilon)^*T(\cdot)^*y\|_{L^2} \cdot \|g(A)T(\cdot)(-A)^{\frac{1}{2}}T(\varepsilon)x\|_{L^2} \\ &= \|(-A^*)^{\frac{1}{2}}T(\varepsilon)^*T(\cdot)^*y\|_{L^2} \cdot \|M_g\left(T(\cdot)(-A)^{\frac{1}{2}}T(\varepsilon)x\right)\|_{L^2} \\ &\leqslant \|(-A^*)^{\frac{1}{2}}T(\varepsilon)^*T(\cdot)^*y\|_{L^2} \cdot \|g\|_\infty \cdot \|T(\cdot)(-A)^{\frac{1}{2}}T(\varepsilon)x\|_{L^2}, \end{aligned} \tag{2.52}$$

where we used Theorem 2.35 and Lemma 2.5. Hence it remains to estimate the two L^2-norms. Since X is a Hilbert space, $\left(e^{A^*t}\right)_{t\geqslant 0}$ is an analytic semigroup as well. Hence both L^2-norms behave similarly. We do the estimate for e^{At}. For $\omega\varepsilon < \frac{1}{4}$,

$$\begin{aligned} \|T(\cdot)(-A)^{\frac{1}{2}}T(\varepsilon)x\|_{L^2}^2 &= \int_0^\infty \|(-A)^{\frac{1}{2}}T(t)T(\varepsilon)x\|^2 dt \\ &= \int_\varepsilon^\infty \|(-A)^{\frac{1}{2}}T(t)x\|^2 dt \\ &\leqslant M^2 \int_\varepsilon^\infty \frac{e^{-2\omega t}}{t}\|x\|^2 dt \\ &= M^2\|x\|^2 \int_1^\infty \frac{e^{-2\varepsilon\omega t}}{t} dt \\ &\leqslant M^2\|x\|^2 m_1 |\log(\varepsilon\omega)|, \end{aligned}$$

where we used (2.49) and where $m_1 > 0$ is an absolute constant (the function $s \mapsto \int_1^\infty \frac{e^{-ts}}{t}\,dt$ is known as *Exponential integral*, see also (3.6) and (3.8) in Chapter 3).
Combining the estimates we find that there exists a constant $m_3 > 0$ depending on ω such that for all $x \in D(A^2)$ and $y \in D(A^*)$ there holds

$$\begin{aligned}|\langle y, g(A)T(2\varepsilon)x\rangle| &\leqslant 2\|(-A^*)^{\frac{1}{2}}T(\varepsilon)^*T(\cdot)^*y\|_{L^2}\cdot\|g\|_\infty\cdot\|T(\cdot)(-A)^{\frac{1}{2}}T(\varepsilon)x\|_{L^2},\\ &\leqslant m_3|\log(\varepsilon)|\|g\|_\infty\|x\|\|y\|. \end{aligned} \tag{2.53}$$

Since $D(A^2)$ and $D(A^*)$ are dense in X, we have proved the estimate (2.50).
We continue with the proof of inequality (2.51). If $(-A^*)^{\frac{1}{2}}$ is admissible, then

$$\|(-A^*)^{\frac{1}{2}}T(\varepsilon)^*T(\cdot)^*y\|_{L^2} \leqslant \|(-A^*)^{\frac{1}{2}}T(\cdot)^*y\|_{L^2} \leqslant m_2\|y\|.$$

Therefore, in (2.53), the logarithmic term gets replaced by $\sqrt{|\log\varepsilon|}$. The case for $(-A)^{\frac{1}{2}}$ is admissible follows analogously. In particular, if $(-A)^{\frac{1}{2}}$ and $(-A^*)^{\frac{1}{2}}$ are both admissible, then we see that the epsilon disappears from the estimate, and since the semigroup is strongly continuous, $g(A)$ extends to a bounded operator. □

In Chapter 3, it is shown that for any $\delta \in (0,1)$ there exists an analytic, exponentially stable semigroup on a Hilbert space, and a $g \in H^\infty(\mathbb{C}_-)$ such that $(-A)^{\frac{1}{2}}$ is admissible and $\|g(A)e^{A\varepsilon}\| \sim (\sqrt{|\log(\varepsilon)|})^{1-\delta}$. Similarly, the sharpness of (2.50) is shown. The fact that the calculus is bounded for analytic semigroups when both $(-A)^{\frac{1}{2}}$ and $(-A^*)^{\frac{1}{2}}$ are admissible, can already be found in **[LM03]**, see also **[BDEM10]**. However, as the admissibility of $(-A)^{\frac{1}{2}}$ is equivalent to A satisfying *square function estimates*, the result is much older and goes back to McIntosh, **[McI86]**, see also Section 3.4.
In Chapter 3, we will prove a much more general version of Theorem 2.36, see Theorems 3.11 and 3.24, allowing for general Banach spaces and functions g bounded and holomorphic on a sector larger than the *sectorality sector* of the operator A.

2.5. Relation to holomorphic functional calculus and discussion

2.5.1. Compatibility with holomorphic $H^\infty(\mathbb{C}_-)$-calculus. In this section we study the relation of our functional calculus with the holomorphic calculus derived by the 'standard' technique when extending a homomorphism (see Section 1.2). In Proposition 2.20 we have already seen a close connection to the *Hille-Phillips calculus*, which we now want to introduce formally.

It is not hard to show that $\mathcal{A}_1 = \mathfrak{L}(M_-)$ (which is the Laplace transforms of measures in M_-) and $\mathcal{A}_2 = \mathcal{H}^2_\perp \cap H^\infty(\mathbb{C}_-)$ are sub-algebras of $H^\infty(\mathbb{C}_-)$. Therefore, it follows by

Proposition 2.20 and Theorem 2.21 that for $i = 1, 2$ the mappings

$$\Psi_i : \mathcal{A}_i \to \mathcal{B}(X) : f = \mathfrak{L}(\nu) \mapsto \int_0^\infty T(s) d\nu(-s),$$

where the integral has to be understood in the strong sense, are homomorphisms. The mapping Ψ_1 is known as the *Hille-Phillips calculus* **[HP57]**.

REMARK 2.37. We remark that Ψ_1 is still well-defined if the semigroup T is only *bounded* rather than exponentially stable. Moreover, for a semigroup T such that $\|T(t)\| \leqslant Me^{t\omega}$ for all $t \geqslant 0$ for some $M \geqslant 1$ and $\omega \in \mathbb{R}$, we can consider

$$\Psi_3 : \mathfrak{L}(e^{-\omega \cdot} M_-) \to \mathcal{B}(X) : f = \mathfrak{L}(\nu) \mapsto \int_0^\infty T(s) d\nu(-s),$$

for $\mathfrak{L}(e^{-\omega \cdot} M_-)$ being the algebra of Laplace transforms of measures $\mu(s) = e^{-\omega s}\nu(s)$, $\nu \in M_-$. It clearly follows by rescaling (consider the bounded semigroup $e^{-t\omega}T(t)$) that Ψ_3 is also homomorphism.

Next, we want to extend Ψ_2 to all functions in $H^\infty(\mathbb{C}_-)$ using the regularization argument as explained in Section 1.2. Thus, we regard $(\mathcal{A}_2, H^\infty(\mathbb{C}_-), \Psi_2)$ as *abstract functional calculus* with primary calculus Ψ_2. The following lemma shows that indeed every function in $H^\infty(\mathbb{C}_-)$ can be regularised by an element in $\mathcal{A}_2$.

LEMMA 2.38. *Let* $r(z) = \frac{1}{1-z}$. *Then* $r \in \mathcal{A}_2 := \mathcal{H}^2_\perp \cap H^\infty(\mathbb{C}_-)$ *and for any* $f \in H^\infty(\mathbb{C}_-)$ *it follows that* $r \cdot f \in \mathcal{A}_2$.

PROOF. Obviously, $r \in H^\infty(\mathbb{C}_-)$. Since r is the Laplace transform of $t \mapsto e^t \mathbb{1}_{\mathbb{R}_-}(t) \in L^2(\mathbb{R}_-)$, it follows that $r \in \mathcal{H}^2_\perp$ by Paley Wiener's theorem. For $f \in H^\infty(\mathbb{C}_-)$, the product $r \cdot f$ is holomorphic and bounded on $\mathbb{C}_-$, and by

$$\|(rf)(x + i\cdot)\|_{L^2(\mathbb{R})} \leqslant \|r\|_\infty \|f(x + i\cdot)\|_{L^2(\mathbb{R})},$$

for all $x < 0$, we conclude that $r \cdot f \in \mathcal{H}^2_\perp \cap H^\infty$, as $\|h\|_{\mathcal{H}^2_\perp} = \sup_{x<0} \|h(x + i\cdot)\|_{L^2(\mathbb{R})}$. □

For $r(z) = \frac{1}{1-z}$ we have that $(r(A))^{-1} = (I - A)$ is injective. Therefore, by the extension procedure of holomorphic functional calculus (see Section 1.2),

$$f \mapsto f_{HP}(A) := r(A)^{-1}(rf)(A) = (I - A)\Psi_2(ef), \quad f \in H^\infty(\mathbb{C}_-), \tag{2.54}$$

defines a functional calculus of closed operators, which extends Ψ_2. Thus, the domain of the operator $f_{HP}(A)$ equals $\{x \in X : \Psi_2(ef)x \in D(A)\}$.

As mentioned before, the uniqueness of a functional calculus is not clear a-priori, see **[Haa06a**, Sections 2.8, 5.3 and 5.7.]. Our goal for the rest of the section is to show that $f_{HP}(A) = f_\Lambda(A)$ for all $f \in H^\infty(\mathbb{C}_-)$. We will need the following elementary result.

LEMMA 2.39. *Let* A *generate a semigroup* T *on* X *and let* B *be a closed operator such that* $D(A) \subset D(B)$ *and* $R(\lambda, A)B \subset BR(\lambda, A)$ *for some* $\lambda \in \rho(A)$. *Then,*

$$B = (\lambda I - A)BR(\lambda, A).$$

PROOF. By $R(\lambda, A)B \subset BR(\lambda, A)$, it follows that $B \subset (\lambda - A)BR(\lambda, A)$. To show the other inclusion, let x be in the domain of $(\lambda - A)BR(\lambda, A)$, which means that $BR(\lambda, A)x \in D(A)$. For $n \in \mathbb{N}$ define

$$D_n = \lambda I - n\left(T\left(\frac{1}{n}\right) - I\right) \in \mathcal{B}(X)$$

and let $y \in D(A)$. Since B is closed and commutes with some resolvent, it follows by the injectivity of the Laplace transform that $T(t)By = BT(t)y$ for all $t > 0$. Thus, $D_n By = BD_n y$. Furthermore, by the definition of the generator A, $\lim_{n\to\infty} D_n y = (\lambda - A)y$. Since $R(\lambda, A)x$ and $BR(\lambda, A)x$ are in $D(A)$, we can apply the above facts to the choices $y = R(\lambda, A)x$ and $y = BR(\lambda, A)x$. Hence,

$$\begin{aligned} z_n := D_n R(\lambda, A)x &\quad\to\quad (\lambda - A)R(\lambda, A)x = x, \quad \text{and} \\ Bz_n = D_n BR(\lambda, A)x &\quad\to\quad (\lambda - A)BR(\lambda, A)x, \end{aligned} \tag{2.55}$$

for $n \to \infty$. Therefore, closedness of B yields that $x \in D(B)$ and that $Bx = (\lambda - A)BR(\lambda, A)x$. This shows that $B \supset (\lambda - A)BR(\lambda, A)$. □

Now, we are able to compare our weakly admissible calculus with the holomorphic calculus derived from Ψ_2.

THEOREM 2.40 (Coincidence of Calculi). *Let* A *generate an exponentially stable semigroup. For all* $f \in H^\infty(\mathbb{C}_-)$,

$$f_\Lambda(A) = f_{HP}(A),$$

where $f_\Lambda(A)$ *is the weakly admissible calculus and* $f_{HP}(A)$ *denotes the calculus derived from the Hille-Phillips calculus via* (2.54).

PROOF. By Proposition 2.20, we know that the calculi coincide on $\mathcal{H}^2_\perp \cap H^\infty(\mathbb{C}_-)$,

$$f_\Lambda(A) = \Psi(f) = f_{HP}(A), \quad f \in \mathcal{H}^2_\perp \cap H^\infty(\mathbb{C}_-)). \tag{2.56}$$

Therefore, by definition (2.54), we get for general $f \in H^\infty(\mathbb{C}_-)$,

$$\begin{aligned} f_{HP}(A) &= (I - A)\Psi(fe) \\ &= (I - A)(fe)_\Lambda(A) \\ &= (I - A)f_\Lambda(A)e_\Lambda(A) \\ &= (I - A)f_\Lambda(A)R(1, A), \end{aligned}$$

where we used that $(fe)_\Lambda(A) = f_\Lambda(A)e_\Lambda(A)$ since $e_\Lambda(A) = R(1, A)$ is bounded, see Theorems 2.19 and 2.21*(iii)*. By the properties of the Lambda extension, $f_\Lambda(A)$

commutes with $R(1, A)$, hence, by Lemma 2.39 with $B = f_\Lambda(A)$, we conclude that

$$(I - A)f_\Lambda(A)R(1, A) = f_\Lambda(A),$$

which finishes the proof. □

REMARK 2.41. By Theorem 2.40, we see that our construction of an $H^\infty(\mathbb{C}_-)$ calculus is actually an alternate way to the classical definition of a holomorphic calculus via an extension procedure of a primary calculus. Let us point out the main difference in the approaches. Schematically speaking, the classical set-up is:

> *Define a homomorphism from a sub-algebra of* $H^\infty(\mathbb{C}_-)$ *to* $\mathcal{B}(X)$ *and then extend this map (purely algebraically) via regularization.*

Whereas our approach can be described as:

> *For general* $f \in H^\infty(\mathbb{C}_-)$, *define operators* $f(A)$ *with domain* $D(A)$ *and then extend the operators (take the closure of each operator).*

However, as eventually both approaches yield the same calculus, it is not surprising that similarities are already observable in the construction. In fact, in the proof of Theorem 2.10 *(iii)*, we encountered already a type of regularization, when considering the operator $M_{\mathfrak{L}(h)g}$ instead of M_g.

From the coincidence of the calculi the following characterization of the domain of $f_{HP}(A)$ by the restriction $f_{HP}(A)|_{D(A)}$ is clear from the definition of $f_\Lambda(A)$.

COROLLARY 2.42. *Let* A *generate an exponentially stable semigroup on* X. *Then, for any* $f \in H^\infty(\mathbb{C}_-)$,

$$D(f_{HP}(A)) = \left\{ x \in X : \lim_{\lambda\to\infty} \lambda f_{HP}(A)|_{D(A)} R(\lambda, A)x \text{ exists} \right\},$$

and for all $x \in D(f_{HP}(A))$, $f_{HP}(A)x = \lim_{\lambda\to\infty} \lambda f_{HP}(A)|_{D(A)} R(\lambda, A)x$.

2.5.2. Concluding remarks. Next, we prove that $f_\Lambda(. - \epsilon)(A + \epsilon I)(A)$ indeed equals $f_\Lambda(A)$, as one could expect.

LEMMA 2.43. *Let* T *be an exponentially stable semigroup with generator* A. *Let* $\epsilon > 0$ *and* $T_\epsilon(t) = e^{\epsilon t}T(t)$ *denote the rescaled semigroup with generator* $A + \epsilon I$. *If* T_ϵ *is also exponentially stable, then for all* $f \in H^\infty(\mathbb{C}_-)$, *it holds that*

$$g_\Lambda(A + \epsilon I) = f_\Lambda(A), \qquad \text{where } g(z) = f(z - \epsilon) \quad \forall z \in \mathbb{C}_-.$$

PROOF. The proof relies on the fact that the projection Π onto $\mathcal{H}^2 \subset L^2(i\mathbb{R})$ is translation-invariant. In fact, for $h \in L^2(\mathbb{R})$,

$$\Big(\Pi\mathcal{F}(h)\Big)(.-\epsilon) = \mathcal{F}(h|_{(0,\infty)})(.-\epsilon) = \mathcal{F}\Big(e^{i\epsilon x}h(x)|_{(0,\infty)}\Big)(.)$$
$$= \Pi\Big(\mathcal{F}(e^{i\epsilon x}\cdot h(x))\Big)(.) = \Pi\Big(\mathcal{F}(h)(.-\epsilon)\Big)(.).$$

Using this and $\mathfrak{L}(h)(.-\epsilon) = \mathfrak{L}\Big(e^{\epsilon \cdot}h(.)\Big)$, we see that for $x \in X, y \in X', t \geqslant 0$

$$\Big[M_g\langle y, T_\epsilon(.)x\rangle\Big](t) = \mathfrak{L}^{-1}\Pi\Big(\langle y, f(i.-\epsilon)\cdot\mathfrak{L}(T(.)x)(i.-\epsilon)\rangle\Big)(t)$$
$$= \mathfrak{L}^{-1}\Big[\Pi\Big(\langle y, f(i.)\cdot\mathfrak{L}(T(.)x)(i.)\rangle\Big)(.-\epsilon)\Big](t)$$
$$= e^{\epsilon t}\Big[M_f\langle y, T(.)x\rangle\Big](t).$$

By (2.23), and letting $t \to 0^+$ yields the assertion. □

Although we have only considered exponentially stable semigroups in this chapter, Lemma 2.43 indicates how to define our calculus for more general (strongly continuous) semigroups.

DEFINITION 2.44. Let A generate a semigroup T such that $\sup_{t>0}\|e^{\omega t}T(t)\| < \infty$, and let $\nu > \omega$. For $f \in H^\infty(\mathbb{L}_\nu)$, where $\mathbb{L}_\nu = \{z \in \mathbb{C} : \operatorname{Re} z < \nu\}$, we define

$$f(A) := f(\cdot + \nu)(A - \nu I),$$

where the right-hand-side is defined since $f(\cdot+\nu) \in H^\infty(\mathbb{C}_-)$ and $A - \nu I$ generates the exponentially stable semigroup $e^{-\nu t}T(t)$.

Theorem 2.40 gives rise to some comments. First of all, we can immediately make use of known consequences of a holomorphic calculus as for instance the following important continuity result.

THEOREM 2.45 (Convergence Lemma, [**BHM13**, Thm. 3.1], [**Mub11**]). *Let A generate a semigroup T such that $\sup_{t>0}\|e^{\omega t}T(t)\| < \infty$. Let $\nu > \omega$ and $(f_n)_{n\in\mathbb{N}} \subset H^\infty(\mathbb{L}_\nu)$, where $\mathbb{L}_\nu = \{z \in \mathbb{C} : \operatorname{Re} z < \nu\}$, such that*

$$f_n(z) \to f(z)\ \forall z \in \mathbb{L}_\nu \text{ as } n \to \infty, \text{ and } \sup_{n\in\mathbb{N}}\|f_n\|_\infty < \infty. \tag{2.57}$$

Then, $f \in H^\infty(\mathbb{L}_\nu)$ and

$$f_n(A)x \to f(A)x \qquad \text{for all } x \in D(A).$$

REMARK 2.46 (to Theorem 2.45).

(i) The Convergence Lemma is not surprising in the view of the Toeplitz operator: By rescaling, we can assume w.l.o.g. that $\nu > 0$. Therefore, the functions f_n converge pointwise on $i\mathbb{R}$ and by Dominated Convergence one can see

that
$$M_{f_n}h \to M_f h \text{ in } L^2(\mathbb{R}_+) \qquad \text{as } n \to \infty,$$
for $h \in L^2(\mathbb{R}_+)$.

(ii) It is well-known, see [**Haa06a**, Prop.F.4], that for any $f \in H^\infty(\mathbb{L}_\nu)$ there exists a sequence of rational functions r_n from
$$\mathcal{R}^\infty(\mathbb{L}_\nu) := \left\{ \frac{p}{q} : p, q \in \mathbb{C}[z], \deg(p) \leqslant \deg(q), \text{poles of } q \text{ are in } \mathbb{C} \setminus \overline{\mathbb{L}_\nu} \right\},$$
such that $r_n \to f$ pointwise on $\mathbb{L}_\nu$ and $\|r_n\|_\infty \leqslant \|f\|_\infty$.

By Theorem 2.45 and Remark 2.46 we observe that the calculus can be built of approximations by simple operators. Thus, it often suffices to restrict to functions in $\mathcal{R}^\infty$ (for $\nu \leqslant 0$, these are Laplace transforms of L^1 functions), to show a property of the calculus.
Finally, let us mention the following representation of the Toeplitz operator M_g,
$$M_g h = \left(\mathcal{F}^{-1}\big(g(i\cdot)\mathcal{F}h\big)\right)\Big|_{(0,\infty)}, \qquad h \in L^2(\mathbb{R}_+), g \in H^\infty(\mathbb{C}_-), \tag{2.58}$$
which, implicitly, has been used a couple of times in the present chapter. This fact shows the relation to (analytic) *Fourier multipliers*, which recently were used in the study of the $H^\infty(\mathbb{C}_-)$-calculus estimates for semigroups by Haase and Rozendaal, [**Haa11, HR13**], see also Chapter 3. Moreover, Fourier multipliers play an important role in the study of maximal regularity, see e.g., [**KW04**].

REMARK 2.47. In the view of the applications of the H^∞-calculus, primarily maximal regularity, it can be argued that the notion of *Fourier multiplier operator* may be more natural to use than the one of *Toeplitz operator*. As both terms coincide in our situation, this is rather a matter of taste. The use of Toeplitz operators is mainly motivated by systems theory, which served as the starting idea for the work presented in this Chapter. However, seeing M_g as Fourier multiplier operator (see, e.g., [**Haa06a**, Appendix E] for an introduction) has some advantages, as it naturally leads to an extension of the *strong admissible calculus* for Hilbert spaces, see Section 2.3.2. In fact, we can consider those $g \in H^\infty(\mathbb{C}_-)$, for which the mapping
$$m_g : h \mapsto \left(\mathcal{F}^{-1}\big(g(i\cdot)\mathcal{F}h\big)\right)|_{(0,\infty)}, \quad h \in \mathcal{S}(\mathbb{R}_+, X),$$
where $\mathcal{S}(\mathbb{R}_+, X)$ denotes the *Schwartz space* of rapidly decreasing X-valued functions, extends to a bounded operator on $L^2(\mathbb{R}_+, X)$. The space of such functions g, equipped with the norm $\|m_g\|_{L^2(\mathbb{R},X)\to L^2(\mathbb{R},X)}$, is called the *analytic (Fourier) multiplier algebra* $\mathcal{AM}_2(X)$. Obviously, this space depends on the Banach space X. If X is a Hilbert space, by Plancherel's theorem, it follows that $\mathcal{AM}_2(X) = H^\infty(\mathbb{C}_-)$. In general, however, $\mathcal{AM}_2(X) \subset H^\infty(\mathbb{C}_-)$.
In other words, if $g \in \mathcal{AM}_2(X)$, then the Toeplitz operator M_g exists in some strong

sense. Therefore, it is not difficult to see that for such g, the operator $g(A)$ is even admissible (not only weakly admissible) and the Hilbert space situation from **[Zwa12]** and Section 2.3.2 carries over to more general Banach spaces X if one replaces $H^\infty(\mathbb{C}_-)$ by the algebra $\mathcal{AM}_2(X)$.
Even more general, considering L^p-*Fourier multipliers*, one can extend the results to L^p-*admissibility* (note also that Lemma 2.3 generalizes to output mappings $\mathfrak{D} : X \to L^p(\mathbb{R}_+, X)$ for $p \geqslant 1$, see **[Wei89]**).

CHAPTER 3

On measuring unboundedness of the H^∞-calculus for generators of analytic semigroups

Abstract. We investigate the boundedness of the H^∞-calculus by estimating the bound $b(\varepsilon)$ of the mapping $H^\infty \to \mathcal{B}(X)$: $f \mapsto f(A)T(\varepsilon)$ for ε near zero. Here, $-A$ generates the analytic semigroup T on Banach space and H^∞ is the space of bounded analytic functions on a domain strictly containing the spectrum of A. We show that $b(\varepsilon) = \mathcal{O}(|\log \varepsilon|)$ in general, whereas $b(\varepsilon) = \mathcal{O}(1)$ for bounded calculi. This generalizes a result by Vitse and complements work by Haase and Rozendaal for non-analytic semigroups. We discuss the sharpness of our bounds and show that single square function estimates yield $b(\varepsilon) = \mathcal{O}(\sqrt{|\log \varepsilon|})$.[1]

3.1. Introduction

Note to the reader: In this chapter, the operator $-A$ (rather than A) will denote the generator of a strongly continuous semigroup. Although this is in contrast to the other chapters in this thesis, this seems to be more natural as we consider sectorial operators.

As we have seen before, a functional calculus can be seen as a pair of operators and (scalar-valued) functions which we want to assign to each other. In this chapter we consider the pair of *sectorial operators* A and functions f which are bounded and analytic on a sector that is containing the spectrum of A.

For $\delta \in (0,\pi)$ define the sector $\Sigma_\delta = \{z \in \mathbb{C} : z \neq 0, |\arg(z)| < \delta\}$ and set $\Sigma_0 = (0,\infty)$. A linear operator A is called *sectorial of angle* $\omega \in [0,\pi)$, if $\sigma(A) \subset \overline{\Sigma_\omega}$ and for all $\delta \in (\omega,\pi)$

$$M(A,\delta) := \sup\left\{\|\lambda(\lambda - A)^{-1}\| : \lambda \in \mathbb{C} \setminus \overline{\Sigma_\delta}\right\} < \infty. \tag{3.1}$$

The minimal ω such that A is sectorial of angle ω will be denoted by ω_A. By $\mathrm{Sect}(\omega)$ we denote the set of sectorial operators of angle ω.

[1]This chapter is slightly adapted from the article:
F.L. SCHWENNINGER, *On measuring unboundedness of the* H^∞*-calculus for generators of analytic semigroups, submitted to Journal of Functional Analysis, 2015*, arXiv: 1502.01535.

The functional calculus for sectorial operators is based on extending the *Riesz–Dunford calculus*, which can be seen as an operator-valued version for the Cauchy formula, see Section 3.1.1 for a brief introduction.
From the very beginnings of this calculus 30 years ago, [**McI86**], it has been known that we cannot expect the H^∞-calculus to be bounded, i.e., that $f(A)$ is a bounded operator for every $f \in H^\infty$, [**MY90**]. Starting with the work by McIntosh, [**McI86**], for sectorial operators on Hilbert spaces, the H^∞-calculus turned out to be very useful in various situations, in particular studying maximal regularity, see [**Haa06a**, Chapter 9], [**KW04**] and the references therein. For a recent survey and open problems of the H^∞-calculus for sectorial operators we refer to [**Fac15**].

The question of boundedness of the calculus in a particular situation remains crucial in the applications and has been subject to research over the last decades, see e.g. [**CDMY96, KW01, KW04**] and [**Haa06a**, Chapter 5] for an overview.
The main goal of this chapter is to investigate and 'measure' the (un)boundedness of the H^∞-calculus.

Functional calculus for subalgebras of H^∞ is of own interest. For instance, in [**Vit05b**] Vitse proves estimates for a Besov space functional calculus for analytic semigroups, (see [**Haa11**] for the case of C_0-semigroup generators on Hilbert spaces). We will discuss this result in Section 3.5 and give a slight improvement. Furthermore, the corresponding framework of H^∞-calculus for C_0-semigroup generators was recently developed in [**BHM13, Haa06b, Mub11**] where *half-plane operators* take over the role of sectorial operators.
Let us state a first observation which can be seen as the starting point for the results to come.

PROPOSITION 3.1. *Let A be a densely defined, invertible, sectorial operator of angle $\omega < \frac{\pi}{2}$ on the Banach space X. Then, for $\phi \in (\omega, \pi)$ the $H^\infty(\Sigma_\phi)$-calculus is bounded if and only if*

$$\forall f \in H^\infty(\Sigma_\phi) \qquad \limsup_{\varepsilon \to 0^+} \|(f \cdot e_\varepsilon)(A)\| =: C_f < \infty, \tag{3.2}$$

where $e_\varepsilon(z) = e^{-\varepsilon z}$.

In Example 3.2 we show that the assumption of A being invertible cannot be weakened in general. Neither can we allow for $\omega = \frac{\pi}{2}$. In fact, for $\omega = \frac{\pi}{2}$, $(fe_\varepsilon)(A)$ need not be a bounded operator. However, it is a remarkable result that by incorporating the geometry of the Banach space, one indeed gets that $(fe_\varepsilon)(A)$ is bounded for, not necessarily analytic, C_0-semigroup generators $-A$ (which are sectorial operators of angle $\frac{\pi}{2}$). Precisely, on Hilbert spaces $(fe_\varepsilon)(A)$ always defines a bounded operator if $-A$ generates an exponentially stable semigroup and if f is bounded and analytic on the right half-plane. This was first proved by Zwart in [**Zwa12**, Theorem 2.5]. Using powerful *transference principles* from [**Haa11**], Haase and Rozendaal generalized this to arbitrary Banach spaces for f in the *analytic multiplier algebra* $\mathcal{AM}_p(X) \subset H^\infty(\mathbb{C}_+)$, $p \geqslant 1$, see [**HR13**]. Note that the latter inclusion is even a strict embedding unless $p = 2$ and X is a Hilbert space (in which case equality holds by Plancherel's theorem). They also showed that, alternatively, one can make additional assumptions on the semigroup rather than on the function space. Namely, by requiring that the (shifted) semigroup is γ-bounded, see [**HR13**, Theorem 6.2]. Again, this result generalizes the Hilbert space case as γ-boundedness coincides with classical boundedness then. Moreover, although norm bounds in terms of ε were already present in [**Zwa12**], they were significantly improved in [**HR13**], see also below. We remark that the definition of functional calculus for non-analytic C_0-semigroups differs by nature from the one for sectorial operators. Using the axiomatics of holomorphic calculus in [**Haa06a**, Chapter 1], this can be done by either directly extending the well-known Hille-Phillips calculus, see [**HR13**], or the above mentioned calculus for half-plane operators, [**BHM13, Haa06b, Mub11**]. In Chapter 2 an alternative definition using notions from systems theory is introduced, see also [**SZ12, Zwa12**]. However, as all these techniques are extensions of the Hille-Phillips calculus, the notions are consistent in the considered situation, see Section 2.5.1.

From Proposition 3.1 we see that the behavior of the norm $\|(fe_\varepsilon)(A)\|$ for ε near zero characterizes the boundedness of the H^∞-calculus for a sectorial operator A of angle less than $\frac{\pi}{2}$ that has 0 in its resolvent set. The negative, $-A$, of such an operator corresponds precisely to the generator of an analytic and exponentially stable C_0-semigroup. Denoting this semigroup by $T(t) = e^{-tA}$, we have $(fe_\varepsilon)(A) = f(A)T(\varepsilon)$. As the H^∞-calculus need not be bounded, in general, we cannot bound $\|(fe_\varepsilon)(A)\|$ uniformly in ε. Therefore, our goal is to establish estimates of the form

$$\|(f \cdot e_\varepsilon)(A)\| \leqslant b(\varepsilon) \cdot \|f\|_\infty, \tag{3.3}$$

for all $f \in H^\infty$ on a sector larger than the sector of sectorality of A. In general, $b(\varepsilon)$ will become unbounded for $\varepsilon \to 0^+$.

In Theorem 3.11 we show that $b(\varepsilon) = \mathcal{O}(|\log \varepsilon|)$ as $\varepsilon \to 0^+$ on general Banach spaces. For $0 \notin \rho(A)$, we derive a similar result for functions $f \in H^\infty$ which are holomorphic at 0, see Theorem 3.4. It turns out that the latter result generalizes a result by Vitse in [**Vit05b**] and improves the dependence on the sectorality constant $M(A, \phi)$ significantly, see Section 3.2.2. Moreover, our techniques seem to be more elementary as we do not employ the Hille-Phillips calculus.
For Hilbert spaces and general exponentially stable C_0-semigroup generators $-A$ an estimate of the form (3.3) $b(\varepsilon) = \mathcal{O}(\varepsilon^{-\frac{1}{2}})$ was derived in [**Zwa12**]. It was subsequently improved to $b(\varepsilon) = \mathcal{O}(|\log \varepsilon|)$ by Haase and Rozendaal, [**HR13**, Theorem 3.3], using an adaption of a lemma due to Haase and Hytönen, [**Haa11**, Lemma A.1]. As mentioned in the lines following Proposition 3.1 above, the techniques rely on the geometry of the Hilbert space and cannot be extended to general Banach spaces without either changing to another function space, [**HR13**, Theorems 3.3 and 5.1], or

strengthening the assumption on the semigroup using γ-boundedness, [**HR13**, Theorem 6.2]. Hence, our results can be seen as additionally requiring analyticity of the semigroup, but dropping any additional assumption on the Banach space. As will be visible in the proofs of Theorems 3.4 and 3.11, in our case the logarithmic dependence on ε is much easier to derive than for general semigroups.
Let us remark that estimates of the form (3.3) reveal information about the domain of $f(A)$. In particular, $b(\varepsilon) = \mathcal{O}(|\log\varepsilon|)$ implies that $D(A^\alpha) \subset D(f(A))$ for $\alpha > 0$, see [**HR13**, Theorem 3.7]. For instance, this can be used to derive convergence results for numerical schemes, see, e.g., [**ER13**].

In Section 3.3.1 it is shown that the logarithmic behavior is essentially optimal on Hilbert spaces by means of a scale of examples of Schauder basis multipliers. More precisely, in Theorem 3.19, we see that for any $\gamma < 1$ we can find a suitable sectorial operator on $L^2(-\pi,\pi)$ such that $b(\varepsilon)$ grows like $|\log(\varepsilon)|^\gamma$. Moreover, in the examples we also focus on tracking the dependence on the sectorality constant.

Square function estimates or *quadratic estimates* play a crucial role in characterizing bounded H^∞-calculi for sectorial operators, see [**CDMY96, GMY11, KW01, KW04, McI86**]. On Hilbert spaces this means that an estimate of the form

$$\int_0^\infty \|f(tA)x\|^2 \frac{dt}{t} \leqslant K^2\|x\|^2, \quad \forall x \in X,$$

and an analogous one for the adjoint A^* have to hold for some f . By an example of Le Merdy [**LM03**], it is known that the validity of such an estimate for only one of A or A^* is not sufficient for a bounded calculus. However, we show in Section 3.4 that a single estimate does improve the situation in the way that $b(\varepsilon) = \mathcal{O}(\sqrt{|\log\varepsilon|})$ then. Again, by means of an example it is shown that this behavior is essentially sharp.
In Section 3.5 we compare our result with the one by Haase and Rozendaal in the case of an analytic semigroup on a Hilbert space. Furthermore, using the results of Section 3.2, we derive a slightly improved estimate for the Besov space functional calculus introduced by Vitse in [**Vit05b**].

3.1.1. The functional calculus for sectorial operators. For a C_0-semigroup T on X, $-A$ denotes its generator. T is called an *analytic* C_0-semigroup if it can be extended to a sector in the complex plane, see Section 1.4 for the definition.
We recall that there is a one-to-one correspondence between densely-defined sectorial operators of angle strictly less than $\frac{\pi}{2}$ and generators of bounded analytic C_0-semigroups, namely, $A \in \operatorname{Sect}(\omega)$ with $\omega < \frac{\pi}{2}$ and $\overline{D(A)} = X$ if and only if $-A$ generates a bounded analytic C_0-semigroup, see e.g., [**EN00**, Theorem II.4.6].

We will now briefly introduce the *holomorphic functional calculus for sectorial operators*. Recall that $H^\infty(\Omega)$ denotes the Banach algebra of bounded analytic functions on the open set Ω, equipped with $\|f\|_{\infty,\Omega} := \sup_{z\in\Omega}|f(z)|$. As we will mainly use sectors

$\Omega = \Sigma_\delta$, we abbreviate $\|f\|_{\infty,\Sigma_\delta}$ by $\|f\|_{\infty,\delta}$ or write $\|f\|_\infty$ if the set is clear from the context. For $\delta = \frac{\pi}{2}$ we will write $H^\infty(\Sigma_\delta) = H^\infty(\mathbb{C}_+)$. Furthermore, let us define

$$H^\infty_{(0)}(\Sigma_\delta) = \left\{f \in H^\infty(\Sigma_\delta) : |f(z)| \leqslant C|z|^{-s} \text{ for some } C, s > 0\right\},$$
$$H^\infty_0(\Sigma_\delta) = \left\{f \in H^\infty(\Sigma_\delta) : |f(z)| \leqslant C\frac{|z|^s}{1+|z|^{2s}} \text{ for some } C, s > 0\right\},$$

which are the bounded analytic functions which decay polynomially at ∞ (and 0).

Let A be a sectorial operator of angle ω. Then, the *Riesz-Dunford integral*

$$f(A) = \frac{1}{2\pi i}\int_\Gamma f(z)R(z,A)\,dz, \tag{3.4}$$

is well-defined in $\mathcal{B}(X)$ in each of the following situations, with $\omega < \delta' < \delta < \pi$,

(i) $f \in H^\infty_0(\Sigma_\delta)$ and $\Gamma = \partial\Sigma_{\delta'}$, where $\partial\Sigma_\delta$ denotes the boundary of Σ_δ,

(ii) $f \in H^\infty_{(0)}(\Sigma_\delta) \cap H(B_{r'}(0))$ for some $r > 0$ and $\Gamma = \partial\left(B_{r'}(0) \cup \Sigma_{\delta'}\right)$ for $r \in (0, r')$,

(iii) $f \in H^\infty_{(0)}(\Sigma_\delta)$, $0 \in \rho(A)$ and $\Gamma = \partial\left(B_r(0)^c \cap \Sigma_{\delta'}\right)$ for $r > 0$ sufficiently small,

where $B_r(0) = \{z \in \mathbb{C} : |z| < r\}$. The above paths Γ are orientated positively and by Cauchy's theorem it follows that the definitions are consistent and independent of the choice of δ' and r'.
The mapping $f \mapsto f(A)$ is an algebra homomorphism from $H^\infty_0(\Sigma_\delta)$ to $\mathcal{B}(X)$. It is straight-forward to extend it to a homomorphism Φ from $\mathcal{E} = H^\infty_0(\Sigma_\delta) \oplus \langle 1\rangle \oplus \langle \frac{1}{1+z}\rangle$ to $\mathcal{B}(X)$. The *abstract functional calculus* $(\mathcal{E}, H(\Sigma_\delta), \Phi)$ has *primary calculus* Φ, which, by the *regularization argument* shown in Section 1.2, can be extended to more general $f \in H(\Sigma_\delta)$. This algebraic procedure yields an, in general unbounded, calculus of closed operators. For the ease of the presentation, we recap the sketch of the regularization argument. The set of *regularizers* is defined as

$$\mathrm{Reg}_A = \{e \in \mathcal{E} : e(A) \text{ is injective}\}$$

and the functions that can be *regularized* by elements in Reg_A are

$$\mathcal{M}_A = \{f \in H(\Sigma_\delta) : \exists e \in \mathrm{Reg} \text{ with } (ef) \in \mathcal{E}\},$$

where $H(\Omega)$ denotes the analytic functions on Ω. Then, for any $f \in \mathcal{M}_A$, we can define $f(A) = e(A)^{-1}(ef)(A)$ which turns out to be independent of the choice of e. If A is injective, it holds that $H^\infty(\Sigma_\delta) \subset \mathcal{M}_A$ and that $e(z) = \frac{z}{(1+z)^2}$ is a regularizer for every $f \in H^\infty(\Sigma_\delta)$. One can show that the extension procedure is in conformity with the Riesz-Dunford integral definition in items 2 and 3 above. Clearly, for invertible A one can do the analogous construction with a primary calculus on $H^\infty_{(0)}(\Sigma_\delta)$, which extends the previous calculus. For more details about the construction of the calculus for sectorial operators we refer to Chapter 1 and 2 in **[Haa06a]**.
Let $(\mathcal{F}, \|\cdot\|_\mathcal{F})$ be a Banach algebra such that $\mathcal{F}$ is a subalgebra of $H^\infty(\Sigma_\delta)$ and that $f(A)$

is defined by the above calculus for all $f \in \mathcal{F}$. Following Haase [**Haa06a**, Chapter 5.3], we say that the $\mathcal{F}$-calculus is *bounded* if $f(A)$ is bounded for all $f \in \mathcal{F}$ and

$$\exists C > 0 : \quad \|f(A)\| \leqslant C\|f\|_{\mathcal{F}}, \quad \forall f \in \mathcal{F}. \tag{3.5}$$

The infimum over all possible C is called the bound of the calculus. Note that for $\mathcal{F}$ closed in $H^\infty(\Sigma_\delta)$ with $\|\cdot\|_{\mathcal{F}} = \|\cdot\|_{\infty,\delta}$ and A injective, (3.5) follows already if $f(A)$ is bounded for all $\mathcal{F}$ by the Convergence Lemma, [**Haa06a**, Proposition 5.1.4] and the Closed Graph Theorem.

By e_ε we denote the function $z \mapsto e^{-\varepsilon z}$ which lies in $H^\infty_{(0)}(\Sigma_\delta)$ for $\delta < \frac{\pi}{2}$ and $\varepsilon > 0$. In the following the *exponential integral* function

$$\mathrm{Ei}(x) = \int_1^\infty \frac{e^{-xt}}{t}\,dt, \quad x > 0, \tag{3.6}$$

will be used several times. It is clear that $\mathrm{Ei}(x)$ is decreasing. The asymptotic behavior of $\mathrm{Ei}(x)$ is reflected in the estimates

$$\tfrac{1}{2}e^{-x}\log\left(1 + \tfrac{2}{x}\right) < \mathrm{Ei}(x) < e^{-x}\log\left(1 + \tfrac{1}{x}\right), \quad x > 0, \tag{3.7}$$

which go back to Gautschi [**Gau60**] and can also be found in [**AS64**, 5.1.20]. Moreover, it is easy to prove that

$$\mathrm{Ei}(x) < \log\left(\tfrac{1}{x}\right), \quad x \in \left(0, \tfrac{1}{2}\right). \tag{3.8}$$

Thus, by (3.7), $\mathrm{Ei}(x) \sim |\log x|$ for $x < \frac{1}{2}$.

3.2. Main results

Unless explicitely stated, the space X will always denote a general Banach space.

3.2.1. Sectorial operators and functions holomorphic at 0. We first give a proof for Proposition 3.1

PROOF (Proof of Proposition 3.1). Since A is invertible, $f(A)$ is defined as a closed operator for every $f \in H^\infty(\Sigma_\phi)$ and $D(A) \subset D(f(A))$ because $\frac{z}{(1+z)^2}$ is a regularizer for f. Since for every $\delta < \pi/2$, $e_\varepsilon \in H^\infty_{(0)}(\Sigma_\delta)$ we have that $(fe_\varepsilon) \in H^\infty_{(0)}(\Sigma_\delta)$ for some $\delta < \pi/2$. Hence $e_\varepsilon(A)$ and $(fe_\varepsilon)(A)$ are bounded operators. If the calculus is bounded, (3.5) holds with $\mathcal{F} = H^\infty(\Sigma_\phi)$. Thus,

$$\|(fe_\varepsilon)(A)\| = \|f(A)e_\varepsilon(A)\| \leqslant C\|e_\varepsilon(A)\| \cdot \|f\|_{\infty,\phi} \leqslant \tilde{C}\|f\|_{\infty,\phi},$$

where the last inequality follows by [**Haa06a**, Proposition 3.4.1c] (or, also by our Corollary 3.13) and $\tilde{C}$ does not depend on ε. Therefore, (3.2) holds.Conversely, let (3.2) be satisfied. Since $e_\varepsilon(A) \in \mathcal{B}(X)$ we have for $x \in D(A)$ that

$$\|f(A)x\| \leqslant \|f(A)x - e_\varepsilon(A)f(A)x\| + \|e_\varepsilon(A)f(A)x\|.$$

For $\varepsilon \to 0^+$, the first term on the right-hand-side tends to zero as $(e^{-\varepsilon\cdot})(A)$ converges to I strongly on $\overline{D(A)} = X$, see [**Haa06a**, Proposition 3.4.1.f)]. Since $e_\varepsilon(A)f(A)x = (e_\varepsilon f)(A)x$ for $x \in D(A)$, see [**Haa06a**, Theorem 1.3.2.c)], the second term can be estimated by the assumption of (3.2). Thus, we get $f(A) \in \mathcal{B}(X)$ for all $f \in H^\infty(\Sigma_\phi)$ because $D(A)$ is dense. Hence, the calculus is bounded. The norm inequality then follows automatically, see the remark after (3.5). ☐

The following example shows that the assumption on the invertibility of A cannot be neglected, if we restrict to functions on $H^\infty(\mathbb{C}_+)$.

EXAMPLE 3.2. Let $-B$ be the generator of the bounded analytic semigroup S with $0 \in \rho(B)$. Assume that the $H^\infty(\mathbb{C}_+)$-caclulus is not bounded, thus, there exists $f \in H^\infty(\mathbb{C}_+)$ such that $f(B)$ is unbounded. Such examples exist even on Hilbert spaces, see e.g., [**BC91**] or Section 3.3.1. Then, $A = B^{-1}$ is bounded, sectorial of the same angle as B, see [**Haa06a**], and has dense range. Thus $g(A)$ is defined by the H^∞-calculus for sectorial operators for $g \in H^\infty$ in some sector. Furthermore, by the *composition rule*, see [**Haa06a**, Proposition 2.4.1], we have that for $h = (z \mapsto z^{-1})$,

$$(f \circ h)(A) = f(B),$$

where $(f \circ h) \in H^\infty(\mathbb{C}_+)$. Since A is bounded, it even generates a group T. Hence, $(f \circ h)(A)T(t) = f(B)T(t)$ cannot be bounded for any $t \geqslant 0$.

The reason why we cannot expect $(f \cdot e_\varepsilon)(A)$ to be a bounded operator if $0 \notin \rho(A)$ is that the integrand in (3.4) may have a singularity at 0. However, instead of making the resolvent exist at 0, we can pass over to functions that are holomorphic at 0.

PROPOSITION 3.3. *Let* A *be a densely defined, sectorial operator of angle* $\omega < \frac{\pi}{2}$ *on the Banach space* X *with dense range. Then, for* $\phi \in (\omega, \pi)$ *the* $H^\infty(\Sigma_\phi)$*-calculus is bounded if and only if*

$$\exists C > 0\ \forall g \in H^\infty(\Sigma_\phi), g \textit{ hol. at 0}: \quad \limsup_{\varepsilon \to 0^+} \|(g e_\varepsilon)(A)\| < C\|g\|_{\infty,\phi}. \tag{3.9}$$

PROOF. The proof is essentially the same as for Proposition 3.1 with the following adaptions: Note that A is injective as it is a sectorial operator with dense range, see [**Haa06a**, Proposition 2.1.1]. Thus, the calculus is defined for $H^\infty(\Sigma_\phi)$. For g holomorphic at 0, $(\frac{g(z)}{1+z})(A)$ is defined by (3.4), and hence bounded. Thus, $D(A) \subset D(g(A))$. Because $D(A)$ is dense, it follows analogously to the proof of Proposition 3.1 that

$$\|g(A)\| \leqslant \limsup_{\varepsilon \to 0^+} \|(g e_\varepsilon)(A)\| \leqslant C\|g\|_{\infty,\phi}, \tag{3.10}$$

where the last inequality holds if (3.9) holds. For arbitrary $f \in H^\infty(\Sigma_\phi)$ take a sequence $g_n \in H^\infty(\Sigma_\phi)$ which are holomorphic at 0 and converge to f pointwise in Σ_ϕ with $\sup_n \|g_n\|_{\infty,\phi} < \infty$. Applying the Convergence Lemma [**Haa06a**, Proposition 5.1.4b)], using (3.10) and the fact that $D(A) \cap R(A)$ is dense, yields that $f(A)$ is bounded. □

In the following theorem we estimate $\|(f \cdot e_\varepsilon)(A)\|$. In Section 3 we show that this estimate is sharp.

THEOREM 3.4. *Let* $A \in \mathrm{Sect}(\omega)$, $0 < \omega < \phi < \frac{\pi}{2}$ *and* $\varepsilon, r_0 > 0$. *Further, let* $f \in H^\infty(\Omega_{\phi,r_0})$ *with* $\Omega_{\phi,r_0} := \Sigma_\phi \cup B_{r_0}(0)$. *Then* $(fe_\varepsilon)(A)$ *is bounded and*

$$\|(f \cdot e_\varepsilon)(A)\| \leqslant \frac{M(A,\phi)}{\pi} \cdot b(\varepsilon, r_0, \phi) \cdot \|f\|_{\infty,\Omega_{\phi,r_0}}, \tag{3.11}$$

with

$$b(\varepsilon, r_0, \phi) = \begin{cases} \mathrm{Ei}(\varepsilon r_0 \cos\phi) + e^{\varepsilon r_0}(\pi - \phi), & 2\varepsilon r_0 \leqslant 1, \\ \mathrm{Ei}\left(\frac{\cos\phi}{2}\right) + \sqrt{e}(\pi-\phi), & 2\varepsilon r_0 > 1. \end{cases} \tag{3.12}$$

Here, $\mathrm{Ei}(x)$ *is the the exponential integral, see (3.6)–(3.8), therefore,*

$$b(\varepsilon, r_0, \phi) \sim \begin{cases} |\log(\varepsilon r_0 \cos\phi)|, & \varepsilon r_0 < \frac{1}{2}, \\ |\log\frac{\cos\phi}{2}|, & \varepsilon r_0 \geqslant \frac{1}{2}. \end{cases} \tag{3.13}$$

PROOF. Since $fe_\varepsilon \in H^\infty_{(0)}(\Sigma_\phi) \cap H^\infty(\Omega_{\phi,r_0})$, we get (see (3.4))

$$(fe_\varepsilon)(A) = \frac{1}{2\pi i}\int_{\Gamma_r} f(z)e^{-\varepsilon z}R(z,A)\,dz \in \mathcal{B}(X), \tag{3.14}$$

where the integration path is $\Gamma_r = \Gamma_{1,r} \cup \Gamma_{2,r} \cup \Gamma_{3,r}$ with

$$\Gamma_{1,r} = \left\{\tilde{r}e^{i\delta}, \tilde{r} > r\right\}, \Gamma_{2,r} = \left\{re^{is}, |s| \geqslant \delta\right\}, \Gamma_{3,r} = \left\{\tilde{r}e^{-i\delta}, \tilde{r} > r\right\},$$

$r \in (0, r_0)$, $\delta \in (\omega, \phi)$, orientated counter-clockwise. Since $f \in H^\infty(\Omega_{\phi,r_0})$, we can estimate

$$\|(fe_\varepsilon)(A)\| \leqslant \frac{\|f\|_{\infty,\Omega_{\phi,r_0}}}{2\pi}\int_{\Gamma_r} \|e^{-\varepsilon z}R(z,A)\|\,|dz|. \tag{3.15}$$

The rest of the proof is similar to a standard argument to show that sectorial operators of angle $< \frac{\pi}{2}$ are generators of bounded analytic semigroup, see e.g. [**EN00, Paz83, Vit05b**]. Splitting up the integral, for $z \in \Gamma_{1,r}$,

$$\|e^{-\varepsilon z}R(z,A)\| \leqslant e^{-\varepsilon\,\mathrm{Re}\,z} \cdot \frac{M(A,\delta)}{|z|} = \frac{e^{-\varepsilon|z|\cos\delta}}{|z|}M(A,\delta).$$

On $\Gamma_{3,r}$ the same estimate holds.
For $z \in \Gamma_{2,r}$,

$$\|e^{-\varepsilon z}R(z,A)\| \leqslant e^{-\varepsilon r\,\mathrm{Re}\,z} \cdot \frac{M(A,\delta)}{r} \leqslant e^{\varepsilon r} \cdot \frac{M(A,\delta)}{r}.$$

Therefore,

$$\int_{\Gamma_r} \|e^{-\varepsilon z} R(z,A)\| |dz| \leqslant M(A,\delta) \left(2\int_r^\infty \frac{e^{-\varepsilon \tilde{r}\cos\delta}}{\tilde{r}} d\tilde{r} + \frac{e^{\varepsilon r}}{r} \int_{\Gamma_{2,r}} |dz| \right)$$
$$\leqslant 2M(A,\delta)(\mathrm{Ei}(\varepsilon r \cos\delta) + e^{\varepsilon r}(\pi - \delta)). \tag{3.16}$$

Next, for $n \in \mathbb{N}$ we choose r as

$$r = \begin{cases} r_n = r_0(1-2^{-n}), & 2\varepsilon r_0 \leqslant 1, \\ \frac{1}{2\varepsilon}, & 2\varepsilon r_0 > 1. \end{cases}$$

Clearly, this choice satisfies $r \in (0, r_0)$. Hence, by (3.15) and (3.16),

$$\|(fe_\varepsilon)(A)\| \leqslant \frac{M(A,\delta)}{\pi} \left\{ \begin{array}{ll} \mathrm{Ei}(\varepsilon r_n \cos\delta) + e^{\varepsilon r_n}(\pi-\delta), & 2\varepsilon r_0 \leqslant 1, \\ \mathrm{Ei}\left(\frac{\cos\delta}{2}\right) + \sqrt{e}(\pi - \delta), & 2\varepsilon r_0 > 1 \end{array} \right\} \|f\|_{\infty, \Omega_{\phi, r_0}}.$$

Letting $n \to \infty$ and $\delta \to \phi^-$ shows the assertion. □

As $H^\infty(\Omega_{\frac{\pi}{2}, r_0})$ is continuously embedded in $H^\infty(\mathbb{C}_+)$, and since $\|f\|_{\infty,\mathbb{C}_+} = \|e_\varepsilon f\|_{\infty,\mathbb{C}_+}$ we have the following direct consequence of Theorem 3.4.

COROLLARY 3.5. *Let $A \in \mathrm{Sect}(\omega)$ on the Banach space X and $\omega < \frac{\pi}{2}$. Then, for any $r > 0$ and $\varepsilon > 0$, A has a bounded $e_\varepsilon H^\infty(\Omega_{\frac{\pi}{2},r})$-calculus with $\Omega_{\frac{\pi}{2},r} = \mathbb{C}_+ \cup B_r(0)$.*

Note that $e_\varepsilon H^\infty(\Omega_{\frac{\pi}{2}}, r)$ is a closed ideal in $H^\infty(\mathbb{C}_+)$.

3.2.2. The space $H^\infty[\varepsilon, \sigma]$ and Vitse's result.

In this subsection we show that the result in Theorem 3.4 generalizes Theorem 1.6 in **[Vit05b]**.

For $\varepsilon, \sigma \in \mathbb{R}$ with $0 \leqslant \varepsilon < \sigma \leqslant \infty$ let $H^\infty[\varepsilon, \sigma]$ denote the space of functions which are in $H^\infty(\mathbb{C}_+)$ and are the Laplace-Fourier transform of a distribution supported in $[\varepsilon, \sigma]$. Recall that an entire function g is of *exponential type* $\sigma > 0$ if for any $\epsilon > 0$ there exists $C_\epsilon > 0$ such that $|g(z)| \leqslant C_\epsilon e^{(\sigma+\epsilon)|z|}$ for all $z \in \mathbb{C}$.

For $\sigma < \infty$, the following Paley-Wiener-Schwartz type result holds,

$$g \in H^\infty[\varepsilon, \sigma] \iff g \text{ is entire of exponential type } \sigma \text{ and } ge^{\varepsilon \cdot} \in H^\infty(\mathbb{C}_+). \tag{3.17}$$

For $\sigma = \infty$, we get $H^\infty[\varepsilon, \infty] = e^{-\varepsilon z} H^\infty(\mathbb{C}_+)$. For more details about $H^\infty[\varepsilon, \sigma]$, we refer to **[Vit05b]** and the references therein.
The following lemma is a consequence of the Phragmén-Lindelöf principle, see Theorem A.2, and can be found in Boas [**Boa54**, Theorem 6.2.4, p.82].

LEMMA 3.6. *Let g be an entire function of exponential type σ such that $\|g\|_{\infty, i\mathbb{R}} := \sup_{y\in\mathbb{R}} |g(iy)| < \infty$. Then, for all $x, y \in \mathbb{R}$,*

$$|g(x+iy)| \leqslant e^{\sigma|y|} \|g\|_{\infty, i\mathbb{R}}.$$

Using Lemma 3.6, Theorem 3.4 yields an estimate in the $H^\infty(\mathbb{C}_+)$-norm.

THEOREM 3.7. *Let* $A \in \mathrm{Sect}(\omega)$ *with* $\omega < \frac{\pi}{2}$, *and let* $0 < \varepsilon < \sigma < \infty$. *For* $g \in H^\infty[\varepsilon, \sigma]$,

$$\|g(A)\| \leqslant \|g\|_{\infty,\mathbb{C}_+} \cdot \inf_{\phi \in (\omega, \frac{\pi}{2}), k \geqslant 1} \frac{M(A,\phi)}{\pi} \, b\left(\varepsilon, \tfrac{1}{k\sigma}, \phi\right) e^{\frac{\sigma-\varepsilon}{k\sigma}}. \tag{3.18}$$

where $b(\varepsilon, r, \phi)$ *is defined in (3.12).*

PROOF. Let $f(z) = e^{\varepsilon z} g(z)$. By (3.17), f lies in $H^\infty(\mathbb{C}_+)$ and is entire of exponential type $\sigma - \varepsilon$. Let $k \geqslant 1$. Since f is entire, and bounded on $\mathbb{C}_+$, we can apply Theorem 3.4 with $r_0 = \frac{1}{k\sigma}$. Thus, for $\phi \in (\omega, \frac{\pi}{2})$,

$$\|g(A)\| = \|(fe_\varepsilon)(A)\| \leqslant \inf_{\phi \in (\omega, \frac{\pi}{2})} \frac{M(A,\phi)}{\pi} \cdot b\left(\varepsilon, \tfrac{1}{k\sigma}, \phi\right) \cdot \|f\|_{\infty, \Omega_{\phi, \frac{1}{k\sigma}}},$$

where $\Omega_{\phi, \frac{1}{k\sigma}} = \Sigma_\phi \cup B_{\frac{1}{k\sigma}}(0)$. Clearly, $\|f\|_{\infty,\Omega_{\phi,\frac{1}{k\sigma}}} \leqslant \|f\|_{\infty, \mathbb{C}_+ \cup B_{\frac{1}{k\sigma}}(0)}$. Moreover, as f is entire of exponential type $\sigma - \varepsilon$ and $\sup_{y \in \mathbb{R}} |f(iy)| = \|f\|_{\infty,\mathbb{C}_+}$, we can apply Lemma 3.6 to conclude that

$$\|f\|_{\infty,\Omega_{\phi,\frac{1}{k\sigma}}} \leqslant e^{\frac{\sigma-\varepsilon}{k\sigma}} \|f\|_{\infty,\mathbb{C}_+}.$$

Since $\|g\|_{\infty,\mathbb{C}_+} = \|f\|_{\infty,\mathbb{C}_+}$, the assertion follows. □

Now we write Theorem 3.7 in the terminology used in **[Vit05b]**. This will reveal that the dependence on $M(A,\phi)$ of our approach is improving the corresponding estimate in **[Vit05b]**.

In **[Vit05b]**, for $\theta \in (0, \pi]$, a densely defined closed operator is called θ-*sectorial*, if $\sigma(A)$ is contained in $\Sigma_\theta \cup \{0\}$ (note that in our definition of $\mathrm{Sect}(\theta)$, $\sigma(A)$ is contained in $\overline{\Sigma}_\theta$) and

$$\tilde{M}(A,\theta) = \sup_{z \in \mathbb{C} \setminus (\Sigma_\theta \cup \{0\})} \|zR(z,A)\| < \infty.$$

By $S(\theta)$ let us denote the θ-sectorial operators on X. As pointed out in **[Vit05b**, Section 1.1], $S(\theta) \subset \mathrm{Sect}(\theta) \subset S(\theta + \epsilon)$ for all $\epsilon > 0$ and $S(\theta) = \bigcup_{0<\theta'<\theta} \mathrm{Sect}(\theta')$. Moreover, for $A \in S(\frac{\pi}{2})$ there exists a $\theta < \frac{\pi}{2}$ such that $A \in S(\theta)$, see Lemma 3.8 below. Hence, $A \in \mathrm{Sect}(\theta)$ for some $\theta < \frac{\pi}{2}$ if and only if $A \in S(\frac{\pi}{2})$. Furthermore, for $A \in S(\theta)$, it is a simple consequence of continuity that

$$\tilde{M}(A,\theta) = \sup_{z \in \mathbb{C} \setminus (\Sigma_\theta \cup \{0\})} \|zR(z,A)\| = \sup_{z \in \mathbb{C} \setminus \overline{\Sigma_\theta}} \|zR(z,A)\| = M(A,\theta). \tag{3.19}$$

The following lemma can be found in **[Vit05b**, Lemma 1.1].

LEMMA 3.8. *Let* $A \in S(\frac{\pi}{2})$ *and* $M = \tilde{M}(A, \frac{\pi}{2})$. *Then,* $A \in S(\theta)$ *for*

$$\theta = \arccos \frac{1}{2M} \quad \textit{and} \quad \tilde{M}(A,\theta) = M(A,\theta) \leqslant 2M. \tag{3.20}$$

Note that $S(\theta) = \bigcup_{0<\eta<\theta} \mathrm{Sect}(\eta)$. *Since* $M \geqslant 1$ (**[Haa06a**, Prop.2.1.1]), $\theta \in (\frac{\pi}{3}, \frac{\pi}{2})$.

THEOREM 3.9. *Let* $A \in \mathrm{Sect}(\omega)$ *with* $\omega < \frac{\pi}{2}$, *which is equivalent to* $A \in S(\frac{\pi}{2})$ *(see above). Then, with* $M = M(A, \frac{\pi}{2})$,

(i) For all $t \geqslant 0$,

$$\|e^{-tA}\| \leqslant \frac{2M}{\pi}(\log(M) + 5). \tag{3.21}$$

(ii) For $0 < \varepsilon < \sigma < \infty$ *and* $g \in H^\infty[\varepsilon, \sigma]$,

$$\|g(A)\| \leqslant \left(C_1 + C_2 \log\left(\frac{\sigma}{\varepsilon}\right)\right) \|g\|_{\infty,\mathbb{C}_+} \leqslant C_3 \log\left(\frac{\sigma e}{\varepsilon}\right) \|g\|_{\infty,\mathbb{C}_+}, \tag{3.22}$$

with $C_1 = c_1 M + c_2 M \log(M)$, $C_2 = c_2 M$ *and* $C_3 = c_1 M + c_2 M \log(M)$ *and*

$$c_1 = \tfrac{2e^{\frac{1}{5}}}{\pi}\left(\log(10) + \tfrac{2\pi}{3}\right) \approx 3.42, \; c_2 = \tfrac{2e^{\frac{1}{5}}}{\pi} \approx 0.78.$$

PROOF. Let θ be the defined as in Lemma 3.8, hence, $\theta \in (\frac{\pi}{3}, \frac{\pi}{2})$, $\cos\theta = \frac{1}{2M}$, and $M(A,\theta) \leqslant 2M$. Using Theorem 3.7, we get

$$\|g(A)\| \leqslant \tfrac{2M}{\pi} \cdot \|g\|_{\infty,\mathbb{C}_+} \cdot \inf_{k\geqslant 1} b(\varepsilon, \tfrac{1}{k\sigma}, \theta) e^{\frac{\sigma-\varepsilon}{k\sigma}}. \tag{3.23}$$

It remains to estimate the infimum. For $k \geqslant 2$, $\frac{\varepsilon}{2Mk\sigma} < \frac{\varepsilon}{k\sigma} < \frac{1}{2}$ and thus, by (3.12) and (3.8), we get for $b = b(\varepsilon, \frac{1}{k\sigma}, \theta)$ that

$$b \cdot e^{\frac{\sigma-\varepsilon}{k\sigma}} = \left[\mathrm{Ei}\left(\frac{\varepsilon}{2Mk\sigma}\right) e^{\frac{\sigma-\varepsilon}{k\sigma}} + e^{\frac{1}{k}} \frac{2\pi}{3}\right] \leqslant \left[\log\left(\frac{2Mk\sigma}{\varepsilon}\right) e^{\frac{\sigma-\varepsilon}{k\sigma}} + e^{\frac{1}{k}} \frac{2\pi}{3}\right]. \tag{3.24}$$

To prove (ii), let $t = \varepsilon$ and set $g = e_t \in H^\infty[t, \sigma]$. Then, let $\sigma \to \varepsilon^+ = t^+$ in (3.24) and choose $k = 5$ (alternatively, apply Theorem 3.4 with $f(z) = 1$, $\varepsilon = t$ and $r_0 = \frac{1}{5\varepsilon}$).
To show (i), observe that, using $e^{\frac{\sigma-\varepsilon}{k\sigma}} < e^{\frac{1}{k}}$, the right-hand-side of (3.24) can be further estimated,

$$b \cdot e^{\frac{\sigma-\varepsilon}{k\sigma}} \leqslant \left[\log(M) + \log\left(\frac{\sigma}{\varepsilon}\right) + \log(2k) + \frac{2\pi}{3}\right] \cdot e^{\frac{1}{k}}.$$

Setting $k = 5$, we get the result. $\square$

REMARK 3.10.

(i) In [**Vit05b**, Lemma 1.2 and Theorem 1.6] Vitse derives similar estimates as in Theorem 3.9. However, she uses the Hille-Phillips calculus and considers elements of $H^\infty[\varepsilon,\sigma]$ that are Laplace transforms of $L^1(\varepsilon,\sigma)$-functions first. The approach moreover relies on estimates of derivatives of the (analytic) semigroup. This results in a similar estimate as in (3.22), but with the following constants

$$\tilde{C}_1 = \frac{30}{\pi}M^2, \quad \tilde{C}_2 = \frac{16}{\pi}M^3, \quad \tilde{C}_3 = \frac{30}{\pi}M^3.$$

The dependence on M is strongly improved by our approach, as M^3 gets replaced by $M(1+\log M)$. Moreover, a more careful study even shows that $C_i \leqslant \tilde{C}_i$, $i \in \{1,2,3\}$, for every $M \geqslant 1$.

(ii) We point out that Vitse uses an estimate for the semigroup, [**Vit05b**, Lemma 1.2] (which is slightly improved by (3.21)), to get an estimate for $H^\infty[\varepsilon,\sigma]$ functions. Whereas our estimates all follow directly from Theorem 3.7. In other words, (the estimate for) the dependence on M is the same for any $H^\infty[\varepsilon,\sigma]$ function, including e_ε.

(iii) The constants c_1 and c_2 in Theorem 3.9 can possibly be further improved by optimizing the choice of k in the proof.

3.2.3. Invertible A - exponentially stable semigroups. Theorems 3.4 and 3.7 deal with the situation of *bounded* analytic semigroups and functions f which are holomorphic at 0. As might be expected, a similar result holds for functions f not necessarily holomorphic at 0, but with a sectorial operator A having $0 \in \rho(A)$.

THEOREM 3.11. *Let* $A \in \mathrm{Sect}(\omega)$, $\omega < \phi < \pi/2$, *and* $0 \in \rho(A)$. *Then, for* $\varepsilon > 0$, $f \in H^\infty(\Sigma_\phi)$ *the operator* $(fe_\varepsilon)(A)$ *is bounded and for all* $\kappa \in (0,1)$,

$$\|(f\cdot e_\varepsilon)(A)\| \leqslant \frac{M(A,\phi)}{\pi} \cdot b_\kappa\left(\varepsilon, \tfrac{1}{\|A^{-1}\|}, \phi\right) \cdot \|f\|_{\infty,\phi}. \tag{3.25}$$

Here,

$$b_\kappa(\varepsilon,R,\phi) = \mathrm{Ei}\,(\varepsilon\kappa R\cos\phi) + \frac{\kappa}{1-\kappa}e^{-\varepsilon\kappa R\cos\phi}, \tag{3.26}$$

Hence, $b_\kappa(\varepsilon,R,\phi) \sim C_\kappa|\log(\varepsilon R\cos\phi)|$ *for* $\varepsilon R < \frac{1}{2}$ *and* $\|(fe_\varepsilon)(A)\|$ *goes to zero exponentially as* $\varepsilon \to \infty$ *by the properties of* Ei, *see (3.7) and (3.8).*

PROOF. Since $0 \in \rho(A)$ and $fe_\varepsilon \in H^\infty_{(0)}(\Sigma_\phi)$, $(fe_\varepsilon)(A)$ is well-defined by (3.4),

$$(f\cdot e_\varepsilon)(A) = \frac{1}{2\pi i}\int_{\partial\Sigma_\theta} f(z)e^{\varepsilon z}R(z,A)\,dz,$$

for $\theta \in (\omega, \phi)$ and where $\partial\Sigma_\theta$ denotes the boundary (orientated positively) of Σ_θ. Because $0 \in \rho(A)$, we have that the ball $B_{\frac{1}{\|A^{-1}\|}}(0)$ lies in $\rho(A)$. For $\kappa \in (0,1)$ set $r = \frac{\kappa}{\|A^{-1}\|}$. By Cauchy's theorem, we can replace the integration path $\partial\Sigma_\theta$ by $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ with

$$\Gamma_1 = \left\{se^{i\theta}, s \geqslant r\right\}, \Gamma_2 = \left\{re^{i\theta} - it, t \in (0, 2\Im(re^{i\theta}))\right\}, \Gamma_3 = \left\{-se^{-i\theta}, s \leqslant -r\right\}.$$

Thus,

$$\|(fe_\varepsilon)(A)\| \leqslant \frac{\|f\|_{\infty,\phi}}{2\pi} \int_\Gamma e^{-\varepsilon \operatorname{Re} z} \|R(z,A)\| \, |dz|. \tag{3.27}$$

By the resolvent identity, $\|R(z,A)\| \leqslant \frac{\|A^{-1}\|}{1-|z|\|A^{-1}\|}$, and thus, for $\kappa \in (0,1)$,

$$\|R(z,A)\| \leqslant \frac{\|A^{-1}\|}{1-\kappa} \quad \text{for} \quad |z| \leqslant r = \frac{\kappa}{\|A^{-1}\|}.$$

This yields, since $\Gamma_2 \subset B_r(0)$,

$$\begin{aligned}\int_\Gamma e^{-\varepsilon \operatorname{Re} z}\|R(z,A)\| \, |dz| &\leqslant \tfrac{\|A^{-1}\|}{1-\kappa} \int_{\Gamma_2} e^{-\varepsilon r \cos\theta} \, dt + 2M(A,\theta) \int_r^\infty \frac{e^{-\varepsilon s \cos\theta}}{s} \, ds \\ &= \tfrac{2\|A^{-1}\|}{1-\kappa} \, r \sin\theta \; e^{-\varepsilon r \cos\theta} + 2M(A,\theta)\mathrm{Ei}(\varepsilon r \cos\theta), \\ &\leqslant 2M(A,\theta) \left(\tfrac{\kappa}{1-\kappa} e^{-\varepsilon r \cos\theta} + \mathrm{Ei}(\varepsilon r \cos\theta)\right),\end{aligned}$$

as $M(A,\theta) \geqslant 1$, see e.g. [**Haa06a**, Proposition 2.1.1]. Letting $\theta \to \phi^-$ yields the assertion. □

REMARK 3.12. If A is sectorial and $R > 0$, then clearly RA is sectorial of the same angle. Since $f \mapsto f_R = f(R\cdot)$ is an isometric isomorphism on $H^\infty(\Sigma_\phi)$, and $(fe_\varepsilon)(RA) = (f_R e_{\varepsilon R})(A)$ by the composition rule of holomorphic functional calculus [**Haa06a**, Theorem 2.4.2], we see that it is sufficient to consider $\frac{1}{\|A^{-1}\|} = 1$ in the proof of Theorem 3.11.

Applying Theorem 3.11 to $f \equiv 1$ shows that $\|e_\varepsilon(A)\|$ decays exponentially for $\varepsilon \to \infty$. This behavior is natural as the condition that $0 \in \rho(A)$ implies that the analytic semigroup is exponentially stable. However, for $\varepsilon \to 0$, the theorem gives no bound for the norm. This can be derived by Theorem 3.4 as we will see in the following result.

COROLLARY 3.13. *Let $A \in \mathrm{Sect}(\omega)$ and $0 < \omega < \phi < \frac{\pi}{2}$. If A is invertible, then we define $R = \frac{1}{\|A^{-1}\|}$, otherwise we set R to be zero. Then, for any $\kappa \in [0,1)$, there exists a $C > 0$ such that*

$$\|e_\varepsilon(A)\| \leqslant Ce^{-\varepsilon\kappa R\cos\phi}, \quad \varepsilon > 0, \tag{3.28}$$

with $C \leqslant C_\kappa M(A,\phi)\mathrm{Ei}(\cos\phi)$.

PROOF. Let $f \equiv 1$. If $\varepsilon\kappa R > 1$, by (3.7),

$$\mathrm{Ei}(\varepsilon\kappa R\cos\phi) < e^{-\varepsilon\kappa R\cos\phi}\log\left(1+\tfrac{1}{\cos\phi}\right) < 2e^2 e^{-\varepsilon\kappa R\cos\phi}\mathrm{Ei}(\cos\phi),$$

where we used that $\mathrm{Ei}(2\cos\phi) < \mathrm{Ei}(\cos\phi)$ in the last inequality. Using this, Theorem 3.11 yields

$$\|e_\varepsilon(A)\| \leqslant \tilde{C}_\kappa M(A,\phi)\mathrm{Ei}(\cos\phi)e^{-\varepsilon\kappa R\cos\phi}, \quad \varepsilon\kappa R > 1, \tag{3.29}$$

where $\tilde{C}_\kappa > 0$ only depends on κ.
Now, let $\varepsilon\kappa R \leqslant 1$. We apply Theorem 3.4 with $r_0 = \frac{1}{\varepsilon}$. It implies that there exists an absolute constant C_2 such that $\|e_\varepsilon(A)\| \leqslant C_2 M(A,\phi)\mathrm{Ei}(\cos\phi)$. Together with (3.29) the assertion follows. □

Let us point out that the corollary is interesting in terms of the dependence on the constants $M(A,\phi)$, $\|A^{-1}\|$ and ϕ, whereas the exponential decay is clear for exponentially stable semigroups.
Further note that the use of the scaling variable κ is not so artificial as it might seem: By $B_{\frac{1}{\|A^{-1}\|}}(0) \subset \rho(A)$, we have that the growth bound ω_0 of the semigroup satisfies $\omega_0 \leqslant -\frac{\cos\phi}{\|A^{-1}\|}$. It is well-known that, even in the case of a *spectrum-determined* growth bound, as we have it for analytic semigroups, this rate need not be attained, see e.g., [**EN00**, Example I.5.7]. The κ encodes that we can achieve any exponential decay of rate $\tilde{\omega} \in (-\frac{\cos\phi}{\|A^{-1}\|}, 0]$.

3.3. Sharpness of the result

3.3.1. Diagonal operators on Schauder bases (Schauder multiplier). A typical construction of an unbounded calculus goes back to Baillon and Clement [**BC91**] and has been used extensively since then, see [**Fac15**] and the references therein. The situation is as follows.

Let $\{\phi_n\}_{n\in\mathbb{N}}$ be a Schauder basis of the Banach space X. For the sequence $\mu = (\mu_n)_{n\in\mathbb{N}}$ define the multiplication operator $\mathcal{M}_\mu$ by its action on the basis, i.e. $\mathcal{M}_\mu\phi_n = \mu_n\phi_n$, $n \in \mathbb{N}$, with maximal domain. The choice $\lambda_n = 2^n$ yields a sectorial operator $A = \mathcal{M}_\lambda \in \mathrm{Sect}(0)$ with $0 \in \rho(\mathcal{M}_\lambda)$, and for $f \in H^\infty(\mathbb{C}_+)$,

$$\begin{aligned} &f(A) = f(\mathcal{M}_\lambda) = \mathcal{M}_{f(\lambda)}, \\ &D(\mathcal{M}_{f(\lambda)}) = \left\{x = \textstyle\sum_{n\in\mathbb{N}} x_n\phi_n \in X : \sum_{n\in\mathbb{N}} f(\lambda_n)x_n\phi_n \text{ converges}\right\}. \end{aligned} \tag{3.30}$$

See e.g., [**Haa06a**, Chapter 9] and [**Fac15**].

Because of (3.30), a way of constructing unbounded calculi consists of the following two steps:

(i) Find a sequence $\mu \in \ell^\infty(\mathbb{N}, \mathbb{C})$ such that $\mathcal{M}_\mu \notin \mathcal{B}(X)$.

(ii) Find $f \in H^\infty(\mathbb{C}_+)$ such that $f(\lambda_n) = \mu_n$ for all $n \in \mathbb{N}$.

Since $\{\lambda_n\}$ is interpolating, see **[Gar07]**, the second step is always possible. Note that the first step follows if we can

$$\text{find } x \in X \text{ such that } x = \sum\nolimits_{n\in\mathbb{N}} x_n\phi_n \text{ does NOT converge unconditionally.} \tag{3.31}$$

In fact, then there exists a sequence $\mu_n \subset \{-1, 1\}$ such that $\sum_{n\in\mathbb{N}} \mu_n x_n \phi_n$ does not converge. Thus, $x \notin D(\mathcal{M}_\mu)$, and so $\mathcal{M}_\mu \notin \mathcal{B}(X)$.
Conversely, this indicates that *a bounded* H^∞*-calculus implies a large amount of unconditionality*, **[Haa06a**, p.124], which can be made rigorous, see **[Haa06a**, Section 5.6] and **[KW04]**. For more information about unbounded H^∞-calculi via diagonal operators, see **[Haa06a**, Chapter 9].
Let $\{\phi_n\}_{n\in I}$, $I \subset \mathbb{N}$, be a Schauder basis of a Banach space X. For finite $\sigma \subset I$, P_σ denotes the projection onto $X_\sigma := \{\phi_n\}_{n\in\sigma}$. Let us introduce the following constants,

$$m_\phi = \sup_{n\in I} \|P_{\{n\}}\|, \quad \kappa_\phi = \sup_{k\leqslant\ell} \left\|P_{[k,\ell]\cap I}\right\|, \quad ub_\phi = \sup_{\sigma\subset I, |\sigma|<\infty} \|P_\sigma\|. \tag{3.32}$$

The constant κ_ϕ is called the *basis constant* of $\{\phi_n\}_{n\in\mathbb{N}}$ and ub_ϕ the *uniform basis constant*. Clearly,

$$m_\phi \leqslant \kappa_\phi \leqslant ub_\phi. \tag{3.33}$$

THEOREM 3.14. *Let* $\{\phi_n\}_{n\in\mathbb{N}}$ *be a Schauder basis on a Banach space* X *with* $m_\phi < \infty$. *Let* $\lambda_n = c^n$, $n \in \mathbb{N}$ *for* $c > 1$.
Then $A := \mathcal{M}_\lambda$ *is sectorial of angle* 0, *i.e.,* $A \in \mathrm{Sect}(0)$ *and the following holds.*

(i) $M(A, \psi) \leqslant \kappa_\phi M(\psi)$ *for all* $\psi \in (0, \pi]$, *where* $M(\psi)$ *only depends on* ψ.

(ii) $0 \in \rho(A)$ *and* $\mathrm{dist}(\sigma(A), 0) = c$.

(iii) *For* $\varepsilon > 0$ *and* $N_\varepsilon = \lfloor \frac{2\mathrm{Ei}(\varepsilon)}{\log c} \rfloor$, *there holds*

$$\|(f \cdot e_\varepsilon)(A)\| \leqslant \left(\pi \cdot ub_{\{\phi_n\}_{n=1}^{N_\varepsilon}} + m_\phi e^{-k_\varepsilon} \left(\frac{K_1}{\log c} + 1\right)\right) \|f\|_{\infty,\psi}, \tag{3.34}$$

for all $f \in H^\infty(\Sigma_\psi)$, $\psi \in (0, \frac{\pi}{2})$ *and* $\varepsilon > 0$ *and*

$$k_\varepsilon = \begin{cases} K_0 & \varepsilon \leqslant \varepsilon_c, \\ \max\{K_0, c\varepsilon\} & \varepsilon > \varepsilon_c, \end{cases} \tag{3.35}$$

with absolute constants K_0, $K_1 > 0$ *and* ε_c *such that* $2\mathrm{Ei}(\varepsilon_c) < \log c$.

Here, m_ϕ, κ_ϕ *and* $ub_{\{\phi_n\}_{n=1}^{N_\varepsilon}}$ *are defined in (3.32).*

PROOF. By [**Haa06a**, Lemma 9.1.2 and its proof], $A \in \mathrm{Sect}(0)$ with $M(A,\phi) \leqslant \kappa_\phi M(\psi)$, where $M(\psi)$ only depends on $\psi \in (0,\pi]$. Clearly, $\sigma(A) \subset [\lambda_1, \infty)$. This shows *(i)* and *(ii)*.
To show *(iii)*, note that for $N_\varepsilon = \lfloor \frac{2\mathrm{Ei}(\varepsilon)}{\log c} \rfloor$,

$$h(\varepsilon) := c^{N_\varepsilon+1}\varepsilon \geqslant c^{\frac{2\mathrm{Ei}(\varepsilon)}{\log c}}\varepsilon = e^{2\mathrm{Ei}(\varepsilon)}\varepsilon \overset{(3.7)}{\geqslant} \left(1+\tfrac{1}{\varepsilon}\right)^{e^{-\varepsilon}} \varepsilon > K_0,$$

for some constant $K_0 \in (0,1)$ and all $\varepsilon > 0$. If $N_\varepsilon = 0$, which means that $2\mathrm{Ei}(\varepsilon) < \log c$, then $h(\varepsilon) = c\varepsilon$. Since Ei is bijective and decreasing on $(0,\infty)$, this yields that there exists an $\varepsilon_c > 0$ such that $h(\varepsilon) \geqslant k_\varepsilon$, with k_ε defined in (3.35).
Now,

$$\begin{aligned}\left\| \sum_{n\in\mathbb{N}} f(\lambda_n) e^{-c^n\varepsilon} P_{\{n\}} \right\| &\leqslant \left\| \sum_{n=1}^{N_\varepsilon} f(c^n) e^{-c^n\varepsilon} P_{\{n\}} \right\| + \left\| \sum_{n=N_\varepsilon+1}^{\infty} f(c^n) e^{-c^n\varepsilon} P_{\{n\}} \right\| \\ &\leqslant \pi \cdot \mathrm{ub}_{\{\phi_n\}_{n=1}^{N_\varepsilon}} \cdot \|fe_\varepsilon\|_\infty + \\ &\quad + \sum_{k=0}^{\infty} \left| f(c^{k+N_\varepsilon+1}) e^{-h(\varepsilon)c^k} \right| \left\| P_{\{k+N_\varepsilon+1\}} \right\| \\ &\leqslant \pi \cdot \mathrm{ub}_{\{\phi_n\}_{n=1}^{N_\varepsilon}} \cdot \|f\|_\infty + m_\phi \|f\|_\infty \sum_{k=0}^{\infty} e^{-k_\varepsilon c^k},\end{aligned}$$

where we used [**Nik13**, Lemma 2.9.1] to estimate the first term in the second line. It remains to estimate the sum. By (3.69),

$$\sum_{k=0}^{\infty} e^{-k_\varepsilon c^k} \leqslant e^{-k_\varepsilon} + \frac{\mathrm{Ei}(k_\varepsilon)}{\log c} \overset{(3.7)}{\leqslant} e^{-k_\varepsilon}\left(1 + \frac{\log(1+\frac{1}{k_\varepsilon})}{\log c}\right).$$

Since $k_\varepsilon \geqslant K_0$, we can bound $\log(1+\frac{1}{k_\varepsilon})$ by $K_1 = \log\left(1+\frac{1}{K_0}\right)$. □

REMARK 3.15.

(i) We point out that (3.34) shows that for $\varepsilon \to \infty$, $\|(fe_\varepsilon)(A)\|$ goes to 0 exponentially.

(ii) Using (3.7) it is easy to show that in Theorem 3.14, ε_c can be chosen to be $\frac{1}{\sqrt{c}-1}$.

In (3.34) the ε-dependence for small ε of the right hand side appears only in the term $\mathrm{ub}_{\{\phi_n\}_{n=1}^{N_\varepsilon}}$. The following result shows that this indeed exhibits a logarithmic behavior for $\varepsilon \to 0$, which confirms the result from Theorem 3.4. We also show that on Hilbert spaces the behavior is slightly better.

THEOREM 3.16. *Let* $\{\phi_n\}_{n\in\mathbb{N}}$, X, c, A *be as in Theorem 3.14. Then, the following assertions hold for all* $\psi \in (0,\pi)$, $f \in H^\infty(\Sigma_\psi)$, $\varepsilon > 0$.
If X *is a Banach space, then*

$$\|(f\cdot e_\varepsilon)(A)\| \leqslant \left(\frac{K_2}{\log c}+1\right)\cdot m_\phi \cdot \mathrm{Ei}(\varepsilon)\cdot \|f\|_{\infty,\psi}. \tag{3.36}$$

If X *is a Hilbert space, then*

$$\|(f\cdot e_\varepsilon)(A)\| \leqslant \left(\frac{K_3}{\log c}+1\right)\cdot m_\phi \cdot \mathrm{Ei}(\varepsilon)^{1-\frac{0.32}{\kappa_\phi^2}}\cdot \|f\|_{\infty,\psi}. \tag{3.37}$$

Here the K_2 *and* K_3 *are absolute constants.*

PROOF. By (3.34), it remains to estimate $ub_{\{\phi_n\}_{n=1}^{N_\varepsilon}}$. For a basis $\tilde\phi$ of a general N-dimensional Banach space, it is easy to see that $ub_{\tilde\phi} \leqslant N m_{\tilde\phi}$. Since $N_\varepsilon = \lfloor \frac{2\mathrm{Ei}(\varepsilon)}{\log c}\rfloor$, and $m_{\{\phi_n\}_{n=1}^{N_\varepsilon}} \leqslant m_\phi$, this implies (3.36).
For a basis $\tilde\phi$ of an N-dimensional Hilbert space, we have that

$$ub_{\tilde\phi} \leqslant 2m_{\tilde\phi}\cdot N^{1-\frac{0.32}{\kappa_{\tilde\phi}^2}}. \tag{3.38}$$

This is due to a recent result by Nikolski, [**Nik13**, Theorem 3.1], which is a slight generalization of a classic theorem by McCarthy-Schwartz, [**MS65**]. Hence, because $m_{\{\phi_n\}_{n=1}^{N_\varepsilon}} \leqslant m_\phi$ and $\kappa_{\{\phi_n\}_{n=1}^{N_\varepsilon}} \leqslant \kappa_\phi$,

$$ub_{\{\phi_n\}_{n=1}^{N_\varepsilon}} \leqslant 2m_\phi N_\varepsilon^{1-\frac{0.32}{\kappa_\phi^2}}.$$

By the definition of N_ε, this yields (3.37). □

REMARK 3.17. The key ingredient of the proof of (3.37) in Theorem 3.16 is the McCarthy-Schwartz-type result, (3.38). For general Banach spaces this does not hold. However, there exists a version of McCarthy-Schwartz's result for uniformly convex spaces by Gurarii and Gurarii [**GG71**], see also [**Nik13**, Theorem 3.6.1 and Corollary 3.6.8]. In particular, this enables us to deduce an estimate similar to (3.37) for L^p-spaces with $p > 1$.

3.3.2. A particular example. Apart from functional calculus, the following type of example has been used to construct Schauder multipliers in various situations, e.g., [**BN99, EZ06, Haa12, JPP09, ZJS03**].

DEFINITION 3.18. Let $X = L^2 = L^2(-\pi,\pi)$, $\beta \in (\frac14,\frac12)$. Define $\{\phi_n\}_{n\in\mathbb{N}}$ by

$$\phi_{2k}(t) = w_\beta(t)e^{ikt}, \quad \phi_{2k+1}(t) = w_\beta(t)e^{-ikt},$$

where $k \in \mathbb{N} \cup \{0\}$, $t \in (-\pi, \pi)$ and

$$w_\beta(t) = \begin{cases} |t|^\beta, & |t| \in (0, \frac{\pi}{2}), \\ (\pi - |t|)^{-\beta}, & |t| \in [\frac{\pi}{2}, \pi). \end{cases}$$

$\{\phi_n\}_{n\in\mathbb{N}}$ forms a Schauder basis of L^2, see Lemma 3.32.

THEOREM 3.19. *There exists* $g \in H^\infty(\mathbb{C}_+)$ *such that the following holds. For every* $\delta \in \left(0, \frac{1}{2}\right)$ *there exists* $A \in \mathrm{Sect}(0)$ *on* $H = L^2(-\pi, \pi)$ *with*

(i) $0 \in \rho(A)$ *and* $\mathrm{dist}(\sigma(A), 0) = 2$*,*

(ii) $M(A, \phi) \leqslant \frac{1}{\delta} M(\phi)$ *for all* $\phi \in (0, \pi]$*, where* $M(\phi)$ *only depends on* ϕ*.*

(iii) For all $\varepsilon > 0$*,* $f \in H^\infty(\mathbb{C}_+)$*, and some absolute constant* K_0*,*

$$\|(f \cdot e_\varepsilon)(A)\| \lesssim \frac{1}{\delta} \cdot \mathrm{Ei}(\varepsilon)^{1-K_0\delta^2} \cdot \|f\|_\infty. \tag{3.39}$$

(iv) For $\varepsilon \in (0, \frac{1}{2})$*,*

$$\|(g \cdot e_\varepsilon)(A)\| \gtrsim \frac{1}{\delta} \cdot |\log(\varepsilon)|^{1-\delta}. \tag{3.40}$$

PROOF. Let $\beta = \frac{1}{2} - \frac{\delta}{4} \in (\frac{3}{8}, \frac{1}{2})$ and let $\{\phi_n\}_{n\in\mathbb{N}}$ denote the basis from Definition 3.18 and $\{\phi_n^*\}_{n\in\mathbb{N}}$ its dual basis, see Lemma 3.32. By Lemma 3.32 (i), $\kappa_\phi \lesssim \frac{1}{1-2\beta} = \frac{2}{\delta}$. W.r.t. $\{\phi_n\}_{n\in\mathbb{N}}$, we consider the multiplication operator $A = \mathcal{M}_\lambda$ on $L^2(-\pi, \pi)$, where $\lambda_n = 2^n$. By Theorem 3.14, (i) and (ii) follow.
(iii) follows by (3.37) from Theorem 3.16.
To show (iv) we choose $x(t) = |t|^{-\beta} \mathbb{1}_{(0, \frac{\pi}{2})}(|t|)$ and $y(t) = (\pi - |t|)^{-\beta} \mathbb{1}_{(\frac{\pi}{2}, \pi)}(|t|)$. By Lemma 3.32 (iii), we have that for $x = \sum_n x_n \phi_n$ and $y = \sum_n y_n \phi_n^*$, the coefficients x_n and y_n are real and that

$$x_{2k} = x_{2k+1} \sim \frac{k^{-1+2\beta}}{1-2\beta} \quad \text{and} \quad y_{2k} = y_{2k+1} = (-1)^k 2\pi \cdot x_{2k}. \tag{3.41}$$

Thus, by setting $\mu_{2n} = \mu_{2n+1} = (-1)^n$ for all $n \in \mathbb{N}$, we conclude by using that $\langle \phi_n, \phi_m^* \rangle = \delta_{nm}$,

$$\begin{aligned} |\langle \mathcal{M}_\mu \mathcal{M}_{e^{-\lambda_n \varepsilon}} x, y \rangle| &= 2\pi \sum_{n\in\mathbb{N}} e^{-\lambda_n \varepsilon} |x_n|^2 \\ &\gtrsim \frac{1}{(1-2\beta)^2} \sum_{k\in\mathbb{N}} (e^{\lambda_{2k}\varepsilon} + e^{\lambda_{2k+1}\varepsilon}) k^{-2+4\beta} \\ &\gtrsim \frac{1}{(1-2\beta)^2} |\log(\varepsilon)|^{-1+4\beta}, \end{aligned} \tag{3.42}$$

for $\varepsilon < \frac{1}{2}$, where we have used (3.41) and Lemma 3.30. Since $\|x\| \cdot \|y\| \sim \frac{1}{1-2\beta}$, and $2 - 4\beta = \delta$,

$$\|\mathcal{M}_\mu \mathcal{M}_{e^{-\lambda_n \varepsilon}}\| \gtrsim \frac{1}{\delta} |\log(\varepsilon)|^{1-\delta}, \quad \varepsilon \in (0, \tfrac{1}{2}). \tag{3.43}$$

Since (λ_n) is an interpolating sequence, we can find $g \in H^\infty(\mathbb{C}_-)$ such that $g(\lambda_n) = \mu_n$ for all $n \in \mathbb{N}$. Thus, $g(A) = \mathcal{M}_\mu$ and (3.40) follows. □

The example shows that estimate (3.25) in Theorem 3.11 is sharp in $M(A,\phi)$ and ε as $\delta \to 0^+$.

COROLLARY 3.20. *Let* X *be a Banach space,* $0 < \omega < \phi < \frac{\pi}{2}$. *Then, there exists* K *depending only on* ϕ *such that*

$$\sup\left\{\frac{\|(fe_\varepsilon)(A)\|}{M(A,\phi)\|f\|_\infty} : A \in \mathrm{Sect}_X(\omega), \mathrm{dist}(\sigma(A), 0) \geqslant 1, 0 \neq f \in H^\infty\right\} > K|\log\varepsilon|,$$

for all $\varepsilon < \frac{1}{2}$. *Here,* $H^\infty = H^\infty(\mathbb{C}_+)$ *and* $\mathrm{Sect}_X(\omega)$ *denotes all* $A \in \mathrm{Sect}(\omega)$ *on* X.

REMARK 3.21. *(i)* As the examples are on Hilbert spaces, the sharpness from Corollary 3.20 even holds on Hilbert spaces. However, we point out that in Theorem 3.19 $M(A,\phi) \to \infty$ as $\delta \to 0^+$. Therefore, for fixed $M(A,\phi)$, the behavior in $\varepsilon \to 0^+$ could be better than $|\log\varepsilon|$. For a similar effect we refer to the question of the sharpness of Spijker's result on the *Kreiss-Matrix-Theorem*, see **[Spi91, STW03]** and the recent contribution by Nikolski **[Nik13]**.

(ii) In [**Vit05b**, Theorem 2.1, Remark 2.2], it is shown that estimate (3.22) is indeed sharp in ε and σ on general Banach spaces. Furthermore, Vitse [**Vit05b**, Theorem 2.3 and Remark 2.4] states that for every Hilbert space and every $\delta \in (0,1)$, one can find a sectorial operator A with angle less than $\frac{\pi}{2}$ such that

$$\sup\{\|g(A)\| : g \in H^\infty[\varepsilon,\sigma], \|g\|_{\infty,\mathbb{C}_+} \leqslant 1\} \geqslant a\log\left(\frac{e\sigma}{\varepsilon}\right)^\delta, \tag{3.44}$$

where a depends only on $M(A,\frac{\pi}{2})$. Therefore, item (iii) of Theorem 3.19 and Corollary 3.20 can be seen as a version for $0 \in \rho(A)$ and $\sigma = \infty$. However, Theorem 3.19(iv) shows that the behavior of $\|(fe_\varepsilon)(A)\|$ is indeed better than $|\log(\varepsilon)|$. We remark that Vitse's result, [**Vit05b**, Theorem 2.3] is stated for Banach spaces which *uniformly contain uniformly complemented copies of* ℓ^2, which is more general than for Hilbert spaces.

3.4. Square function estimates improve the situation

The following notion characterizes bounded H^∞-calculus on Hilbert spaces. It was already used in the early work of McIntosh, [**McI86**] and has been investigated intensively since then.

DEFINITION 3.22. Let $A \in \mathrm{Sect}(\omega)$ on the Banach space X. We say that A satisfies *square function estimates* if there exists $\psi \in H_0^\infty(\Sigma_\phi) \setminus \{0\}$, $\phi > \omega$ and $K_\psi > 0$ such that

$$\int_0^\infty \|\psi(tA)x\|^2 \frac{dt}{t} \leqslant K_\psi^2 \|x\|^2, \quad \forall x \in X. \tag{3.45}$$

The property of satisfying square functions estimates does not rely on the particular function ψ. In fact, for $\psi, \tilde{\psi} \in H_0^\infty(\Sigma_\phi) \setminus \{0\}$

$$\exists K > 0\, \forall h \in H^\infty(\Sigma_\phi): \quad \int_0^\infty \|(\psi_t h)(A)x\|^2 \frac{dt}{t} \leqslant K^2 \|h\|_{\infty,\phi}^2 \int_0^\infty \|\tilde{\psi}_t(A)x\|^2 \frac{dt}{t}, \tag{3.46}$$

where $\psi_t(z) = \psi(tz)$ and $\tilde{\psi}_t(z) = \tilde{\psi}(tz)$. We remark that for K can be chosen only depending on $\psi, \tilde{\psi}$ and $M(A, \phi)$. The result can be found in [**ADM96**, Proposition E] for Hilbert spaces, but also holds for general Banach spaces as pointed out in [**Haa05**, Satz 2.1.5], see also [**Haa06a**, Theorem 6.4.2]. The following result goes back to McIntosh in his early work on H^∞-calculus, [**McI86**] and can also be found in [**Haa06a**, Theorem 7.3.1].

THEOREM 3.23 (McIntosh '86). *Let X be a Hilbert space, $A \in \mathrm{Sect}(\omega)$, densely defined and with dense range. Then, the following assertions are equivalent.*

(i) The $H^\infty(\Sigma_\mu)$-calculus for A is bounded for some (all) $\mu \in (\omega, \pi)$.

(ii) A and A^* satisfy square function estimates.

Note that on a Hilbert space, $\overline{D(A)} = X$ follows from sectorality, see [**Haa06a**, Proposition 2.1.1].

Le Merdy showed in [**LM03**, Theorem 5.2] that having square function estimates for only A or A^* is not sufficient to get a bounded calculus. However, we will show that the validity of single square function estimates always yields an improved growth of $\|(fe_\varepsilon)(A)\|$ near zero. Roughly speaking, having 'half of the assumptions' in McIntosh's result indeed interpolates the general logarithmic behavior of $\|(fe_\varepsilon)(A)\|$.

THEOREM 3.24. *Let $\omega < \phi < \frac{\pi}{2}$ and $A \in \mathrm{Sect}(\omega)$ be densely defined on the Banach space X. Assume that*

- *$0 \in \rho(A)$ and that*
- *A satisfies square function estimates.*

Then for every $\kappa \in (0,1)$ there exists $C = C(\kappa, M(A,\phi), \cos(\phi)) > 0$ such that for all $\varepsilon > 0$ and for $f \in H^\infty(\Sigma_\phi)$,

$$\|(fe_\varepsilon)(A)\| \leqslant CK_\psi \cdot \left[\mathrm{Ei}\left(\frac{\kappa\varepsilon\cos\phi}{\|A^{-1}\|}\right)\right]^{\frac{1}{2}} \cdot \|f\|_{\infty,\phi}, \tag{3.47}$$

where K_ψ denotes the constant in (3.45) for $\psi(z) = z^{\frac{1}{2}} e^{-\frac{z}{2}}$.

PROOF. Let $\eta(z) = ze^{-z}$. Since $\sqrt{\eta} \in H_0^\infty(\Sigma_\phi)$, we have by (3.46) that

$$\int_0^\infty \|(fe_\varepsilon\sqrt{\eta_t})(A)x\|^2 \frac{dt}{t} \leqslant K^2 \|fe_\varepsilon\|_{\infty,\phi}^2 \cdot \int_0^\infty \|(\sqrt{\eta_t})(A)x\|^2 \frac{dt}{t}, \tag{3.48}$$

where $K > 0$ only depends on $M(A,\phi)$ and η. The integral on the right-hand side is finite because A satisfies square function estimates (for $\sqrt{\eta}$). It is easy to see that $\int_0^\infty \eta_t(z) \frac{dt}{t} = 1$ for $z \in \Sigma_\phi$, and applying the Convergence Lemma, [**Haa06a**, Proposition 5.1.4], yields $y = \int_0^\infty \eta_t(A)y \frac{dt}{t}$ for $y \in X$. Thus,

$$\begin{aligned}
\|(fe_\varepsilon)(A)x\| &= \left\|\int_0^\infty (fe_\varepsilon\eta_t)(A)x \frac{dt}{t}\right\| \\
&\leqslant \int_0^\infty \left\|\left(e_{\frac{\varepsilon}{2}}\sqrt{\eta_t}\right)(A)\left(fe_{\frac{\varepsilon}{2}}\sqrt{\eta_t}\right)(A)x\right\| \frac{dt}{t} \\
&\leqslant \left(\int_0^\infty \|\left(e_{\frac{\varepsilon}{2}}\sqrt{\eta_t}\right)(A)\|^2 \frac{dt}{t}\right)^{\frac{1}{2}} \left(\int_0^\infty \|\left(fe_{\frac{\varepsilon}{2}}\sqrt{\eta_t}\right)(A)x\|^2 \frac{dt}{t}\right)^{\frac{1}{2}}.
\end{aligned} \tag{3.49}$$

In the last step we used that $t \mapsto (e_{\frac{\varepsilon}{2}}\sqrt{\eta_t})(A)$ is continuous in the operator norm which makes the first integral exist. In fact, $e^{-\frac{\varepsilon z}{2}}\sqrt{\eta_t(z)} = (zt)^{\frac{1}{2}}e^{-z\frac{t+\varepsilon}{2}} \in H_0^\infty(\Sigma_\phi)$, and hence by the functional calculus for sectorial operators,

$$\left[e^{-\frac{\varepsilon z}{2}}\sqrt{\eta_t(z)}\right](A) = t^{\frac{1}{2}}A^{\frac{1}{2}}T\left(\frac{t+\varepsilon}{2}\right). \tag{3.50}$$

For $s > 0$ we have that $A^{\frac{1}{2}}T(s) = A^{-\frac{1}{2}}AT(s) = A^{-\frac{1}{2}}\frac{\partial}{\partial s}T(s)$. Since $s \mapsto T(s)$ is $C^\infty(\mathbb{R}_+, \mathcal{B}(X))$ for analytic semigroups and $A^{-\frac{1}{2}} \in \mathcal{B}(X)$ as $0 \in \rho(A)$, we get indeed that $t \mapsto (e_{\frac{\varepsilon}{2}}\sqrt{\eta_t})(A)$ is continuous in the operator norm.
By (3.48) we can estimate the second integral in (3.49) and find

$$\|(fe_\varepsilon)(A)x\| \leqslant \left(\int_0^\infty \|\left(e_{\frac{\varepsilon}{2}}\sqrt{\eta_t}\right)(A)\|^2 \frac{dt}{t}\right)^{\frac{1}{2}} \cdot K \cdot \|f\|_{\infty,\phi} \cdot K_{\sqrt{\eta}}\|x\|. \tag{3.51}$$

Hence, it remains to study the first term in (3.51). By (3.50) and Lemma 3.34

$$\begin{aligned}
\int_0^\infty \|\left(e^{-\frac{\varepsilon}{2}\cdot}\sqrt{\theta_t}\right)(A)\|^2 \frac{dt}{t} &= \int_{\frac{\varepsilon}{2}}^\infty \|A^{\frac{1}{2}}T(t)\|^2 \, dt \\
&\leqslant \tilde{C}^2 \int_{\frac{\varepsilon}{2}}^\infty t^{-1}e^{-2tR\kappa\cos\omega} \, dt \\
&= \tilde{C}^2 \cdot \mathrm{Ei}\left(\kappa\varepsilon R\cos\phi\right),
\end{aligned} \tag{3.52}$$

for $\kappa \in (0,1)$, $R = \frac{1}{\|A^{-1}\|}$ and $\tilde{C} = C_{\frac{1}{2},\kappa}M(A,\phi)(\cos\phi)^{-\frac{1}{2}} > 0$, see Lemma 3.34. □

REMARK 3.25. In **[GMY11]**, Galé, Miana and Yakubovich draw a connection between the H^∞-calculus for sectorial operators and the theory of functional models for Hilbert space operators. They prove (however, without using this connection) a, as they call it, *logarithmic gap* between the Hilbert space H and H_A. H_A is the space of elements of H such that

$$\|x\|_A^2 = \int_0^\infty \|\psi(tA)x\|^2 \frac{dt}{t} < \infty,$$

for some $\psi \in H_0^\infty(\Sigma_\phi) \setminus \{0\}$. From Theorem 3.23 it is clear that the $H^\infty(\Sigma_\phi)$-calculus is bounded if and only if the norm $\|\cdot\|_A$ is equivalent to the norm of the space H. In the view of **[Haa06a**, Section 6.4], H_A is the *intermediate space* $X_{0,\psi,2}$. This space, in turn, can be shown to be equal to the real interpolation space $\left(H^{(1)}, H^{(-1)}\right)_{\frac{1}{2},2}$, see **[Haa06a**, Theorem 6.4.5], where $H^{(1)}$ and $H^{(-1)}$ are the *homogeneous spaces* for A. In **[GMY11]**, the *logarithmic gap* refers to the result that for all $r > \frac{1}{2}$ there exist $c_r > 0$ such that

$$c_r^{-1}\|\Lambda_1(A)^{-r}x\| \leqslant \|x\|_A \leqslant c_r\|\Lambda_1(A)^r x\|, \tag{3.53}$$

for all $x \in \Lambda_1(A)^{-r}H$, where $\Lambda_1(z) = \mathrm{Log}(z) + 2\pi i$ (here, Log denotes the principle branch of the logarithm) and where $\Lambda_1^{-r}(A)H$ is interpreted as a (dense) subspace of H, see Theorem 2.1 in **[GMY11]**. We learned from D. Yakubovich that it seems that this result can be used to derive estimates of $\|(fe_\varepsilon)(A)\|$ of the form in (3.3), which are slightly weaker than our results presented here.
However, as H_A is an interpolation space, (3.53) should be rather seen as the consequence of the 'idea' that functional calculus properties for A improve in the corresponding interpolation spaces. More generally, this motivates the study of the relation between the results in this Chapter and interpolation spaces. This is subject to future research.

The following theorem proves that the result in Theorem 3.24 is essentially sharp.

THEOREM 3.26. *There exists a Hilbert space* X *and* $g \in H^\infty(\mathbb{C}_+)$ *such that for any* $\delta \in (0, \frac{1}{2})$ *there exists a* $A \in \mathrm{Sect}(0)$ *on* X *with*

(i) $0 \in \rho(A)$,

(ii) A^* *satisfies square function estimates,*

(iii) *for some* $\tilde{C} > 0$,

$$\|(ge_{\varepsilon\cdot})(A)\| \geqslant \tilde{C} \cdot |\log(\varepsilon)|^{\frac{1}{2}-\delta}, \qquad \varepsilon \in (0, \tfrac{1}{2}). \tag{3.54}$$

(iv) *For all* $\varepsilon > 0$ *and* $f \in H^\infty(\mathbb{C}_+)$

$$\|(f \cdot e_\varepsilon)(A)\| \leqslant c_\delta \cdot \mathrm{Ei}(\varepsilon)^{\frac{1}{2}-\frac{\delta}{6}} \cdot \|f\|_\infty, \tag{3.55}$$

where c_δ *depends only on* δ.

PROOF. The example is a multiplication operator w.r.t. to a Schauder basis. It is well-known and easy to see that if the basis is Besselian, the multiplication operator $\mathcal{M}_{\{2^n\}}$ satisfies square function estimates, see e.g., [**LM03**, Proof of Theorem 5.2]. We consider a basis $\{\psi_n\}_{n\in\mathbb{N}}$ such that the dual basis $\{\psi_n^*\}_{n\in\mathbb{N}}$ is Besselian, i.e.

$$\forall y = \sum_{n\in\mathbb{N}} y_n\psi_n^* \in X \;\Rightarrow\; (y_n) \in \ell^2(\mathbb{N}). \tag{3.56}$$

Hence, $A^* = \mathcal{M}_{\{2^n\}}$ w.r.t. to $\{\psi_n^*\}_{n\in\mathbb{N}}$ satisfies square function estimates.
In fact, let $X = L^2(-\pi,\pi)$, $\beta \in (\frac{1}{3}, \frac{1}{2})$, and define ψ_n by

$$\psi_{2k}(t) = |t|^\beta e^{ikt}, \quad \psi_{2k+1}(t) = |t|^\beta e^{-ikt},\; k \in \mathbb{N}_0,$$

see Lemma 3.33. The dual basis $\{\psi_n^*\}_{n\in\mathbb{N}}$ is Besselian, see [**Sin70**, Example 11.2]. Further note that $\psi_n(t) = \phi_n(t)$ for $t \in (-\frac{\pi}{2}, \frac{\pi}{2})$, with ϕ_n from Definition 3.18. Let $x(t) = |t|^{-\beta}\mathbb{1}_{(0,\frac{\pi}{2})}(|t|) \in X$. Since $x(t) = 0$ for $t \in (\frac{\pi}{2}, \pi)$ we get the same coefficients x_n w.r.t to $\{\psi_n\}$ as for the basis $\{\phi_n\}$. Thus, by Lemma 3.32 (iii), the coefficients are positive and $x_{2n} = x_{2n+1} \sim \frac{n^{-1+2\beta}}{1-2\beta}$. Furthermore, let

$$y(t) = |t|^{-\beta}(\pi - |t|)^{-\beta}\mathbb{1}_{(\frac{\pi}{2},\pi)},$$

which lies in $L^2(-\pi,\pi)$ and has, w.r.t. $\left\{\psi_{n,\beta}^*\right\}$, the coefficients y_n,

$$y_{2k} = \langle y, \psi_{2k,\beta}\rangle = \frac{1}{2\pi}\int_{\frac{\pi}{2}<|t|<\pi} (\pi - |t|)^{-\beta} e^{ikt} dt = \frac{(-1)^n}{2\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |t|^{-\beta} e^{-ikt} dt,$$

and $y_{2k+1} = y_{2k}$ which can be seen easily. By Lemma 3.31, we conclude that $|y_{2k}| = |y_{2k+1}| \sim (1-\beta)^{-1}k^{-1+\beta} \sim k^{-1+\beta}$. Thus, because (2^n) is interpolating,we find $g \in H^\infty(\mathbb{C}_+)$ such that $g(2^n) = \operatorname{sgn}(y_n)$ for all $n \in \mathbb{N}$, and we get

$$\begin{aligned}\langle (g\cdot e^{-\varepsilon\cdot})(A)x, y\rangle &= \sum_{n\in\mathbb{N}} g(2^n)e^{-2^n\varepsilon}x_ny_n \\ &= \sum_{n\in\mathbb{N}} e^{-2^n\varepsilon}|x_ny_n| \qquad (3.57)\\ &\gtrsim \tfrac{1}{1-2\beta}\sum_{n\in\mathbb{N}}(e^{-2^n\varepsilon} + e^{-2^{n+1}\varepsilon})n^{-2+3\beta} \gtrsim \tfrac{1}{1-2\beta}|\log(\varepsilon)|^{-1+3\beta},\end{aligned}$$

$$(3.58)$$

where in the last step we used Lemma 3.30. Since $\|x\|_{L^2}\cdot\|y\|_{L^2} \sim \frac{1}{1-2\beta}$ and by defining $\beta = \frac{1}{2} - \frac{\delta}{3}$ the assertion follows.
To show (3.55) let $x, y \in H$ and $x = \sum x_n\psi_n$, $y = \sum y_n\psi_n^*$. For $f \in H^\infty(\mathbb{C}_+)$,

$$\begin{aligned}\langle (fe_\varepsilon)(A)x, y\rangle &= \langle \sum_{n\in\mathbb{N}} f(2^n)e^{-2^n\varepsilon}x_n\psi_n, \sum_{n\in\mathbb{N}} y_n\psi_n^*\rangle \\ &= \sum_{n\in\mathbb{N}} f(2^n)e^{-2^n\varepsilon}x_ny_n, \qquad (3.59)\end{aligned}$$

where we used that $\langle \psi_n, \psi_m^* \rangle = \delta_{nm}$. By the Cauchy-Schwarz inequality

$$|\langle (fe_\varepsilon)(A)x, y\rangle| \leqslant \|f\|_\infty \cdot \|(e^{-2^n\varepsilon/2}x_n)\|_2 \cdot \|(e^{-2^n\varepsilon/2}y_n)\|_2.$$

Since $\{\psi_n^*\}_{n\in\mathbb{N}}$ is Besselian, (3.56) and the uniform boundedness principle imply that there exists a constant C_β such that $\|(y_n)\|_{\ell^2} \leqslant C_\beta\|y\|$ for all $y \in X$. Therefore,

$$|\langle (fe_\varepsilon)(A)x, y\rangle| \leqslant C_\beta \cdot \|f\|_\infty \cdot \|(e^{-2^{n-1}\varepsilon}x_n)\|_2 \cdot \|y\|_{L^2}. \tag{3.60}$$

By (3.80) in Lemma 3.33,

$$|\langle (fe_\varepsilon)(A)x, y\rangle| \lesssim C_\beta K_\beta \cdot \|f\|_\infty \cdot \mathrm{Ei}(\varepsilon)^{\frac{1+2\beta}{4}} \cdot \|x\|_{L^2} \cdot \|y\|_{L^2}.$$

Substituting $\beta = \frac{1}{2} - \frac{\delta}{3}$ and $c_\delta := C_\beta K_\beta$ yields (3.55). □

3.5. Discussion and Outlook

3.5.1. Comparison with a result of Haase & Rozendaal. In **[HR13]**, Haase and Rozendaal derived a result of the type of Theorem 3.4 for Hilbert spaces, but for general bounded, not necessarily analytic, C_0-semigroups. We devote this subsection to compare the results, in particular the dependence on the semigroup bound and the sectorality constant, respectively.

We define the right half-plane $R_\delta = \{z \in \mathbb{C} : \mathrm{Re}\, z > \delta\}$. Using transference principles developed by Haase in **[Haa11]** the following result was proved in **[HR13]**.

THEOREM 3.27 (Haase, Rozendaal, Corollary 3.10 in **[HR13]**). *Let* H *be a Hilbert space and* $-A$ *generate a bounded semigroup* T *on* H *and define* $B = \sup_{t>0}\|T(t)\|$. *Then, there exists an absolute constant* $c > 0$ *such that for all* $\varepsilon, \delta > 0$ *the following holds.*
For $f \in H^\infty(R_\delta)$*, the operator* $(fe_\varepsilon)(A) = f(A)T(\varepsilon)$ *is bounded and*

$$\|(fe_\varepsilon)(A)\| \leqslant B^2 \cdot \eta(\delta, \varepsilon) \cdot \|f\|_{\infty, R_\delta}, \tag{3.61}$$

where

$$\eta(\delta,\varepsilon) = \begin{cases} c|\log(\varepsilon\delta)|, & \delta\varepsilon \leqslant \frac{1}{2}, \\ 2c, & \delta\varepsilon > \frac{1}{2}. \end{cases}$$

We can now compare Theorems 3.4 and 3.27 by setting $r_0 = \delta$. Then $\Omega_{\phi,\delta} \subset R_\delta$ for all $\phi \in (0, \frac{\pi}{2}]$ and thus, for functions $f \in H^\infty(R_\delta)$, we have $\|f\|_{\infty,\Omega_{\phi,\delta}} \leqslant \|f\|_{\infty,R_\delta}$. Hence, Theorem 3.4 yields

$$\|(fe_\varepsilon)(A)\| \leqslant M(A,\phi) \cdot b(\varepsilon,\delta,\phi) \cdot \|f\|_{\infty,R_\delta}, \tag{3.62}$$

for all $\phi \in (\omega_A, \frac{\pi}{2})$ and $f \in H^\infty(R_\delta)$, where

$$b(\varepsilon,\delta,\phi) \sim \begin{cases} |\log(\varepsilon\delta\cos\phi)|, & \varepsilon\delta < \frac{1}{2}, \\ |\log\frac{\cos\phi}{2}|, & \varepsilon\delta \geqslant \frac{1}{2}. \end{cases}$$

Let us collect the key observations when comparing (3.61) and (3.62).

(i) We see that the square of the semigroup bound B gets replaced by the sectorality constant $M(A,\phi)$ in our result.

(ii) Our estimate depends on another parameter ϕ that accounts for the fact that the spectrum is truly lying in a sector rather than the half-plane. Taking the infimum over all $\phi \in (\omega_A, \frac{\pi}{2})$ in (3.62) yields an optimized estimate. However, then the constant dependence on $M(A,\phi)$ becomes unclear. See also Theorem 3.7.

(iii) The dependence on ϕ also explains how the estimate explodes when considering A's with sectorality angle ω_A tending to $\frac{\pi}{2}$. However, one can cover this behavior in terms of the constant $M = M(A, \frac{\pi}{2})$: Taking $\phi = \arccos\frac{1}{2M}$, we get by (3.20) that $M(A,\phi) \leqslant 2M$ and thus (3.62) becomes

$$\|(fe_\varepsilon)(A)\| \leqslant M \cdot b(\varepsilon, \delta, \arccos\tfrac{1}{2M}) \cdot \|f\|_{\infty,R_\delta}. \tag{3.63}$$

Therefore, we get an M-dependence of the form $\mathcal{O}(M(\log(M)+1))$.

(iv) By Theorem 3.9, the semigroup bound of $e_t(A)$ is of order $\mathcal{O}(M(\log(M)+1))$. Whether $B \sim M(\log(M)+1))$ in general is still an open problem, see also [**Vit05b**, Remark 1.3]. However, it is easy to see that, in general, $M(A,\pi) \leqslant B$. Therefore,

$$M(A,\pi) \leqslant B \lesssim M(\log(M)+1). \tag{3.64}$$

3.5.2. The Besov calculus. We give a brief introduction to the following homogenous Besov space and refer to [**Vit05b**, Section 1.7] and the references therein for details, see also [**Haa11**]. The notation follows [**Vit05b**]. The space $B^0_{\infty,1}$ can be defined as the space of holomorphic functions f on $\mathbb{C}_+$ such that

$$\|f\|_B := \|f\|_\infty + \int_0^\infty \|f'(t+i\cdot)\|_\infty dt < \infty.$$

Clearly, $B^0_{\infty,1}$, equipped with the above norm, is continuously embedded in $H^\infty(\mathbb{C}_+)$. Moreover, $\cup_{0<\varepsilon<\sigma} H^\infty[\varepsilon,\sigma]$, see Section 3.2.2, lies dense in $B^0_{\infty,1}$ and the following norm is equivalent to $\|\cdot\|_B$, see [**Vit05b**, Theorem A.1],

$$\|f\|_{*B} = |f(\infty)| + \sum_{k\in\mathbb{Z}} \|f * \widehat{\phi_k}\|_\infty,$$

where ϕ_k is the continuous, triangular-shaped function that is linear on the intervals $[2^{k-1}, 2^k]$ and $[2^k, 2^{k+1}]$, vanishes outside $[2^{k-1}, 2^{k+1}]$, and such that $\phi_k(2^k) = 1$. Thus, $\{\phi_k\}_{k\in\mathbb{N}}$ is a partition of unity with $\sum_{k\in\mathbb{Z}} \phi_k \equiv 1$ locally finite on $(0,\infty)$, see [**Haa11, Vit05b**]. Obviously, the (inverse) Fourier-Laplace transform of $f * \widehat{\phi_k}$ has support in $[2^{k-1}, 2^{k+1}]$, hence, $f * \widehat{\phi_k} \in H^\infty[2^{k-1}, 2^{k+1}]$. Therefore, it follows directly from Theorem 3.9 that for $f \in B^0_{\infty,1}$

$$\|(f * \widehat{\phi_k})(A)\| \leqslant cM(\log(M)+1) \cdot 4 \cdot \|f * \widehat{\phi_k}\|_\infty, \tag{3.65}$$

where c is an absolute constant and $M = M(A, \frac{\pi}{2})$. The following Theorem is a slight improvement of Theorem 1.7 in **[Vit05b]**, see also **[Haa11**, Corollary 5.5].

THEOREM 3.28. *Let $A \in \mathrm{Sect}(\omega)$ on the Banach space X with $\omega < \frac{\pi}{2}$. Let $M = M(A, \frac{\pi}{2})$. Then,*

$$\|f(A)\| \leqslant cM(\log(M)+1)\|f\|_{*B},$$

for all $f \in B^0_{\infty,1}$, where $c > 0$ is an absolute constant.
Thus, the $B^0_{\infty,1}$-calculus is bounded.

PROOF. It is easy to see that for $g \in H^\infty[\varepsilon, \sigma]$ with $0 < \varepsilon < \sigma < \infty$,

$$g(z) = \sum_{k \in \mathbb{Z}} (\widehat{\phi_k} * g)(z), \quad z \in \mathbb{C}_+ \tag{3.66}$$

because the inverse Fourier transform of g has compact support. Let $f \in B^0_{\infty,1}$. Since $\cup_{0<\varepsilon<\sigma} H^\infty[\varepsilon, \sigma]$ is dense in $B^0_{\infty,1}$, see **[Vit05b]**, we find a sequence $g_n \in H^\infty[\frac{1}{n}, n]$ such that $g_n \to (f - f(\infty))$ in $B^0_{\infty,1}$ as $n \to \infty$. Thus, $g_n \to f - f(\infty)$ in $\|\cdot\|_\infty$ and $\|\cdot\|_{*B}$. Therefore, by (3.66) and the fact that $\widehat{\phi_k} * (f - f(\infty)) = \widehat{\phi_k} * f$ we have that

$$f(z) = f(\infty) + \sum_{k \in \mathbb{Z}} (\widehat{\phi_k} * f)(z), \quad z \in \mathbb{C}_+. \tag{3.67}$$

Since $\|\sum_{|k| \leqslant N} (\widehat{\phi_k} * f)\|_\infty \leqslant \|f\|_{*B}$ for any $N \in \mathbb{N}$, the Convergence Lemma, **[Haa06a**, Proposition 5.1.1], implies

$$f(A) = f(\infty) + \sum_{k \in \mathbb{Z}} (\widehat{\phi_k} * f)(A)$$

and the assertion follows from (3.65). □

REMARK 3.29. *(i)* In **[Vit05b**, Theorem 1.7], it is already shown that the $B^0_{\infty,1}$-calculus is bounded where the bound of the calculus was estimated by $31M^3$. Like in our proof, she derived the result from an H^∞-calculus estimate for $H^\infty[\varepsilon, \sigma]$.

(ii) In **[Haa11]** Haase showed that for (polynomially) bounded semigroups on Hilbert spaces, one can consider more general homogenous Besov spaces $B^s_{\infty,1}$, $s \geqslant 0$. $B^s_{\infty,1}$ consists of functions f, holomorphic on $\mathbb{C}_+$, and such that $\lim_{z\to\infty} f(z)$ exists and

$$\|f\|_{*B^s} := |f(\infty)| + \sum_{k<0} \|\widehat{\phi_k} * f\|_\infty + \sum_{k \geqslant 0} 2^{ks} \|\widehat{\phi_k} * f\|_\infty < \infty.$$

It is easy to see that Theorem 3.28 holds for $B^s_{\infty,1}$ with the analogous proof as for $B^0_{\infty,1}$.

3.5.3. Final remarks and outlook. Let us mention the well-known relation between analytic semigroup generators and *Tadmor-Ritt operators*, see e.g., [**Haa06a, Vit05a, Vit05b**]. In Chapter 4, we will study these operators using 'discrete' versions of the techniques developed in the present chapter.

We point out that in Theorems 3.4 and 3.11 the operator A need not be densely defined. Thus, in the view of analytic semigroups, $e_t(A)$ need not be strongly continuous at 0, see [**Haa06a**, Chapter 3.3].

Looking back to Propositions 3.1 and 3.3 which served as a starting point to study $\|(fe_\varepsilon)\|$ to quantify the (un)boundedness, we can ask ourselves which other functions g_ε with $g_\varepsilon \to 1$ as $\varepsilon \to 0^+$ can be studied in order to characterize a bounded calculus. For instance, one could consider $g_\varepsilon(z) = z^\varepsilon e^{-\varepsilon z}$ which yields that $fg_\varepsilon \in H_0^\infty(\Sigma_\delta)$ for $f \in H^\infty(\Sigma_\delta)$.

An interesting question is how Theorem 3.24 generalizes to general Banach spaces. As Theorem 3.23 is not true on general Banach spaces, one has to use generalized square function estimates to characterize bounded H^∞-calculus then. These generalized square function estimates where introduced in[**CDMY96**] for L^p-spaces and for general Banach spaces by Kalton and Weis [**KW01**]. See also [**KW04**] for an detailed overview.
It is subject to future work to find a version of Theorem 3.24 for general square function estimates. We refer also to Section 4.4 of the following chapter, where a similar result is proved for Tadmor-Ritt operators and *general discrete square function estimates*.

3.6. Appendix - some technical results

LEMMA 3.30 (Growth Lemma). *Let* $b > 1$ *and* $\gamma < 0$.

(i) *For* $0 < \varepsilon < \frac{1}{b}$,

$$e^{-1}F_\gamma(\varepsilon, b) \leqslant \sum_{n=1}^{\infty} n^\gamma e^{-b^n \varepsilon} \leqslant F_\gamma(\varepsilon, b) + 1 + \tfrac{\mathrm{Ei}(1)}{\log(b)}, \tag{3.68}$$

where

$$F_\gamma(\varepsilon, b) = \begin{cases} \frac{\log(1/\varepsilon)^{1+\gamma} - \log(b)^{1+\gamma}}{\log(b)^{1+\gamma}(1+\gamma)}, & \gamma \neq -1, \\ \log\log(1/\varepsilon) - \log\log(b), & \gamma = -1. \end{cases}$$

Moreover, for all $\gamma_0 \in (-1, 0)$ *there exists* $C_{\gamma_0,b} > 0$ *such that*

$$\forall \gamma \in (\gamma_0, 0): \quad F_\gamma(\varepsilon, b) \geqslant C_{\gamma_0,b} \log\left(\tfrac{1}{\varepsilon}\right)^{1+\gamma}. \tag{3.69}$$

(ii) *For all* $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} e^{-b^n \varepsilon} \leqslant \frac{\mathrm{Ei}(\varepsilon)}{\log(b)}.$$

PROOF. We estimate $\int_1^\infty x^\gamma e^{-b^x \varepsilon} dx$. Substitute $y = b^x \varepsilon$, thus, $x = \frac{\log(y/\varepsilon)}{\log(b)}$,

$$\begin{aligned} \int_1^\infty x^\gamma e^{-b^x \varepsilon}\, dx &= \frac{1}{\log(b)^{1+\gamma}} \int_{\varepsilon b}^\infty \log(y/\varepsilon)^\gamma \frac{e^{-y}}{y}\, dy \\ &= \frac{1}{\log(b)^{1+\gamma}} \Big(\int_{\varepsilon b}^1 \log(y/\varepsilon)^\gamma \frac{e^{-y}}{y}\, dy + \underbrace{\int_1^\infty \log(y/\varepsilon)^\gamma \frac{e^{-y}}{y}\, dy}_{\leqslant \log(\frac{1}{\varepsilon})^\gamma \mathrm{Ei}(1) < \log(b)^\gamma \mathrm{Ei}(1)} \Big). \end{aligned}$$

Because $e^{-1} \leqslant e^{-y} \leqslant 1$ for $y \in (\varepsilon b, 1)$ and since the primitive of $\frac{\log(y/\varepsilon)^\gamma}{y}$ is

$$\begin{cases} \frac{(\log(y/\varepsilon))^{1+\gamma}}{1+\gamma}, & \gamma \neq -1, \\ \log\log(y/\varepsilon), & \gamma = -1, \end{cases}$$

we get

$$e^{-1}F_\gamma(\varepsilon, b) \leqslant \int_1^\infty x^\gamma e^{-b^x \varepsilon}\, dx \leqslant F_\gamma(\varepsilon, b) + \tfrac{\mathrm{Ei}(1)}{\log(b)}.$$

Next we use the fact that for the decreasing, integrable function $f : [1, \infty) \to (0, \infty)$, $x \mapsto x^\gamma e^{-b^x \varepsilon}$ there holds

$$\int_1^\infty f(x)\, dx \leqslant \sum_{n=1}^\infty f(n) \leqslant f(1) + \int_1^\infty f(x)\, dx, \tag{3.70}$$

and so we conclude (3.68). (3.69) can be easily seen by the definition of $F_\gamma(\varepsilon, b)$. Finally, *(ii)*. follows by

$$\sum_{n=1}^{\infty} e^{-b^n \varepsilon} \leqslant \int_0^{\infty} e^{-b^x \varepsilon}\, dx = \frac{1}{\log(b)} \int_{\varepsilon}^{\infty} \frac{e^{-y}}{y}\, dy = \frac{\mathrm{Ei}(\varepsilon)}{\log(b)}.$$

□

LEMMA 3.31. *Let $\alpha \in (-1,1)$. Then, for all $n \in \mathbb{N}$,*

$$c_{n,\alpha} := \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |t|^{\alpha} e^{int}\, dt = C_{1,\alpha} n^{-1-\alpha} + B_n, \tag{3.71}$$

where $C_{1,\alpha} = -2\sin\left(\alpha\frac{\pi}{2}\right)\Gamma(\alpha+1)$, $|B_n| \leqslant C_2 n^{-1}$ and C_2 is an absolute constant. Moreover, $c_{n,\alpha} \in \mathbb{R}$ and for $\alpha \in (-1,0]$,

$$d_{3,\alpha} n^{-1-\alpha} \leqslant c_{n,\alpha} \leqslant d_{1,\alpha} n^{-1-\alpha}, \quad n \in \mathbb{N}, \tag{3.72}$$

where $d_{k,\alpha} = 2\int_0^{\frac{k\pi}{2}} t^{\alpha} \cos t\, dt \sim \frac{1}{1+\alpha}$, for $k \in \{1,3\}$. If $\alpha \in (-1, -\frac{1}{2}]$, then $d_{3,\alpha} > 0$, hence $c_{n,\alpha} > 0$ for all $n \in \mathbb{N}$.

PROOF. By

$$c_{n,\alpha} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |t|^{\alpha} e^{int} dt = 2\,\mathrm{Re} \int_0^{\frac{\pi}{2}} t^{\alpha} e^{int} dt,$$

it is clear that $c_{n,\alpha}$ is a real number and we can consider

$$\int_0^{\frac{\pi}{2}} t^{\alpha} e^{int} dt = n^{-1-\alpha} \int_0^{n\frac{\pi}{2}} t^{\alpha} e^{it} dt. \tag{3.73}$$

Consider the contour consisting of the lines segments $[\varepsilon, n\frac{\pi}{2}]$ and $i[\varepsilon, n\frac{\pi}{2}]$ connected via quarter circles with radii $n\frac{\pi}{2}$ and ε respectively, orientated counterclockwise. Then, since $h(z) = z^{\alpha} e^{iz}$ is holomorphic on $\mathbb{C} \setminus \{z \in \mathbb{R} : z \leqslant 0\}$,

$$\int_{\varepsilon}^{n\frac{\pi}{2}} h(t) dt = \int_{\varepsilon}^{n\frac{\pi}{2}} h(it) i dt - i \int_0^{\frac{\pi}{2}} (n\frac{\pi}{2} e^{i\theta})^{\alpha+1} e^{in\frac{\pi}{2} e^{i\theta}} d\theta + i \int_0^{\frac{\pi}{2}} (\varepsilon e^{i\theta})^{\alpha+1} e^{i\varepsilon e^{i\theta}} d\theta. \tag{3.74}$$

The last two integrals can both be estimated using the fact that $|e^{ire^{i\theta}}| = e^{-r\sin\theta} \leqslant e^{-r\frac{2\theta}{\pi}}$ for $\theta \in [0, \frac{\pi}{2}]$, $r > 0$. This yields

$$\left| \int_0^{\frac{\pi}{2}} (re^{i\theta})^{\alpha+1} e^{ire^{i\theta}} d\theta \right| \leqslant \frac{\pi}{2} r^{\alpha} (1 - e^{-r}).$$

Therefore, the integral for $r = \varepsilon$ goes to zero as $\varepsilon \to 0^+$ because $\alpha > -1$. The integral for $r = n\frac{\pi}{2}$ can be estimated by $\left(\frac{\pi}{2}\right)^{\alpha+1} n^{\alpha}$. It remains to consider

$$\lim_{\varepsilon \to 0^+} i \int_{\varepsilon}^{n\frac{\pi}{2}} h(it) dt = i \int_0^{n\frac{\pi}{2}} h(it) dt = e^{i(\alpha+1)\frac{\pi}{2}} \int_0^{n\frac{\pi}{2}} t^{\alpha} e^{-t} dt$$

$$= e^{i(\alpha+1)\frac{\pi}{2}} \left[\Gamma(\alpha+1) - \int_{n\frac{\pi}{2}}^{\infty} t^\alpha e^{-t} dt \right].$$

It is easily seen that there exists a constant C such that $\int_n^\infty t^\alpha e^{-t} dt \leqslant Cn^\alpha e^{-n}$ for all $\alpha \in (-1,1)$. Altogether we get by (3.73) and the estimates for the terms in (3.74) that

$$\int_0^{\frac{\pi}{2}} t^\alpha e^{int} dt = e^{i(\alpha+1)\frac{\pi}{2}} \Gamma(\alpha+1) n^{-1-\alpha} + B_{n,\alpha},$$

with $|B_{n,\alpha}| \leqslant \frac{1}{n} \left[\left(\frac{\pi}{2}\right)^{\alpha+1} + Ce^{-n} \right]$. This yields (3.71).
To show (3.72) for $\alpha \in (-1,0)$, note that with (3.73)

$$c_{n,\alpha} = n^{-1-\alpha} 2 \int_0^{\frac{n\pi}{2}} t^\alpha \cos(t) dt.$$

We define $d_{n,\alpha} = 2\int_0^{\frac{n\pi}{2}} t^\alpha \cos t dt$. It remains to show that $d_{3,\alpha} \leqslant d_{n,\alpha} \leqslant d_{1,\alpha}$ for all $n \in \mathbb{N}$. Since $t \mapsto t^\alpha$ is positive and decreasing on $(0,\infty)$, it follows by the periodicity of the cosine that for all for all $m \in \mathbb{N}_0$

(i) since $\cos(\frac{t\pi}{2}) < 0$ on $((4m+1),(4m+3))$, $d_{4m+1,\alpha} > d_{4m+2,\alpha} > d_{4m+3,\alpha}$,

(ii) since $\cos(\frac{t\pi}{2}) > 0$ on $((4m+3),(4m+5))$, $d_{4m+3,\alpha} < d_{4m+4,\alpha} < d_{4m+5,\alpha}$,

(iii) since $t \mapsto t^\alpha$ is decreasing, $d_{4m+5,\alpha} < d_{4m+1,\alpha}$ and $d_{4m+3,\alpha} < d_{4(m+1)+3,\alpha}$.

Inductively, this shows that $\max_n d_{n,\alpha} = d_{1,\alpha}$ and $\min_n d_{n,\alpha} = d_{3,\alpha}$. Finally we check that $d_{3,\alpha} > 0$ if $\alpha \in (-1,-\frac{1}{2}]$,

$$d_{3,\alpha} = \int_0^{\frac{3\pi}{2}} t^\alpha \cos(t) dt \geqslant \cos(1) \int_0^1 t^\alpha dt + \left(\frac{\pi}{2}\right)^\alpha \int_1^{\frac{\pi}{2}} \cos(t) dt + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} t^\alpha \cos(t) dt$$
$$\geqslant 2\cos(1) + \frac{2}{\pi}(1-\sin(1)) + \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{\cos(t)}{\sqrt{t}} dt > 0,$$

where the last integral can be computed via a Fresnel integral and is approximately 0.0314. □

LEMMA 3.32. *Let* $\beta \in (\frac{1}{4}, \frac{1}{2})$, $X = L^2(-\pi,\pi)$ *and* $\{\phi_n\}_{n\in\mathbb{N}} \subset X$ *as in Definition 3.18. The following assertions hold. (See (3.32) for the definitions of* m_ϕ *and* κ_ϕ*.)*

(i) $\{\phi_n\}_{n\in\mathbb{N}}$ *forms a bounded Schauder basis of* X *with* $\kappa_\phi \sim \frac{1}{1-2\beta}$.

(ii) The family $\{\phi_n^*\}_{n\in\mathbb{N}} \subset X$ *given by*

$$\phi_{2k}^*(t) = \tfrac{1}{2\pi w_\beta(t)} e^{ikt}, \quad \phi_{2k+1}^*(t) = \tfrac{1}{2\pi w_\beta(t)} e^{-ikt},$$

satisfies $\langle \phi_n^*, \phi_m\rangle_{L^2} = \delta_{nm}$ *and forms a Schauder basis of* X *with* $\kappa_{\phi^*} \sim \frac{1}{1-2\beta}$*. Here,* w_β *is defined as in Definition 3.18.*

(iii) The coefficients of $x(t) = |t|^{-\beta}\mathbb{1}_{(0,\frac{\pi}{2})}(|t|)$, $x = \sum_n x_n\phi_n$ *are positive and satisfy*

$$x_{2k} = x_{2k+1} \sim \tfrac{k^{-1+2\beta}}{1-2\beta}, \qquad k \in \mathbb{N}. \tag{3.75}$$

For the coefficients of $y(t) = (\pi - |t|)^{-\beta}\mathbb{1}_{(\frac{\pi}{2},\pi)}(|t|)$, $y = \sum_n y_n\phi_n^*$*, we have that*

$$y_{2k} = (-1)^k 2\pi\cdot x_{2k}, \quad y_{2k+1} = (-1)^k 2\pi \cdot x_{2k+1}, \qquad k\in\mathbb{N}. \tag{3.76}$$

PROOF. Lemma 3.32 (i)-3.32 (ii) follow from **[Sin70**, Example 11.2].
To see 3.32 (iii) we point out that for all $x = \sum_k x_k\phi_{k,\beta} \in X$ there holds

$$x_n = \langle x, \phi_n^*\rangle_{L^2}, \qquad n \in \mathbb{N}. \tag{3.77}$$

Thus, for $x = (t \mapsto |t|^{-\beta}\mathbb{1}_{(0,\frac{\pi}{2})}(|t|))$, $k\in\mathbb{N}$,

$$x_{2k} = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} |t|^{-2\beta} e^{-ikt}dt = \frac{c_{k,-2\beta}}{2\pi}, \quad x_{2k+1} = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}|t|^{-2\beta}e^{ikt}dt = x_{2k}, \tag{3.78}$$

where $c_{k,-2\beta}$ are the coefficients from Lemma 3.31. Since $-2\beta \in (-1, -\frac{1}{2})$, they are even positive and (3.75) follows. The assertion for y follows similarly. □

LEMMA 3.33. *Let* $X = L^2 = L^2(-\pi,\pi)$, $\beta\in(\frac{1}{3},\frac{1}{2})$*. Then* $\{\psi_n\}_{n\in\mathbb{N}}$ *defined by*

$$\psi_{2k}(t) = |t|^\beta e^{ikt}, \quad \psi_{2k+1}(t) = |t|^\beta e^{-ikt},\ k\in\mathbb{N}_0,$$

is a bounded Schauder basis. For $x = \sum_{n\in\mathbb{N}} x_n\psi_n \in X$*, we have that* $\{x_n\}\in \ell^r$ *for* $r > \frac{2}{1-2\beta}$ *and*

$$\|(x_n)\|_r \lesssim \|x\|_{L^2}\cdot\|n^{-1+\beta}\|_q, \qquad \tfrac{1}{q} = \tfrac{1}{2} + \tfrac{1}{r}. \tag{3.79}$$

Furthermore,

$$\|(e^{-2^n\varepsilon}x_n)\|_{\ell^2} \lesssim K_\beta \cdot \mathrm{Ei}(\varepsilon)^{\frac{1+2\beta}{4}}\|x\|_{L^2}, \tag{3.80}$$

with $K_\beta = \|(n^{\beta-1})\|_{\frac{3-2\beta}{4}}$.

PROOF. That $\{\psi_n\}$ is a Schauder basis can e.g., be found in [**Sin70**, Example 11.2]. Let $w_\beta(t) = |t|^\beta$ on $(-\pi, \pi)$. Since $\{e^{int}\}_{n\in\mathbb{Z}}$ is an orthogonal basis of L^2 it follows that for $x = \sum_{n\in\mathbb{N}} x_n\psi_n \in X$,

$$x_{2k} = \frac{1}{2\pi}\langle x w_\beta^{-1}, e^{ik\cdot}\rangle_{L^2} = \mathcal{F}(x w_\beta^{-1})[k],$$

where $\mathcal{F}$ denotes the discrete Fourier transform. Thus,

$$x_{2k} = \left(\mathcal{F}(x) * \mathcal{F}(w_\beta^{-1})\right)[k]. \tag{3.81}$$

By $x \in L^2$, $\{\mathcal{F}(x)[n]\} \in \ell^2$. From [**Haa12**, Proof of Theorem 2.4, p.861] (see also Lemma 3.31) we have that

$$\int_{-\pi}^{\pi} |t|^{\gamma-1}e^{-int}\,dt = 2n^{-\gamma}\cos(\gamma\tfrac{\pi}{2})\Gamma(\gamma) + \mathcal{O}(\tfrac{1}{n}),$$

for $\gamma > 0$. Thus, with $\gamma = 1-\beta \in (\frac{1}{2}, \frac{2}{3})$, $\mathcal{F}(w_\beta^{-1})[n] \in \ell^q$ with $q > q_0 := \frac{1}{1-\beta}$ and

$$\|\mathcal{F}(w_\beta^{-1})[n]\|_{\ell^q} \lesssim \|(n^{-1+\beta})\|_{\ell^q}.$$

We use Young's inequality with $\frac{1}{2}+\frac{1}{q} = 1+\frac{1}{r}$ and $q \in (q_0, 2)$ to estimate the right-hand-side of (3.81). Hence, $\{x_{2k}\} \in \ell^r$ for $r > \frac{2}{1-2\beta}$. Analogously, $\{x_{2k+1}\} \in \ell^r$. Eq. (3.79) then follows since the discrete Fourier transform is isometric from L^2 to ℓ^2.

To show (3.80), we use Hölder's inequality and (3.79),

$$\begin{aligned}\|(e^{-2^{n-1}\varepsilon}x_n)\|_2^2 &= \|(e^{-2^n\varepsilon}|x_n|^2)\|_1\\ &\leqslant \|(e^{-2^n\varepsilon})\|_{r_0'} \cdot \|(x_n)\|_{2r_0}^2\\ &\lesssim \|(e^{-2^n\varepsilon})\|_{r_0'} \cdot \|(n^{-1+\beta})\|_q^2 \cdot \|x\|_{L^2}^2\end{aligned} \tag{3.82}$$

for $r_0' = (1-\frac{1}{r_0})^{-1} = \frac{2}{1+2\beta}$ and $\frac{1}{q} = \frac{1}{2}+\frac{1}{2r_0} = \frac{3-2\beta}{4}$. By Lemma 3.30,

$$\|(e^{-2^n\varepsilon})\|_{r_0'=\frac{2}{1+2\beta}} \lesssim \mathrm{Ei}(r_0'\varepsilon)^{\frac{1+2\beta}{2}} \overset{(3.6)}{\leqslant} \mathrm{Ei}(\varepsilon)^{\frac{1+2\beta}{2}}, \tag{3.83}$$

where we used that $r_0' > 1$. Thus, (3.82) shows (3.80). □

A version of the following Lemma can be found in [**Paz83**, Theorem 6.13], however, the constant dependence is unclear there.

LEMMA 3.34. *Let* $A \in \mathrm{Sect}(\omega)$ *with* $\omega < \phi < \frac{\pi}{2}$ *and* $\alpha \in (0,1]$. *Set* $R = 0$ *if* $0 \notin \rho(A)$, *and* $R = \frac{1}{\|A^{-1}\|}$ *otherwise. Then, for every* $\kappa \in [0,1)$

$$\|A^\alpha T(t)\| \leqslant Ct^{-\alpha}e^{-t\kappa R\cos\phi} \qquad \forall t > 0, \tag{3.84}$$

with $C = C_{\alpha,\kappa}M(A,\phi)(\cos\phi)^{-\alpha}$.
Note that by the assumptions, the growth bound ω_0 *of* T *satisfies* $\omega_0 \leqslant -R\cos\phi$.

PROOF. Let $r = \eta R$ for $\eta \in [0,1)$. Thus, if $R = 0$, then $r = 0$. We define the path $\Gamma^r = \Gamma_{1,r} \cup \Gamma_{2,r} \cup \Gamma_{3,r}$ with

$$\Gamma_1^r = \left\{\tilde{r}e^{i\phi}, \tilde{r} \geqslant r\right\}, \Gamma_2^r = \left\{re^{i\phi} - it, t \in (0, 2\Im(re^{i\phi})\right\}, \Gamma_3^r = \left\{-\tilde{r}e^{-i\phi}, \tilde{r} \leqslant -r\right\},$$

orientated counter-clockwise. Note that $\Gamma^0 = \partial\Sigma_\phi$. Since $z \mapsto z^\alpha e^{-tz} \in H_0^\infty(\Sigma_\phi)$, and $\Gamma \subset \rho(A)$ if $0 \in \rho(A)$, we get (see (3.4))

$$A^\alpha T(t) = \frac{1}{2\pi i}\int_{\Gamma^r} z^\alpha e^{-tz} R(z,A)\, dz. \tag{3.85}$$

Splitting up the integral and taking norms, we derive for Γ_1^r,

$$\begin{aligned}\int_{\Gamma_1^r} |z|^\alpha e^{-t\,\mathrm{Re}\,z}\|R(z,A)\|\,|dz| &\leqslant M(A,\phi)\int_r^\infty s^{\alpha-1}e^{-ts\cos\phi}\, ds\\ &= M(A,\phi)(t\cos\phi)^{-\alpha}\int_{rt\cos\phi}^\infty s^{\alpha-1}e^{-s}\, ds.\end{aligned}$$

By the definition of the Gamma function it follows that $\int_a^\infty s^{\alpha-1}e^{-s}\,ds \leqslant \Gamma(\alpha)e^{-a}$ for $a \geqslant 0$ and $\alpha \in (0,1]$. Thus,

$$\left\|\int_{\Gamma_1^r} z^\alpha e^{-tz}R(z,A)dz\right\| \leqslant M(A,\phi)(t\cos\phi)^{-\alpha}\Gamma(\alpha)e^{-tR\eta\cos\phi}. \tag{3.86}$$

The estimate Γ_3^r can be derived analogously. Since $\Gamma_2^0 = \{\}$, it remains to consider Γ_2^r for $r > 0$. Thus, $R, \eta > 0$ which means that $0 \in \rho(A)$. By the resolvent identity follows

$$\|R(z,A)\| \leqslant \frac{\|A^{-1}\|}{1-\eta} \quad \text{for} \quad |z| \leqslant r = \eta R,$$

see also the proof of Theorem 3.11. Therefore,

$$\begin{aligned}\int_{\Gamma_2^r} |z|^\alpha e^{-t\,\mathrm{Re}\,z}\|R(z,A)\|dz &\leqslant e^{-tr\cos\phi}\frac{\|A^{-1}\|}{1-\eta}r^\alpha\int_{\Gamma_{2,r}}|dz|\\ &\leqslant e^{-tr\cos\phi}\frac{\|A^{-1}\|}{1-\eta}r^{\alpha+1}(2\sin\phi)\\ &= r^\alpha e^{-tr\cos\phi}\frac{2\eta}{1-\eta}.\end{aligned}$$

Since $r \mapsto f(r) = r^\alpha e^{-rb}$ attains its maximum $(\frac{\alpha}{b})^\alpha e^{-\alpha}$ at $r = \frac{\alpha}{b}$, we conclude that

$$r^\alpha e^{-rb} = r^\alpha e^{-rb(1-\eta)}e^{-rb\eta} \leqslant \left(\tfrac{\alpha}{b(1-\eta)}\right)^\alpha e^{-\alpha}e^{-rb\eta},$$

and thus

$$\left\|\int_{\Gamma_2^r} z^\alpha e^{-tz}R(z,A)dz\right\| \leqslant M(A,\phi)\frac{2\eta}{1-\eta}\left(\tfrac{e^{-1}\alpha}{1-\eta}\right)^\alpha(t\cos\phi)^{-\alpha}e^{-tR\eta^2\cos\phi}. \tag{3.87}$$

where we use that $M(A,\phi) \geqslant 1$, see **[Haa06a**, Prop. 2.1.1]. Since $e^{-tR\eta^2\cos\phi} \geqslant e^{-tR\eta\cos\phi}$, combining (3.85), (3.86) and (3.87) yields the assertion by setting $\eta = \sqrt{\kappa}$. □

CHAPTER 4

Functional calculus estimates for Tadmor–Ritt operators

Abstract. We prove H^∞-functional calculus estimates for Tadmor–Ritt operators. These generalize and improve results by Vitse and are in conformity with the best known power-bounds for Tadmor–Ritt operators in terms of the constant dependence. We furthermore show the effect of having discrete square function estimates on these estimates[1]

4.1. Tadmor–Ritt and Kreiss operators

For this chapter, by convention all operators T on a Banach space X are assumed to be bounded. Let us start with the definition of a Tadmor–Ritt operator. Unless stated otherwise, X will denote a general Banach space.

DEFINITION 4.1. A bounded operator T on X, i.e., $T \in \mathcal{B}(X)$, is called a *Tadmor–Ritt operator* if $\sigma(T) \subset \overline{\mathbb{D}}$ and if

$$C(T) := \sup_{|z|>1} \|(z-1)R(z,T)\| < \infty. \tag{4.1}$$

Let $TR(X)$ denote the set of all Tadmor–Ritt operators on X.

Tadmor–Ritt operators, in the literature also sometimes referred to as *Ritt operators*, were, with a slightly different but equivalent definition, first studied in **[Rit53]**. See **[Bak03, EFR02, Vit04b]** for a detailed discussion of these two definitions.
Tadmor–Ritt operators form a class consisting of operators satisfying *Kreiss' resolvent condition*,

$$\sigma(T) \subset \overline{\mathbb{D}}, \quad \text{and} \quad C_{\mathrm{Kreiss}}(T) = \sup_{|z|>1} \|(|z|-1)R(z,T)\| < \infty. \tag{4.2}$$

We will call operators satisfying (4.2) *Kreiss operators* and denote the set of all such operators on X by $KR(X)$. Obviously, $TR(X) \subset KR(X)$. The most prominent question related to these operators is the one of *power-boundedness*, i.e. whether

$$Pb(T) := \sup_{n\in\mathbb{N}} \|T^n\| < \infty \tag{4.3}$$

[1]This chapter is a slight adaptation of the article:
F. L. SCHWENNINGER, *Functional calculus estimates for Tadmor–Ritt operators*, submitted, 2015

holds. Originally, and of more importance for numerical analysis, the question was studied for finite-dimensional spaces X. In this case the answer is positive for operators in $\mathrm{KR}(X)$. This follows from a result by Kreiss from 1962, **[Kre62]**, which states that

$$\mathrm{Pb}(T) \leqslant g(C_{\mathrm{Kreiss}}(T), N), \tag{4.4}$$

for a function g depending on $C_{\mathrm{Kreiss}}(T)$ and the dimension N of the space X. Kreiss' originial estimate (of the function g) was improved steadily in the following decades ending up with the final result proved by Spijker in 1991, **[Spi91]**,

$$\forall T \in \mathrm{KR}(X): \quad \mathrm{Pb}(T) \leqslant e C_{\mathrm{Kreiss}}(T) N. \tag{4.5}$$

For the detailed history of the result we refer to the monograph **[TE05]** and the recent work **[Nik13]**. See also **[SW97]** for a study of infinite-dimensional versions. By **[LT84]**, estimate (4.5) is sharp in the sense that there exists a sequence of matrices $T_N \in \mathrm{KR}(\mathbb{C}^{N\times N})$ such that

$$\lim_{N\to\infty} \frac{\mathrm{Pb}(T_N)}{C_{\mathrm{Kreiss}}(T_N)N} = e.$$

However, for this sequence, $C_{\mathrm{Kreiss}}(T_N) \to \infty$, hence, for $C_{\mathrm{Kreiss}}(T) \leqslant C$ with a fixed constant C, the behavior could theoretically be better. Indeed, a recent result by Nikolski shows that for T having *unimodular* spectrum, i.e. $\sigma(T) \subset \partial\mathbb{D}$, and a basis of eigenvectors, one gets a *sublinear* growth in the dimension.

THEOREM 4.2 (Nikolski 2013, **[Nik13]**). *Let X be a Hilbert space of dimension $N < \infty$. Let T be a Kreiss operator on X such that $\sigma(T) \subset \partial\mathbb{D}$ and such that T has a basis of eigenvectors $\mathcal{X}_N = (x_j)_{j=1}^N$. Then*

$$\mathrm{Pb}(T) \leqslant 2\pi C_{\mathrm{Kreiss}}(T) N^{1-\varepsilon},$$

where $\varepsilon = \frac{0.32}{b(\mathcal{X}_N)^2}$ and $b(\mathcal{X}_N)$ denotes the basis constant of $\mathcal{X}_N$, i.e.

$$b(\mathcal{X}_N) = \sup_{k\leqslant l} \|P_{(x_j)_{j=l}^k}\|,$$

where $P_{(x_j)_{j=l}^k}$ denotes the projection onto the span of the vectors $(x_j)_{j=l}^k$.

The proof of the result is based on a classic theorem by McCarthy and Schwartz **[MS65]**. We remark that Nikolski also shows a corresponding result on more general Banach spaces using a generalization of McCarthy and Schwartz' result by Gurari and Gurari **[GG71]**. By using well-known techniques from Spijker, Tracogna, Welfert **[STW03]**, he further proves that the sublinear behavior is sharp. As indicated by Nikolski, in order to get an estimate in the spirit of the Kreiss Matrix Theorem, one has to close the loop by estimating $b(\mathcal{X}_N)$ in terms of $C_{\mathrm{Kreiss}}(T)$. This still remains open.

When we turn to general infinite-dimensional spaces X, the power-boundedness of general Kreiss operators, even on Hilbert spaces, is no longer true. We refer to [**Fog64**] and [**Hal64**] for counterexamples. In the conference paper [**Tad86**], E. Tadmor states that the growth of $\|T^n\|$ can at most be logarithmically in n under the additional assumption that the spectrum of T 'is not too dense in the neighbourhood of the unit circle'. This condition is in particular ensured if (4.1) holds. Moreover, the existence of an example is stated confirming the sharpness of the growth. As both the proof and the example are unfortunately not published, we are indebted to E. Tadmor for sharing them with us, [**Tad14**].

Knowing that general Kreiss operators are not power-bounded, the same question for Tadmor–Ritt operators remained open until 1999 when Lyubich [**Lyu99**] and Nagy & Zemanek [**NZ99**] used a preceding result of O. Nevanlinna [**Nev93**] to prove that they are indeed power-bounded.
We remark that in 1993, C. Palencia [**Pal93**] and, independently, Crouzeix, Larsson, Piskarev and Thomée [**CLPT93**] showed that the *Crank–Nicolson-scheme* is stable for sectorial operators. In particular, this shows that the *Cayley transform* $\mathrm{Cay}(A) := (I - A)(A + I)^{-1}$ of a sectorial operator A is power-bounded. As the mapping $A \mapsto \mathrm{Cay}(A)$ establishes a one-to-one correspondence between sectorial operators A with $0 \in \rho(A)$ and Tadmor–Ritt operators, Palencia's result already shows the power-boundedness of Tadmor–Ritt operators. This fact seems to be unnoticed in the literature. Moreover, Palencia's result shows that any bounded operator S with $\sigma(S) \subset \overline{\mathbb{D}}$ and such that there exists a constant $M(S) > 0$ and

$$\|R(z, S)\| \leqslant M(S)(|z+1|^{-1} + |z-1|^{-1}), \quad |z| > 1,$$

is power-bounded. See also Remark 5.18 in Chapter 4 for further discussion.
In 2002, El-Fallah and Ransford, [**EFR02**] showed that for a Tadmor–Ritt operator T, it holds that $\mathrm{Pb}(T) \leqslant C(T)^2$, which was subsequently improved by Bakaev [**Bak03**] to

$$\forall T \in \mathrm{TR}(X): \quad \mathrm{Pb}(T) \leqslant aC(T)\log(aC(T)), \tag{4.6}$$

for some absolute constant $a > 0$ (which was not determined). The latter result seems to be not so well-known. In [**Vit05a**, Remark 2.2] an alternative proof for the quadratic dependence on $C(T)$ is sketched. A careful study of this sketch reveals that it is based on a similar approach as in Bakaev's proof, which, with a sharper estimation and some additional work, actually yields (4.6). We will encounter a similar approach in the proof of Theorem 4.7, which was actually motivated by a result of the author for analytic semigroups, [**Sch15b**], see Chapter 3.

In [**Vit04a, Vit05a**], Vitse investigated the more general setting of a functional calculus for Tadmor–Ritt operators and proved that for $1 \leqslant m \leqslant n$ and any polynomial $p(z) = \sum_{j=m}^{n} a_j z^j$

$$\|p(T)\| \leqslant c(C(T), m, n) \cdot \sup_{z \in \mathbb{D}} |p(z)|, \tag{4.7}$$

with $c(C(T), m, n) = 191C(T)^5 \log\left(\frac{e(n+1)}{m}\right)$. We also remark that Le Merdy showed in [**LM98**] that a Tadmor–Ritt operator on a Hilbert space has bounded polynomially calculus, i.e.

$$\sup\{\|p(T)\| : p \text{ is polynomial}, \|p\|_{\infty,\mathbb{D}} \leqslant 1\} < \infty, \tag{4.8}$$

if and only if T is similar to a contraction.
Obviously, (4.7) implies power-boundedness of T, however, with a $C(T)$-dependence worse than in (4.6). We will show that this dependence can be improved significantly, coupling it to the, so-far known, optimal constant for the power-bound of T in (4.6). Precisely, in Theorem 4.9 we will show that for $p(z) = \sum_{j=m}^{n} a_j z^j$, $0 \leqslant m \leqslant n$,

$$\|p(T)\| \leqslant aC(T) \log\left(C(T) + b + \log\frac{n+1}{m+1}\right) \cdot \|p\|_{\infty,\mathbb{D}}, \tag{4.9}$$

with absolute constants $a, b > 0$. The proof is significantly shorter and more direct than the one for (4.7) in [**Vit05b**]. Moreover, the result is actually a consequence of a more general functional calculus result for Tadmor–Ritt operators, see Theorem 4.7. As a direct consequence our results improve the constant dependence on $C(T)$ for the Besov-calculus derived by Vitse in [**Vit04a, Vit05a**], see Theorem 4.27.

Finally, motivated by the result for analytic semigroup generators (or sectorial operators), which can be seen as the continuous counterparts of Tadmor–Ritt operators, we discuss the influence of *discrete square function estimates* on the calculus estimates, see also [**LM14b**]. For Hilbert spaces, it is known that if a Tadmor–Ritt operator and its dual operator satisfy square function estimates, then the corresponding H^∞-functional calculus is bounded. As for the more known continuous counterpart of sectorial operators (see Section 3.4), here, it is essential to have square function estimates for both T and T^*. We show that having only T (or alternatively T^*) satisfying square function estimates however improves the functional calculus estimate (4.9), see Theorem 4.17. In Section 4.4, we generalize the result about square function estimates to general Banach spaces. This involves a refined definition of square function estimates using Rademacher means and $\mathcal{R}$-*boundedness*. These abstract square function estimates are the discrete counterpart to the ones for sectorial operators, which were introduced by Kalton and Weis [**KW01**] and have proved very useful in the study of L^p-maximal regularity for parabolic evolution equations since then.

In Section 4.5, we discuss sharpness of the derived estimates. We conclude by a result about a Besov-space calculus for Tadmor–Ritt operators, which is a refinement of [**Vit05a**, Theorem 2.5].

4.1.1. Properties of Tadmor–Ritt operators. Unless stated otherwise, X will always denote a, in general infinite-dimensional, Banach space.
From (4.1) it follows that for Tadmor–Ritt operators the only possible spectral point on $\mathbb{T}$ is 1. Moreover, it is well-known that the spectrum is contained in the *Stolz type* domain $\mathcal{B}_\theta$, which is the interior of the convex hull of $\{\{1\}, B_{\sin\theta}(0)\}$ for some $\theta \in (0, \frac{\pi}{2})$, see Figure 4.1. Here, $B_r(z_0)$ denotes the open ball centred at z_0 with radius r. For this and a proof of the following lemma we refer to Vitse **[Vit04b, Vit05a]** and Le Merdy **[LM14b]**, which improves earlier results in **[Lyu99, NZ99]** and **[Nev93]**.

LEMMA 4.3. *Let T be a Tadmor–Ritt operator on a Banach space X. Then, there exists $\theta \in [0, \frac{\pi}{2})$ such that*

(i) $\sigma(T) \subset \overline{\mathcal{B}_\theta}$, *and*

(ii) for all $\eta \in (\theta, \frac{\pi}{2}]$,

$$C_\eta(T) = \sup_{z \in \mathbb{C} \setminus \overline{\mathcal{B}_\eta}} \|(z-1)R(z,T)\| \leqslant \frac{C(T)}{1 - \frac{\cos\eta}{\cos\theta}}. \tag{4.10}$$

We say that T is of type θ.
Moreover, θ can always chosen to be $\theta = \arccos \frac{1}{C(T)}$.

Note that $\mathcal{B}_\alpha \subset \mathcal{B}_\beta$ for $\alpha < \beta$. The previous lemma tells us that for η going to θ, the right-hand-side of (4.10) explodes whereas for $\eta = \frac{\pi}{2}$ it becomes $C(T)$. We further remark that the converse of Lemma 4.3 also holds: If there exists $\theta \in (0, \frac{\pi}{2})$ such that $\sigma(T) \subset \overline{\mathcal{B}_\theta}$ and $C_\eta < \infty$ for all $\eta \in (\theta, \frac{\pi}{2})$, then T is Tadmor–Ritt, see **[LM14b**, Lemma 2.1**]**.
We further need the following well-known characterization, which can be found e.g., in **[LM14b, Lyu99, NZ99, Vit05a]**.

LEMMA 4.4. *Let T be an operator on a Banach space X. The following assertions are equivalent.*

(i) T *is Tadmor–Ritt.*

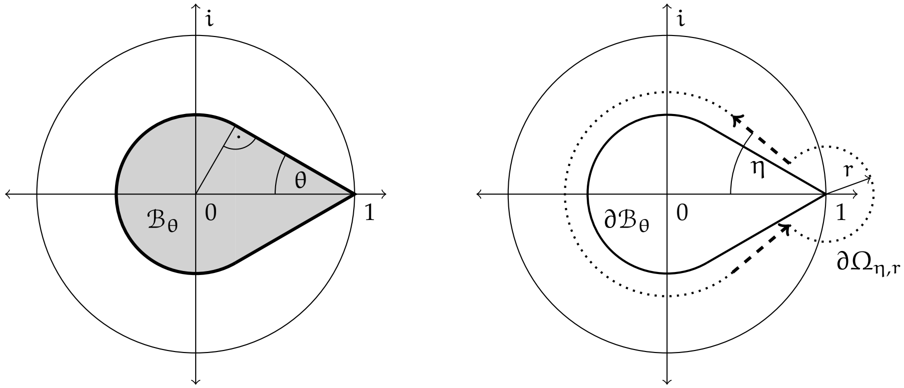
(ii) The sets $\{T^n : n \in \mathbb{N}\}$ *and* $\left\{n(T^n - T^{n-1}) : n \in \mathbb{N}\right\}$ *are bounded, i.e.*

$$Pb(T) = \sup_{n\in\mathbb{N}} \|T^n\| < \infty, \text{ and } c_{1,T} := \sup_{n\in\mathbb{N}} \|n(T^n - T^{n-1}\| < \infty. \tag{4.11}$$

4.2. A functional calclulus result for Tadmor–Ritt operators

By Lemma 4.3 we know that the spectrum of a Tadmor–Ritt operator is contained in the Stolz type domain $\overline{\mathcal{B}_\theta}$, with $\theta = \arccos \frac{1}{C(T)}$. Let $\Omega \supset \overline{\mathcal{B}_\theta}$ be an open, bounded and simply connected subset of $\mathbb{C}$. Then for any function holomorphic on Ω, the

FIGURE 4.1. The sets $\mathcal{B}_\theta$ and $\Omega_{\eta,r}$ with $\eta \in \left(\theta, \frac{\pi}{2}\right)$.



operator $f(T)$ can be defined via the Riesz-Dunford integral

$$f(T) = \frac{1}{2\pi i} \int_\Gamma f(z)\, R(z, T)\, dz, \tag{4.12}$$

where Γ is a rectifiable, positively orientated, simple contour inside Ω which encircles $\overline{\mathcal{B}_\theta}$. Let $H^\infty(\Omega)$ denote the bounded holomorphic functions on Ω.

REMARK 4.5. Let $H_0^\infty(\mathcal{B}_\delta)$ be the functions f in $H^\infty(\mathcal{B}_\delta)$ for which exist constants $c, s > 0$ such that $f(z) \leqslant c|1-z|^s$ for all $z \in \mathcal{B}_\delta$. For $\delta \in (\theta, \frac{\pi}{2})$ and $f \in H_0^\infty(\mathcal{B}_\delta)$, $f(T)$ can still be defined by (4.12) with Γ equal to the boundary of $\partial\mathcal{B}_{\delta'}$ of $\mathcal{B}_{\delta'}$ with $\delta' \in (\theta, \delta)$. Analogously to the situation for sectorial operators, see e.g., **[Haa06a]**, it can be shown that the mapping $f \mapsto f(A)$ becomes an algebra homomorphism from $H_0^\infty(\mathcal{B}_\delta)$ to $\mathcal{B}(X)$, see **[LM14b**, Section 2] for more details.

For $0 < r < 1$ and $\eta \in (0, \frac{\pi}{2}]$, we define the 'keyhole-shaped' set,

$$\Omega_{\eta,r} := \mathcal{B}_\eta \cup B_r(1), \tag{4.13}$$

see Figure 4.1.
The function

$$\mathrm{Ei}(s) = \int_s^\infty \frac{e^{-x}}{x}\, dx \tag{4.14}$$

is known as the *Exponential integral*. It holds that

$$\tfrac{1}{2} e^{-s} \log\left(1 + \tfrac{2}{s}\right) < \mathrm{Ei}(s) < e^{-s} \log(1 + \tfrac{1}{s}),\ s > 0, \tag{4.15}$$

$$\mathrm{Ei}(s) < \log(\tfrac{1}{s}),\ s \in (0, \tfrac{1}{2}], \tag{4.16}$$

see **[Gau60]** and Section 3.1.1 for more details.
The following lemma outsources technicalities in the proofs of the results to come. Estimates of this kind for deriving functional calculus estimates can already be found in **[Bak03, Pal93, Vit05a]**. The technique to compute power-bounds for matrices by

using functional calculus methods can already be traced back to Laptev [**Lap75**], Tadmor [**Tad81**], Le Veque and Trefethen [**LT84**] and Spijker [**Spi91**]. Here, the focus is laid on deriving estimates explicitly in the used constants.

LEMMA 4.6. *For* $0 < r < 1$, $m \geqslant 0$ *and* $\eta \in (0, \frac{\pi}{2})$, *we have that*

$$G(m, \eta, r) := \int_{\partial\Omega_{\eta,r}} \frac{|z|^m}{|z-1|} |dz| \leqslant \mathcal{C}(r, m, \eta) \tag{4.17}$$

where $\partial\Omega_{\eta,r}$ *denotes the boundary of the set defined in (4.13) and*

$$\mathcal{C}(r, m, \eta) := 4(\sin\eta)^{m+1} \log \tfrac{4}{\cos\eta} + 4\mathrm{Ei}\left(r\tfrac{m+1}{2}\cos\eta\right) + 2\pi(1+r)^m,$$

where Ei *is defined in (4.14).*
If $r \leqslant \frac{1}{m+1}$, *by (4.16),*

$$\mathcal{C}(r, m, \eta) \leqslant -8\log\cos\eta - 4\log(r(m+1)) + 2\pi(1+r)^m + 12\log 2.$$

PROOF. Let us first assume that $r < \cos\eta$. We split up the path $\partial\Omega_{\eta,r} = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$, where Γ_2 denotes the union of the two straight line segments of $\partial\Omega_{\eta,r}$ (dashed lines in Figure 4.1), whereas Γ_1, Γ_3 denote the part of $\partial\Omega_{\eta,r}$ that lies on the circles $B_{\sin\eta}(0)$ and $B_r(1)$, respectively (dotted lines in Figure 4.1). Precisely,

$$\Gamma_1 = \left\{(\sin\eta)e^{i\delta}, |\delta| \in (\tfrac{\pi}{2} - \eta, \pi]\right\}, \Gamma_2 = \left\{1 - te^{\pm i\eta}, t \in (r, \cos\eta]\right\},$$
$$\Gamma_3 = \left\{1 + re^{i\delta}, |\delta| \in [0, \pi - \eta)\right\},$$

Next we estimate $G_i := \int_{\Gamma_i} \frac{|z|^m}{|z-1|}|dz|$ for $i = 1, 2, 3$.
For Γ_1, we see that

$$G_1 = 2(\sin\eta)^{m+1} \int_{\frac{\pi}{2}-\eta}^{\pi} \frac{dx}{|e^{ix}\sin\eta - 1|}. \tag{4.18}$$

Since $2|Re^{ix} - 1| \geqslant |e^{ix} - 1|$ for all $R, x \geqslant 0$,

$$G_1 \leqslant 4(\sin\eta)^{m+1} \int_{\frac{\pi}{2}-\eta}^{\pi} \frac{dx}{|e^{ix} - 1|} = 2\sqrt{2}(\sin\eta)^{m+1} \int_{\frac{\pi}{2}-\eta}^{\pi} \frac{dx}{\sqrt{1-\cos x}}. \tag{4.19}$$

Since $\sqrt{2}\log\tan\frac{x}{4}$ is a primitive of $\frac{1}{\sqrt{1-\cos x}}$ for $x \in (0, \pi)$, we derive

$$G_1 \leqslant -4(\sin\eta)^{m+1} \log\tan\frac{\frac{\pi}{2}-\eta}{4} \leqslant 4(\sin\eta)^{m+1} \log\tfrac{4}{\frac{\pi}{2}-\eta},$$

where in the last step we used that $\tan x \geqslant x$ for $x \in [0, \frac{\pi}{4}]$, which follows from the Taylor series of tan. Since $\sin x \leqslant x$ for all $x \geqslant 0$, we finally get

$$G_1 \leqslant 4(\sin\eta)^{m+1} \log\frac{4}{\cos\eta}. \tag{4.20}$$

To estimate G_2, note that $|1-te^{i\eta}|^2 \leqslant (1-t\cos\eta)$ for $t \in [0,\cos\eta]$ and thus,

$$G_2 = \int_{\Gamma_2} \frac{|z|^m |dz|}{|z-1|} = 2\int_r^{\cos\eta} |1-te^{i\eta}|^m \frac{dt}{t} \leqslant 2\int_r^{\cos\eta} |1-t\cos\eta|^{\frac{m}{2}} \frac{dt}{t}$$

Since $1-x \leqslant e^{-x}$ and $e^{-\frac{x}{2}} e^{\frac{1}{2}} \geqslant 1$ for $x \in [0,1]$,

$$G_2 \leqslant 2e^{\frac{1}{2}} \int_{r\cos\eta}^{\cos^2\eta} e^{-x\frac{m}{2}-\frac{x}{2}} \frac{dx}{x} \leqslant 2e^{\frac{1}{2}} \int_{r\frac{m+1}{2}\cos\eta}^{\infty} \frac{e^{-x}}{x}\, dx = 4\mathrm{Ei}\left(r\tfrac{m+1}{2}\cos\eta\right).$$

Finally, G_3 can be estimated by

$$G_3 = 2\int_0^{\pi-\eta} |1+re^{i\delta}|^m \, d\delta \leqslant 2\pi(1+r)^m. \tag{4.21}$$

This shows (4.17) for $r < \cos\eta$.

If $r \geqslant \cos\eta$, then $\partial\Omega_{\eta,r} = \partial(B_{\sin\eta}(0) \cup B_r(1))$. Hence, we choose Γ_1 and Γ_3 to be the convenient parts of the circles $\partial B_{\sin\eta}(0)$, $\partial B_r(1)$ such that $\Gamma_1 \cup \Gamma_3 = \partial\Omega_{\eta,r}$, i.e. $\Gamma_1 = \left\{\sin\eta e^{i\delta}, |\delta| \in (\alpha,\pi]\right\}$ and $\Gamma_3 = \left\{1+re^{i\delta}, |\delta| \in [0,\pi-\beta)\right\}$ for certain angles α, β depending on η. Since $r \geqslant \cos\eta$ it is easy to see that $\alpha > \frac{\pi}{2}-\eta$ and hence, we can estimate similarly as in (4.20) and (4.21),

$$G(m,\eta,r) = \int_{\Gamma_1} + \int_{\Gamma_3} \leqslant G_1 + 2\pi(1+r)^m \leqslant 4(\sin\eta)^{m+1}\log\tfrac{4}{\cos\eta} + 2\pi(1+r)^m,$$

which concludes the proof as the right hand side is smaller than $\mathcal{C}(r,m,\eta)$. □

THEOREM 4.7. *Let* T *be a Tadmor–Ritt operator on* X. *Let* $\theta = \arccos\frac{1}{C(T)}$. *Then, for* $m \in \mathbb{N}_0$, $r \in (0,1)$ *and* $\eta \in (\theta,\frac{\pi}{2})$ *we have, with* $\tau_m(z) = z^m$, *that*

$$\|(f\cdot\tau_m)(T)\| \leqslant c(T,m,r,\eta)\cdot\|f\|_{\infty,\Omega_{\eta,r}} \tag{4.22}$$

for $f \in H^\infty(\Omega_{\eta,r})$. *Here,*

$$c(T,m,r,\eta) \leqslant \frac{C_\eta(T)}{2\pi}\mathcal{C}(r,m,\eta)$$

where $C_\eta(T) = \sup_{z\in\mathbb{C}\setminus\overline{\mathcal{B}_\eta}} \|(z-1)R(z,T)\|$, *and* $\mathcal{C}$ *as in Lemma 4.6.*

PROOF. Let $\eta \in (\theta,\frac{\pi}{2})$ and $r > 0$. By Lemma 4.3 we know that $\sigma(T) \subset \Omega_{\eta,r}$. Let $f \in H(\Omega_{\eta,r})$. Since $f\tau_m$ is holomorphic on $\Omega_{\eta,r}$,

$$(f\tau_m)(T) = \frac{1}{2\pi i}\int_{\partial\Omega_{\tilde\eta,\tilde r}} f(z)z^m R(z,T)\, dz,$$

where $\tilde\eta \in (\theta,\eta)$ and $\tilde r \in (0,r)$. Since $\Omega_{\tilde\eta,\tilde r} \subset \Omega_{\eta,r}$,

$$\begin{aligned}\|(f\tau_m)(T)\| &\leqslant \frac{C_{\tilde\eta}(T)}{2\pi}\|f\|_{\infty,\Omega_{\eta,r}} \int_{\partial\Omega_{\tilde\eta,\tilde r}} \frac{|z|^m}{|z-1|}\, dz \\ &\leqslant \frac{C_{\tilde\eta}(T)}{2\pi}\|f\|_{\infty,\Omega_{\eta,r}}\cdot\mathcal{C}(\tilde r,m,\tilde\eta).\end{aligned}$$

Therefore, Lemma 4.6 and letting $(\tilde{\eta}, \tilde{r}) \to (\eta, r)$ give the assertion (that $C_{\tilde{\eta}}(T) \to C_\eta(T)$ can be seen by the maximum principle, see Theorem A.3, the rest follows since $\mathcal{C}$ is continuous). □

The following inequality is a direct consequence of the maximum principle. The disc case ($\eta = \frac{\pi}{2}$) can be traced back to S. Bernstein, and can be found in [**Rie16**, p. 346], or [**PS25**, Problem III. 269, p.137].

LEMMA 4.8. *Let $\mathcal{B}_\alpha$, $\alpha \in (0, \frac{\pi}{2}]$, be the Stolz type domain defined in Sec. 4.1.1. The following assertions hold.*

(i) For a polynomial p of degree n, and $r \geqslant 1$,

$$\|p\|_{\infty, r\mathcal{B}_\alpha} \leqslant \left(\frac{r}{\sin\alpha}\right)^n \cdot \|p\|_{\infty, \mathcal{B}_\alpha}. \tag{4.23}$$

(ii) For $f \in H(\mathcal{B}_\alpha)$ and continuous on $\overline{\mathcal{B}_\alpha}$, $m \in \mathbb{N}$ and $\tau_m(z) = z^m$,

$$\|f \cdot \tau_m\|_{\infty, \mathcal{B}_\alpha} \leqslant \|f\|_{\infty, \mathcal{B}_\alpha} \leqslant \frac{1}{(\sin\alpha)^m} \|f \cdot \tau_m\|_{\infty, \mathcal{B}_\alpha}. \tag{4.24}$$

PROOF. The first assertion is a consequence of the maximum principle applied to $p(z)z^{-n}$. In fact, let $z \in \mathbb{C} \setminus \mathcal{B}_\alpha$. Then, since $z \mapsto p(z)z^{-n}$ is analytic at ∞, by the maximum principle,

$$|p(z)z^{-n}| \leqslant \max_{z \in \partial\mathcal{B}_\alpha} |p(z)z^{-n}| \leqslant \max_{z \in \partial\mathcal{B}_\alpha} |z^{-n}| \cdot \|p\|_{\infty, \mathcal{B}_\alpha}. \tag{4.25}$$

It is easy to see that $\max_{z \in \partial\mathcal{B}_\alpha} |z^{-1}| = \frac{1}{\sin\alpha}$. Hence, multiplying (4.25) by $|z|^n$ and noting that $|z| \leqslant r$ for $z \in \partial(r\mathcal{B}_\alpha) \subset \mathbb{C} \setminus \mathcal{B}_\alpha$ yields

$$|p(z)| \leqslant \left(\frac{r}{\sin\alpha}\right)^n \|p\|_{\infty, \mathcal{B}_\alpha}, \; z \in \partial(r\mathcal{B}_\alpha).$$

Therefore, (4.23) follows by the maximum principle.
It is easy to see that $\sin\alpha \leqslant |z|$ for $z \in \partial\mathcal{B}_\alpha$. Therefore, by the maximum principle,

$$\|f\|_{\infty, \mathcal{B}_\alpha} = \sup_{z \in \partial\mathcal{B}_\alpha} |f(z)| \leqslant \frac{1}{(\sin\alpha)^m} \sup_{z \in \partial\mathcal{B}_\alpha} |z^m f(z)| = \frac{1}{(\sin\alpha)^m} \|f\tau_m\|_{\infty, \mathcal{B}_\alpha}$$

The other inequality of (4.24) is clear as $\mathcal{B}_\alpha \subset \mathbb{D}$. □

THEOREM 4.9. *Let T be a Tadmor–Ritt operator on X and let $m, n \in \mathbb{N}$ such that $0 \leqslant m \leqslant n$. Then, for any $p(z) = \sum_{k=m}^{n} a_k z^k$, we have that*

$$\|p(T)\| \leqslant aC(T)\left(2\log C(T) + b + \log\frac{n+1}{m+1}\right) \cdot \|p\|_{\infty, \mathbb{D}}, \tag{4.26}$$

with absolute constants a, b, that can be chosen as

$$a = \tfrac{2e}{\pi(1-s)}, \quad b = -2\log(s) + 6, \quad s \in (0, 1).$$

PROOF. Let $p(z) = \sum_{k=m}^{n} a_k z^k = z^m p_0(z)$ with $0 \leqslant m \leqslant n$ and p_0 is a polynomial of degree $n - m$. For $s \in (0, 1]$ let $\eta(s) = \arccos \frac{s}{C(T)}$. By Theorem 4.7 we have for $s, r \in (0, 1)$ that

$$\|p(T)\| \leqslant c(T, m, r, \eta(s)) \cdot \|p_0\|_{\infty,\Omega_{\eta(s),r}}, \tag{4.27}$$

where $p(z) = z^m p_0$. Since $\Omega_{\eta(s),r} \subset (1 + r)\mathbb{D}$, Lemma 4.8 (i) (with $\alpha = \frac{\pi}{2}$) yields

$$\|p_0\|_{\infty,\Omega_{\eta(s),r}} \leqslant \|p_0\|_{\infty,(1+r)\mathbb{D}} \leqslant (1 + r)^{n-m}\|p_0\|_{\infty,\mathbb{D}}. \tag{4.28}$$

By the maximum principle, $\|p_0\|_{\infty,\mathbb{D}} = \|p\|_{\infty,\mathbb{D}}$. Hence, by choosing $r = \frac{\tau}{n+1}$ with $\tau \in (0, 1)$, Eq. (4.27) becomes

$$\|p(T)\| \leqslant c(T, m, \tfrac{\tau}{n}, \eta(s)) \cdot (1 + \tfrac{\tau}{n+1})^{n-m}\|p\|_{\infty,\mathbb{D}}. \tag{4.29}$$

It remains to estimate the right hand side. Clearly, $(1 + \frac{\tau}{n+1})^{n-m} \leqslant e$. Theorem 4.7 yields that

$$c(T, m, \tfrac{\tau}{n+1}, \eta(s)) \leqslant \frac{C_{\eta(s)}}{2\pi}\mathcal{C}(\tfrac{\tau}{n+1}, m, \eta(s)).$$

We can further estimate $\mathcal{C}$ using Lemma 4.6. Since $r = \frac{\tau}{n+1} \leqslant \frac{1}{m+1}$,

$$c(T, m, \tfrac{\tau}{n+1}, \eta(s)) \leqslant \frac{2C_{\eta(s)}}{\pi}(-2\log\cos\eta(s) + \log\frac{n+1}{m+1} - \log\tau + \frac{\pi}{2}e^{\tau} + 3\log 2).$$

By Lemma 4.3, $C_{\eta(s)}(T) \leqslant \frac{C(T)}{1-s}$ for $s \in (0, 1)$. Since $\cos\eta(s) = \frac{s}{C(T)}$,

$$c(T, m, \tfrac{\tau}{n}, \eta(s)) \leqslant \frac{2C(T)}{\pi(1-s)}(2\log C(T) - 2\log s + \log\frac{n+1}{m+1} - \log\tau + \frac{\pi}{2}e^{\tau} + 3\log 2).$$

As $\min_{\tau\in(0,1)} \log\frac{1}{\tau} + \frac{\pi}{2}e^{\tau} + 3\log 2 < 6$, together with (4.29), this yields (4.26). □

COROLLARY 4.10. *Let T be a Tadmor–Ritt operator. Then T is power-bounded,*

$$\sup_{n\in\mathbb{N}} \|T^n\| \leqslant aC(T)\,(2\log C(T) + b)$$

with absolute constants $a, b > 0$ as in Theorem 4.9.

REMARK 4.11.

(i) Theorem 4.9 shows that any Tadmor–Ritt operator has a bounded $H^\infty[m, n]$-calculus, where

$$H^\infty[m, n] = \left\{ p(z) = \sum_{k=m}^{n} a_k z^k : a_k \in \mathbb{C} \right\} \tag{4.30}$$

and $m \leqslant n$. With different techniques, such a result was proved by Vitse in **[Vit05a]**, see also (4.7). However, in **[Vit05a]** the bound of the calculus depends on a factor $C(T)^5$, whereas in our Theorem 4.9, this gets improved to a behavior of $C(T)(\log C(T) + 1)$. Moreover, Corollary 4.10 shows that the same dependence holds true for the power-bound of a Tadmor–Ritt operator. This confirms the result by Bakaev [**Bak03**], which seems not so well-known, and improves the better known quadratic dependence $C(T)^2$, see [**EFR02**], [**Vit05a**].

(ii) It is a natural question to ask if the $\|\cdot\|_{\infty,\mathbb{D}}$-norm in Theorem 4.9 can be replaced by the sharper $\|\cdot\|_{\infty,\mathcal{B}_\eta}$-norm for some $\eta < \frac{\pi}{2}$. Indeed, Lemma 4.8 allows us to do this, see also (4.28). However, this leads to an additional factor $(\sin\eta)^{-n}$, which therefore destroys the logarithmic behavior in n.
Let us further remark that a polynomially bounded Tadmor–Ritt operator T (see (4.8)) on a Hilbert space implies an estimate of the form

$$\|p(T)\| \lesssim \|p\|_{\infty,\mathcal{B}_\eta},$$

for some $\eta < \frac{\pi}{2}$. In other words, T *allows for a bounded* $H^\infty(\mathcal{B}_\eta)$*-calculus*. However, this is not true for general Banach spaces, see [**LM13**]. More generally, including the Hilbert space case, if one assumes that T is R-Ritt (see Section 4.4), then polynomial-boundedness does indeed imply a bounded $H^\infty(\mathcal{B}_\eta)$-calculus, see [**LM14b**, Proposition 7.6] on arbitrary Banach spaces.

4.3. The effect of discrete square function estimates - Hilbert space

In the following we will show that discrete square function estimates improve the dependence in the way that $\log\frac{n+1}{m+1}$ in (4.26) gets replaced by its square root.

DEFINITION 4.12 (Hilbert space square function estimate). Let T be a bounded operator on a Hilbert space X. We say that T *satisfies square function estimates* if there exists a $K > 0$ such that

$$\|x\|_T^2 := \sum_{k=1}^{\infty} k\|T^k x - T^{k-1}x\|^2 \leqslant K^2\|x\|^2, \quad \forall x \in X. \tag{4.31}$$

Square function estimates are a well-known tool characterizing bounded H^∞-calculi for sectorial operators, going back to McIntosh's seminal work in the 80s [**McI86**]. From the 90s on, H^∞-calculus has proved very useful in the study of maximal regularity. In [**CDMY96**] a suitable L^p-version of square function estimates was introduced which then got further adapted to general Banach spaces by Kalton and Weis in the unpublished note [**KW01**], see also [**KW04**] and the references therein. Maximal regularity for discrete-time difference equations were investigated in [**Blu01b, Blu01a**]. Discrete square function estimates for Tadmor–Ritt operators were studied in [**KP08**]. We mention that in the literature there exists a whole scale of square functions, see [**LM14b**, Section 3], whereas we only use the specific form in Definition 4.12.

As for sectorial operators, for non-Hilbert (typically, L^p-) spaces suitable square function estimates have to be redefined for Tadmor–Ritt operators using Rademacher means. For the moment we will restrict ourselves to the Hilbert space case and leave the general Banach space case for Section 4.4.
The following characterization of bounded H^∞-calculus for Tadmor–Ritt operators was recently proved in **[LM14b]**. For the rest of the section we want to emphasize that on Hilbert spaces the notions of R-Ritt and Tadmor–Ritt operator coincide, whereas on general Banach spaces R-Ritt is stronger than Tadmor–Ritt. For a definition of R-Ritt operators and square function estimates on general Banach spaces, we refer to Definition 4.21 and Section 4.4.

THEOREM 4.13 (Le Merdy 2014, [**LM14b**, Corollary 7.5]). *Let T be a Tadmor–Ritt operator on a Banach space X. Consider the assertions*

(i) T is R-Ritt (Def. 4.21) and both T and T^* satisfy square function estimates.

(ii) For some $\eta \in (0, \frac{\pi}{2})$,

$$\|f(T)\| \lesssim \|f\|_{\infty,\mathcal{B}_\eta} \quad \forall f \in H_0^\infty(\mathcal{B}_\eta), \tag{4.32}$$

where $H_0^\infty(\mathcal{B}_\eta)$ is defined in Remark 4.5.

Then, (i) ⇒ (ii).
If X is a UMD space (in particular, a Hilbert space), then (ii) ⇒ (i).

The assumption on (the geometry of) the Banach space for the direction (i) to (ii) can be further generalized to X *having property* (Δ), see **[LM14b, KW01]**. In [**LM14b**, Proposition 8.1] it is further shown that there exist Tadmor–Ritt operators (even on Hilbert spaces) such that (only) T satisfies square function estimates, but (4.32) does not hold. However, we will see that having square function estimates for T (or T^*) does improve the functional calculus estimate in Theorem 4.9. Note that for a Tadmor–Ritt operator T of type θ and $r \in (0,1)$, rT is again Tadmor–Ritt with

$$C(rT) = \sup_{|\lambda|>1} \left\| (\lambda-1)\tfrac{1}{r}\left(\tfrac{\lambda}{r} - T\right)^{-1} \right\| \leqslant C(T) \sup_{|\lambda|>1} \left| \frac{\lambda-1}{\lambda-r} \right| = \frac{2C(T)}{1+r}. \tag{4.33}$$

We remark that moreover $\lim_{r\nearrow 1} f(rT) = f(T)$ for $f \in H_0^\infty(\mathcal{B}_\eta)$ with $\eta \in (\theta, \frac{\pi}{2})$, see [**LM14b**, Lemma 2.3].

LEMMA 4.14. *Let T be a Tadmor–Ritt operator on a Hilbert space X. For $m \in \mathbb{N}\cup\{0\}$, $r \in (0,1)$,*

$$\|(rT)^m x\|_{rT} \leqslant a r^m \sqrt{b + \log\left(1 - \frac{1}{2(m+1)\log r}\right)}\, \|x\| \quad \forall x \in X, \tag{4.34}$$

with $a = \sqrt{2}c_{1,T}$ and $b = 1 + \frac{Pb(T)^2}{c_{1,T}^2}$, where $c_{1,T}$ and $Pb(T)$ are defined in (4.11).

PROOF. Clearly, rT is a Tadmor–Ritt operator. By definition,

$$\begin{aligned}\|(rT)^m x\|_{rT}^2 &= r^{2m}\sum_{k=1}^{\infty} k\|r^k T^{k+m}x - r^{k-1}T^{k-1+m}x\|^2\\ &\leqslant r^{2m}\sum_{k=1}^{\infty} kr^{2(k-1)}\left(2\|T^{k+m}x - T^{k-1+m}x\|^2 + 2\|(1-r)T^{k+m}x\|^2\right)\\ &\leqslant r^{2m}\sum_{k=1}^{\infty} kr^{2(k-1)}\left(\frac{2c_{1,T}^2}{(k+m)^2} + 2(1-r)^2 Pb(T)^2\right)\|x\|^2, \qquad (4.35)\end{aligned}$$

where $c_{1,T} = \sup_{n\in\mathbb{N}}\|n(T^n - T^{n-1})\|$ which is finite by Lemma 4.4. Since $\frac{k}{(k+m)^2} \leqslant \frac{1}{k+m}$,

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{kr^{2(k-1)}}{(k+m)^2} &\leqslant \sum_{k=0}^{\infty}\frac{r^{2k}}{k+1+m}\\ &\leqslant \frac{1}{m+1} + \int_0^{\infty}\frac{e^{2x\log r}}{x+1+m}\,dx\\ &= \frac{1}{m+1} + r^{-2(m+1)}\mathrm{Ei}(-2(m+1)\log r)\\ &\leqslant \frac{1}{m+1} + \log\left(1 - \frac{1}{2(m+1)\log r}\right), \qquad (4.36)\end{aligned}$$

where the last step follows by (4.15). Using this and the fact that $\sum_{k=1}^{\infty} kr^{2(k-1)} = \frac{1}{(1-r^2)^2}$, we can conclude in (4.35) that

$$\begin{aligned}\|(rT)^m x\|_{rT}^2 &\leqslant r^{2m}\left[2c_{1,T}^2\left(\frac{1}{m+1} + \log\left(1 - \frac{1}{2(m+1)\log r}\right)\right) + \frac{2Pb(T)^2}{(1+r)^2}\right]\|x\|^2\\ &\leqslant 2c_{1,T}^2 r^{2m}\left(b + \log\left(1 - \frac{1}{2(m+1)\log r}\right)\right)\|x\|^2,\end{aligned}$$

for $b = 1 + \frac{Pb(T)^2}{c_{1,T}^2}$. □

Another lemma, we will need, is the following result relating square function estimates for T and rT as $r \nearrow 1$. This can be seen as a discrete analog of [**LM03**, Proposition 3.4].

LEMMA 4.15. *Let T be a Tadmor–Ritt operator on a Hilbert space. Then, the following are equivalent*

(i) *T satisfies square function estimates.*

(ii) *rT satisfies square function estimates uniform in $r \in (0,1)$, i.e.,*

$$\exists K > 0\ \forall r \in (0,1)\forall x \in X: \quad \|x\|_{rT} \leqslant K\,\|x\|.$$

PROOF. This follows from the more general Lemma 4.24 in Section 4.4. □

The following theorem is essentially Le Merdy's key argument to prove that (i) implies (ii) in Theorem 4.13. As we need its precise form, we state it explicitly. For a proof we refer to [**LM14b**, Proof of Theorem 7.3]. As before, for a definition of R-*Ritt operator of* R-*type* θ we refer to Definition 4.21 in Section 4.4. For the moment it suffices to remark that on Hilbert spaces this notion is equivalent to the one of a Tadmor–Ritt operator of type θ, see Section 4.4.

THEOREM 4.16 (Le Merdy 2014). *Let* T *be a* R-*Ritt operator of* R-*type* θ *on a Banach space* X *(if* X *is a Hilbert space, this is equivalent to* T *being a Tadmor-Ritt operator of type* θ*). Let* $0 < \theta < \eta < \frac{\pi}{2}$*. Then, there exists* $c = c(\eta, C(T)) > 0$ *such that*

$$|\langle y, p(T)x\rangle| \leqslant c \cdot \|p\|_{\infty,\mathcal{B}_\eta} \cdot \|x\|_T \cdot \|y\|_{T^*},$$

for any polynomial p, $x \in X$ *and* $y \in X^*$.
(Note that the right-hand-side is allowed to be ∞*).*

Combining Theorem 4.16 and Lemma 4.14 yields the following refinement of Theorem 4.9.

THEOREM 4.17. *Let* T *be a Tadmor–Ritt operator on a Hilbert space* X*. Assume that either* T *or* T^* *satisfies square function estimates. Then, for integers* $0 \leqslant m \leqslant n$ *and* $p(z) = \sum_{j=m}^{n} a_j z^j$,

$$\|p(T)\| \leqslant acKe^{\frac{1}{2}} \cdot \sqrt{b + \log \frac{n+2}{m+1}} \cdot \|p\|_{\infty,\mathbb{D}},$$

with K, a, b, c *defined in* (4.31)*, Lemma 4.14 and Theorem 4.16, respectively.*

PROOF. Since X is a Hilbert space, T is R-Ritt of type $\theta = \arccos \frac{1}{C(T)}$. Let $r \in (0,1)$ and choose $\eta \in (\theta, \frac{\pi}{2})$. Define $p_{\frac{1}{r}}(z) = p(\frac{z}{r})$. It is easy to see that $p_{\frac{1}{r}}(rT) = p(T)$ since p is a polynomial. Furthermore, we write $p(z) = z^m q(z)$ for q having degree $n - m$. Therefore, for all $x \in X$,

$$p(T)x = q_{\frac{1}{r}}(rT)(rT)^m x. \tag{4.37}$$

W.l.o.g. let T^* satisfy square function estimates. Hence, by Lemma 4.15, $\|y\|_{rT^*} \leqslant K\|y\|$ for all $y \in X^*$ and all $r \in (0,1)$. Applying Theorem 4.16 for rT and $p = q_{\frac{1}{r}}$ yields

$$|\langle y, q_{\frac{1}{r}}(rT)(rT)^m x\rangle| \leqslant cK \cdot \|q_{\frac{1}{r}}\|_{\infty,\mathbb{D}} \cdot \|(rT)^m x\|_{rT} \cdot \|y\|, \tag{4.38}$$

for $x \in X, y \in X^*$ where we used that $\mathcal{B}_\eta \subset \mathbb{D}$. By Lemma 4.8 (i) and the maximum principle, $\|q_{\frac{1}{r}}\|_{\infty,\mathbb{D}} \leqslant r^{m-n}\|q\|_{\infty,\mathbb{D}} = r^{m-n}\|p\|_{\infty,\mathbb{D}}$. Therefore, and by Lemma 4.14, Eq. (4.38) yields

$$\|q_{\frac{1}{r}}(rT)(rT)^m\| \leqslant acK\sqrt{b + \log\left(1 - \frac{1}{2(m+1)\log r}\right)} \cdot r^{2m-n} \cdot \|p\|_{\infty,\mathbb{D}}.$$

Choose $r = e^{-\frac{1}{2(n-m+1)}}$. Then $1 - \frac{1}{2(m+1)\log r} = \frac{n+2}{m+1}$ and $r^{2m-n} = e^{\frac{n-2m}{2(n-m+1)}} < e^{\frac{1}{2}}$. Thus, by (4.37),

$$\|p(T)\| \leqslant acKe^{\frac{1}{2}}\sqrt{b + \log\frac{n+2}{m+1}} \cdot \|p\|_{\infty,\mathbb{D}}.$$

□

REMARK 4.18.

(i) The proof idea of Theorem 4.17 can also be used for an alternative proof of the logarithmic behavior in Theorem 4.9, if we do a similar computation for $\|T^m y\|_{rT^*}$ (instead of assuming square function estimates $\|y\|_T \lesssim \|y\|$). This finally yields another factor of the form $\sqrt{\tilde{b} + \log\frac{n+2}{m+1}}$.

(ii) As explained in Remark 4.11, in Theorem 4.17 we can also derive 'sharper' estimates in the $\|\cdot\|_{\infty,\mathcal{B}_\eta}$-norm at the price that additional factors of the form $(\sin\eta)^{-n}$ enter the estimate.

4.4. Discrete square function estimates on general Banach spaces

As indicated in Section 4.3, for non-Hilbert spaces, Definition 4.12 is not suitable for characterizing boundedness of the H^∞-calculus. For L^p-spaces the proper replacement is given by

$$\|x\|_T := \left\| \left(\sum_{k=1}^{\infty} k|T^k x - T^{k-1}x|^2 \right)^{\frac{1}{2}} \right\|_{L^p} \lesssim \|x\| \tag{4.39}$$

where T is a Tadmor–Ritt operator on $L^p(\Omega)$, $p \in [1,\infty)$, for some measure space (Ω,μ), see **[KP08]**, **[LM14b]** and the references therein. By Fubini's theorem, this definition coincides with Definition 4.12 if $p = 2$.
However, to cover general Banach spaces, we need the following generalization using Rademacher averages. This approach (for sectorial operators), paving the way for a lot of research in this field, was introduced by Kalton and Weis in their 'famous' unpublished note, see the preprint **[KW01]**. For an excellent overview on the topic we refer to **[KW04]**. The discrete version of these general square function estimates for Tadmor–Ritt operators recently appeared in **[LM14b]**.

We briefly recap the definition of the needed Rademacher norms. For more details, we refer to **[LM14b, KW04]**. For $k \geqslant 1$, we define the Rademacher function $\varepsilon_k(t) = \operatorname{sgn}(\sin(2^k\pi t))$. It is easy to see that $(\varepsilon_k)_{k\geqslant 1}$ forms an orthonormal basis in $L^2(I)$ with $I = [0,1]$. For a Banach space X let us consider the linear span of elements $\varepsilon_k \otimes x = (t \mapsto \varepsilon_k(t)x)$, $k \geqslant 0$, $x \in X$, in the Bochner space $L^2(I,X)$. Denote the closure of this set, w.r.t. the norm in $L^2(I,X)$, by $\operatorname{Rad}(X)$. Hence, $\operatorname{Rad}(X)$ becomes a Banach space with the norm

$$\|\tilde{x}\|_{\mathrm{Rad}(X)} = \left(\int_I \left\| \sum_k \varepsilon_k(t) x_k \right\|^2 dt \right)^{\frac{1}{2}},$$

for elements $\tilde{x} = \sum_k \varepsilon_k \otimes x_k$ with $(x_k)_k$ being a finite family in X. By orthonormality of the Rademacher functions it follows that

$$\mathrm{Rad}(X) = \left\{ \sum_{k=1}^{\infty} \varepsilon_k \otimes x_k : x_k \in X, \text{the sum converges in } L^2(I, X) \right\}. \tag{4.40}$$

Now we can define a general square function by

$$\|x\|_T = \left\| \sum_{k=1}^{\infty} \varepsilon_k \otimes k(T^k x - T^{k-1} x) \right\|_{\mathrm{Rad}(X)}, \tag{4.41}$$

where we set $\|x\|_T = \infty$ if $\sum_k \varepsilon_k \otimes k(T^k x - T^{k-1} x) \notin \mathrm{Rad}(X)$.

DEFINITION 4.19 (Square function estimates for Tadmor–Ritt operators). Let T be a Tadmor–Ritt operator on a Banach space X. We say that T satisfies *(abstract) square function estimates*, if there exists $K_T > 0$ such that for all $x \in X$,

$$\|x\|_T = \left\| \sum_{k=1}^{\infty} \varepsilon_k \otimes k^{\frac{1}{2}} (T^k x - T^{k-1} x) \right\|_{\mathrm{Rad}(X)} \leqslant K_T \|x\|. \tag{4.42}$$

Note that if X is a Hilbert space, as a consequence of Parseval's identity, this definition of square function estimates coincides with the one given in Definition 4.12. Precisely, for any finite sequence $(x_k)_k \in X$,

$$\left\| \sum_k \varepsilon_k \otimes x_k \right\|_{\mathrm{Rad}(X)} = (\sum_k \|x_k\|^2)^{\frac{1}{2}}, \tag{4.43}$$

which shows that both definitions of square functions estimates coincide. Further, it can be shown that for $X = L^p = L^p(\Omega, \mu)$ ($p \in [1, \infty)$ and (Ω, μ) being σ-additive),

$$\left\| \sum_k \varepsilon_k \otimes x_k \right\|_{\mathrm{Rad}(L^p)} \sim \left\| \left(\sum_k |x|^2 \right)^{\frac{1}{2}} \right\|_{L^p},$$

see [**KW04**, Remark 2.9]. Hence, (4.39) is equivalent to having square function estimates using Rademacher averages.

The notion of R-boundedness emerges naturally when considering the space Rad(X). After being introduced in [**BG94**], it has been proved very useful in the study of maximal regularity, see [**KW04**] for a detailed introduction.

DEFINITION 4.20. Let X be a Banach space and $\mathcal{T} \subset \mathcal{B}(X)$ a set of bounded operators. Then, $\mathcal{T}$ is called R-**bounded** if there exists a constant M such that for any finite family $(T_k))k \in \mathcal{T}$, and finite sequence $(x_k)_k \subset X$,

$$\left\| \sum_k \varepsilon_k \otimes T_k x_k \right\|_{\mathrm{Rad}(X)} \leqslant M \left\| \sum_k \varepsilon_k \otimes x_k \right\|_{\mathrm{Rad}(X)}. \tag{4.44}$$

The smallest possible constant C is called the R-bound.

By (4.43), it follows that for Hilbert spaces the notion of R-boundedness of $\mathcal{T}$ coincides with (uniform) boundedness of $\mathcal{T}$ in the operator norm. However, in general, R-boundedness only implies boundedness, see [**AB02**].

Now we are able to introduce R-Ritt operators, which first appeared in [**Blu01b, Blu01a**]. Nonetheless the notion R-*Tadmor–Ritt* would be more consistent in this Chapter, we use the name R-*Ritt* following Le Merdy [**LM14b**]. For Hilbert spaces, the following notion is equivalent to the one of a Tadmor–Ritt operator, see Lemma 4.4.

DEFINITION 4.21. An operator T on a Banach space X is called R-Ritt if the sets

$$\{T^n : n \in \mathbb{N}\} \text{ and } \left\{n(T^n - T^{n-1}) : n \in \mathbb{N}\right\} \tag{4.45}$$

are R-bounded. We denote the bounds by $\mathrm{Pb}^R(T)$ and $c^R_{1,T}$, respectively.

By Lemma 4.4, an R-Ritt operator is always a Tadmor–Ritt operator and the notions coincide on Hilbert spaces. Moreover, the following R-Ritt version of Lemmata 4.3 and 4.4 holds. For a proof, see [**LM14b**, Lemma 5.2] and [**Blu01b**].

LEMMA 4.22. *Let* T *be a bounded operator on a Banach space* X*. The following assertions are equivalent.*

(i) T *is* R-*Ritt.*

(ii) $\sigma(T) \subset \overline{\mathcal{B}_\theta}$ *for some* $\theta \in [0, \frac{\pi}{2})$ *and for all* $\eta \in (\theta, \frac{\pi}{2}]$

$$\left\{(z-1)R(z,T) : z \in \mathbb{C} \setminus \overline{\mathcal{B}_\eta}\right\} \text{ is R-bounded.} \tag{4.46}$$

In this case, we say that T *is of* R-*Ritt type* θ*.*

Now we are ready to prove the corresponding R-Ritt version of the results in Section 4.3 for general Banach spaces.

LEMMA 4.23. *Let* T *be a* R*-Ritt operator on a Banach space* X. *For* $m \in \mathbb{N} \cup \{0\}$, $r \in (0,1)$,

$$\|(rT)^m x\|_{rT} \leqslant a r^m \sqrt{b_R + \log\left(1 - \frac{1}{2(m+1)\log r}\right)}\,\|x\| \quad \forall x \in X, \tag{4.47}$$

with $a_R = \sqrt{2} c^R_{1,T}$ *and* $b_R = 1 + \frac{Pb^R(T)^2}{(c^R_{1,T})^2}$, *where* $c^R_{1,T}, Pb^R(T)$ *are defined in Def. 4.21.*

PROOF. The proof technique is very similar to the proof of Lemma 4.14. Therefore, we will focus on the arguments involving R-boundedness. Since rT is a Tadmor–Ritt operator, we have, see 4.42,

$$\begin{aligned}\|T^m x\|_{rT} &= \left\| \sum_{k=1}^{\infty} \varepsilon_k \otimes k^{\frac{1}{2}} (r^k T^{k+m} x - r^{k-1} T^{k-1+m} x) \right\|_{\mathrm{Rad}(X)} \\ &\leqslant \left\| \sum_{k=1}^{\infty} \varepsilon_k \otimes \left[(T^{k+m} - T^{k-1+m}) + (1-r) T^{k+m} \right] k^{\frac{1}{2}} r^{k-1} x \right\|_{\mathrm{Rad}(X)} \\ &\leqslant c^R_{1,T} \left\| \sum_{k=1}^{\infty} \varepsilon_k \otimes \frac{k^{\frac{1}{2}} r^{k-1}}{k+m} x \right\|_{\mathrm{Rad}(X)} + \\ &\qquad + (1-r) Pb^R(T) \left\| \sum_{k=1}^{\infty} \varepsilon_k \otimes k^{\frac{1}{2}} r^{k-1} x \right\|_{\mathrm{Rad}(X)}, \end{aligned} \tag{4.48}$$

where the last step follows since T is R-Ritt. By the definition of the $\mathrm{Rad}(X)$-norm, and Parseval's identity (for $L^2[0,1]$), the first norm in (4.48) equals

$$\begin{aligned}\left\| \sum_{k=1}^{\infty} \varepsilon_k \otimes \frac{k^{\frac{1}{2}} r^{k-1}}{k+m} x \right\|^2_{\mathrm{Rad}(X)} &= \int_0^1 \left\| \sum_{k=1}^{\infty} \varepsilon_k(t) \tfrac{k^{\frac{1}{2}} r^{k-1}}{k+m} x \right\|^2_X dt \\ &= \|x\|^2 \int_0^1 \left| \sum_{k=1}^{\infty} \varepsilon_k(t) \tfrac{k^{\frac{1}{2}} r^{k-1}}{k+m} \right|^2 dt \\ &= \|x\|^2 \sum_{k=1}^{\infty} \left| \frac{k^{\frac{1}{2}} r^{k-1}}{k+m} \right|^2. \end{aligned}$$

The remaining series can be estimated as in the Hilbert space proof. Analogously, the second norm in (4.48) can be computed. Therefore, we derive,

$$\begin{aligned}\|(rT)^m x\|_{rT} &\leqslant r^m \left[c^R_{1,T} \left(\frac{1}{m+1} + \log\left(1 - \frac{1}{2(m+1)\log r}\right)\right)^{\frac{1}{2}} + \frac{Pb^R(T)}{(1+r)} \right] \|x\| \\ &\leqslant \sqrt{2} c^R_{1,T} r^{2m} \left(b_R + \log\left(1 - \frac{1}{2(m+1)\log r}\right)\right)^{\frac{1}{2}} \|x\|, \end{aligned}$$

for $b_R = 1 + \frac{Pb^R(T)^2}{(c^R_{1,T})^2}$. □

We further need the generalization of Lemma 4.15 to (abstract) square function estimates.

LEMMA 4.24. *Let* T *be a* R*-Ritt operator on a Banach space* X*. Then, the following are equivalent.*

(i) T *satisfies (abstract) square function estimates.*

(ii) rT *satisfies (abstract) square function estimates uniform in* $r \in (0,1)$*,*

$$\exists K > 0\ \forall r \in (0,1) \forall x \in X : \quad \|x\|_{rT} \leqslant K\,\|x\|.$$

PROOF. The proof is similar to one for the continuous time analog **[LM03**, Proposition 3.4] and is based on using the identity

$$(I - T)T^k x = (I - rT)T^k x + (1 - r)T^{k+1}x. \tag{4.49}$$

This yields, using that T is R-Ritt,

$$\begin{aligned}\left\|\sum_{k=1}^{\infty} \varepsilon_k \otimes k^{\frac{1}{2}} r^k (I - T)T^k x\right\|_{\mathrm{Rad}(X)} &\leqslant \left\|\sum_{k=1}^{\infty} \varepsilon_k \otimes k^{\frac{1}{2}} (I - rT)(rT)^k x\right\|_{\mathrm{Rad}(X)} + \\ &\quad + Pb^R(T)(1 - r)\left\|\sum_{k=1}^{\infty} \varepsilon_k \otimes k^{\frac{1}{2}} r^k x\right\|_{\mathrm{Rad}(X)}.\end{aligned} \tag{4.50}$$

By Parseval's identity, the second $\mathrm{Rad}(X)$-norm equals $(\sum_{k=1}^{\infty} kr^{2k})^{\frac{1}{2}}\|x\| = \frac{\|x\|}{1-r^2}$. Therefore, it is easy to see that the second term in (4.50) is bounded in $r \in (0,1)$. Hence, by Fatou's lemma, we get that (ii) implies (i). The other direction also follows, with a similar estimation, from (4.49). □

The Banach space version of Theorem 4.17 now follows completely analogously to the Hilbert space proof with Lemmata 4.23 and 4.24 (instead of Lemmata 4.14 and 4.15).

THEOREM 4.25. *Let* T *be a* R*-Ritt operator on a Banach space* X*. Assume that either* T *or* T^* *satisfies (abstract) square function estimates. Then, for integers* $0 \leqslant m \leqslant n$ *and* $p(z) = \sum_{j=m}^{n} a_j z^j$*,*

$$\|p(T)\| \leqslant a_R c K_T e^{\frac{1}{2}} \cdot \sqrt{b_R + \log\frac{n+2}{m+1}} \cdot \|p\|_{\infty,\mathbb{D}},$$

with K_T, a_R, b_R *and* c *defined in* (4.42)*, Lemma 4.23 and Theorem 4.16, respectively.*

4.5. Sharpness of the estimates

It is natural to ask whether the deduced functional calculus estimates from Theorems 4.9 and 4.17,

$$\|p(T)\| \leqslant aC(T)\left(\log C(T) + b + \log \frac{n+1}{m+1}\right)\|p\|_{\infty,\mathbb{D}}, \tag{4.51}$$

and

$$\|p(T)\| \leqslant a_2 c K_T e^{\frac{1}{2}} \cdot \sqrt{b_2 + \log \frac{n+2}{m+1}} \cdot \|p\|_{\infty,\mathbb{D}}, \tag{4.52}$$

for $p \in H^\infty[m, n]$, that is $p(z) = \sum_{k=m}^{n} a_k z^k$, are sharp. Clearly, here 'sharpness' has different aspects depending on the variables $C(T), m, n$ it is referring to. For a clear discussion, we distinguish between the following questions.

(A) Is (4.51) sharp in the variables m, n, with $0 \leqslant m \leqslant n$?

(B) Is (4.51) sharp in the variable $C(T)$ for (some) fixed m, n?

(C) Question (A) for (4.52).

(D) Question (B) for (4.52).

To answer these questions, we introduce the quantity

$$C(T, m, n) = \sup\{\|p(T)\| : p \in H^\infty[m, n], \|p\|_{\infty,\mathbb{D}} \leqslant 1\}. \tag{4.53}$$

Question (A) was discussed Vitse in [**Vit05a**, Remark 2.6] using the prior works [**Vit04b, Vit05b**]. In particular, she showed that if X contains a complemented isomorphic copy of ℓ^1 or ℓ^∞ (e.g., some infinite-dimensional L^1 or $C(K)$ spaces), then there exists a Tadmor–Ritt operator on X such that

$$C(T, m, n) \gtrsim \log \frac{ne}{m},$$

where the involved constant only depends on X and is thereby linked with constant $C(T)$. However, the precise dependence on $C(T)$ is not apparent there. If X is an (infinite-dimensional) Hilbert space (more, generally if the Banach space X contains a complemented isomorphic copy of ℓ^2), then for any $\delta \in (0, 1)$, there exists a Tadmor–Ritt operator such that

$$C(T, m, n) \gtrsim \left(\log \frac{ne}{m}\right)^\delta.$$

These statements can be generalized to more general spaces X that *uniformly contain uniform copies* of ℓ^1_n (or ℓ^2_n respectively). We refer to [**Vit04a, Vit05a**] for details.

Question (B) can be split up in several cases. If $m = 0$, hence p is an arbitrary polynomial of degree n, (4.51) implies that $C(T, 0, n) \lesssim C(T)(\log C(T) + \log(n + 1))$. Hence, we observe 'linear' asymptotic behavior in $C(T)$ as $n \to \infty$. In fact, in [**Vit04a,** Theorem 2.1] it is shown that it is indeed linear, namely

$$C(T,0,n) \leqslant (C(T)+1)\log(e^2 n), \tag{4.54}$$

and there exists a T on some Banach space X such that $C(T,0,n) \sim \log(e^2 n)$. We point out that the proof technique, [**Vit04a**, Theorem 2.1], requires $m = 0$.

However, for $m = n$, Question (B) reduces to the prominent question of the optimal power-bound for T. As mentioned in Corollary 4.10, (4.51) yields

$$C(T,n,n) = \|T^n\| \lesssim C(T)(\log C(T)+1),$$

for all n. This is so-far the best known power-bound for Tadmor–Ritt operators, see also [**Bak03**]. It remains open whether this can be replaced by a linear $C(T)$-dependence. Furthermore, motivated by the Kreiss Matrix Theorem (4.5), it is not clear whether for N-dimensional spaces X, an estimate of the form

$$Pb(T) \leqslant C(T)g(N) \tag{4.55}$$

for some scalar function g can be achieved, where $g(N) \in o(N)$. Note that the estimate for $g(N) = eN$ trivially holds by (4.5) and the fact that $C_{Kreiss}(T) \leqslant C(T)$.

Let us turn to Question (C) now. We want to show sharpness of

$$C(T,m,n) \lesssim \sqrt{\log \frac{n+1}{m+1}} \tag{4.56}$$

under the assumption that T satisfies square function estimates. Therefore, we construct T as a Schauder basis multiplier, which is a well-known technique to construct unbounded calculi, see e.g., Section 3.3.1 and [**BC91**], where it was introduced. Let X be a separable infinite-dimensional Hilbert space with a bounded Schauder basis $\{\psi_k\}$. For a sequence $(\lambda_n) \subset [0,1]$, define the bounded operator $T = \mathcal{M}_\lambda$ by

$$Tx = \left(\sum_k x_k \psi_k\right) = \sum_k \lambda_k x_k \psi_k,$$

for finite sequences $(x_k) \subset \mathbb{C}$. Let $\lambda_n = 1 - 2^{-n}$, then T is Tadmor–Ritt, see [**LM14b**, Proposition 8.2]. With this setting we can use the following argument from [**Vit05b**, Proof of Theorem 2.1]. Let $\delta \in (0,1)$. If for the uniform basis constant $ub(\{\psi_k\}_{k=1}^N)$ it holds that $ub(\{\psi_k\}_{k=1}^N) \gtrsim N^\delta$, i.e.

$$\exists c > 0\ \forall N \in \mathbb{N}: \quad \sup\left\{\left\|\sum_{k=1}^N \alpha_k x_k \psi_k\right\| : |\alpha_k| \leqslant 1, \left\|\sum_{k=1}^N x_k \psi_k\right\| \leqslant 1\right\} \geqslant cN^\delta, \tag{4.57}$$

then $C(T,m,n) \gtrsim \left(\log \frac{n+1}{m+1}\right)^\delta$.

As for sectorial Schauder multipliers, it holds that T satisfies square function estimates if the basis is *Besselian*, i.e., $\exists c_\psi > 0$

$$c_\psi \left(\sum_k |x_k|^2 \right)^{\frac{1}{2}} \leqslant \left\| \sum_k x_k \psi_k \right\|, \tag{4.58}$$

for finite sequences $(x_k) \subset \mathbb{C}$, see [**LM03**, Theorem 5.2] and [**LM14b**, Theorem 8.2]. Note that (4.58) already implies that $\mathrm{ub}(\{\psi_k\}_{k=1}^N) \leqslant c_\psi m(\psi)\sqrt{N}$, where $m(\psi) = \sup_k \|\psi_k\|$. It remains to find a Besselian basis $\{\psi_k\}$ such that (4.57) is fulfilled for $\delta \in (0, \frac{1}{2})$. Indeed, such an example can be constructed for an L^2-space on the unit circle with suitable weight, see [**LM03**, Theorem 5.2], Theorem 3.26, and [**STW03**, Section 4.3]. In fact, the example in Theorem 3.26 in Chapter 3 gives a basis

$$\psi_{2k}(t) = |t|^{-\beta} e^{ikt}, \quad \psi_{2k+1}(t) = |t|^{-\beta} e^{-ikt}, \; k \in \mathbb{N}_0,$$

(there, the notation is ψ^*) with $\beta \in (\frac{1}{3}, \frac{1}{2})$. Moreover, it is shown that there exist elements $x, y \in L^2$ such that

$$|x_n| \sim n^{3\beta-1}, \qquad |y_n| \sim n^{\beta-1}, \; n \in \mathbb{N},$$

where $x = \sum_n x_n \psi_n$ and $y \in \sum_n y_n \psi_n^*$, and where $\{\psi_n^*\}$ denotes the dual basis such that $\langle \psi_k^*, \psi_n \rangle = \delta_{nk}$. Choosing $|\alpha_n| = 1$ such that $\alpha_n x_n y_n \in \mathbb{R}_{\geqslant 0}$, we deduce

$$|\langle y, \sum_{n=1}^N \alpha_n x_n \psi_n \rangle| = \sum_{n=1}^N \alpha_n x_n y_n \gtrsim \sum_{n=1}^N n^{3\beta-2} \sim N^{3\beta-1}.$$

Since $\| \sum_{n=1}^N x_n \psi_n \| \leqslant b(\psi)\|x\|$, (4.57) follows for $\delta = 3\beta - 1 \in (0, \frac{1}{2})$. Therefore, we have proved the following result, which answers (C) for Hilbert spaces.

THEOREM 4.26. *There exists a Hilbert space such that for any* $\delta \in (0, \frac{1}{2})$ *there exists a Tadmor–Ritt operator* T *which satisfies square function estimates and*

$$C(T, m, n) \gtrsim \left(\log \frac{n+1}{m+1} \right)^{\delta}$$

holds, where $C(T, m, n)$ *is defined in* (4.53). *Note that the involved constants depend on* δ.

An open question is whether there exists an R-Ritt operator on a Banach space such that T satisfies square function estimates and $C(T, m, n) \gtrsim \left(\log \frac{n+1}{m+1}\right)^{\frac{1}{2}}$.

By $c_{1,T} \lesssim C(T)^3$, see [**Vit05a**], and $c \lesssim Pb(T)^3 c_{1,T}$, see [**LM14b**, Proof of Theorem 7.3], we can track $C(T)$ in the constants of the estimate in Theorem 4.17. This yields a $C(T)$-dependence, which seems far from being sharp. Hence, the answer to (D) is probably 'no'.

4.6. Further results

As a direct corollary of the improvements of Vitse's result, we get the following result for the Besov space functional calculus of T, which in turn is a slight improvement of **[Vit05a**, Theorem 2.2]. For details of the following notions and facts see **[Vit05a]** and the references therein. Recall that the *Besov space* $B_{\infty,1}(\mathbb{D})$ is defined by the functions $f \in H(\mathbb{D})$ such that

$$\|f\|_B := \|f\|_{\infty,\mathbb{D}} + \int_0^1 \max_\alpha |f'(re^{i\alpha})| dr < \infty.$$

It is well known that there exists an equivalent definition via the dyadic decomposition $f = \sum_{n=0}^\infty W_n * f$, where W_n, $n \geqslant 1$ are shifted Fejer type polynomials, whose Fourier coefficients $\hat{W}_n(k)$ are the integer values of the triangular-shaped function supported in $[2^{n-1}, 2^{n+1}]$ with peak $\hat{W}_n(2^n) = 1$ and $W_0(z) = 1 + z$. Here $(g * f)(z) = \sum_{k=0}^\infty \hat{g}(k)\hat{f}(k)z^k$. Then,

$$f \in B_{\infty,1}(\mathbb{D}) \iff f \in H(\mathbb{D}) \text{ and } \|f\|_* = \sum_{n=0}^\infty \|W_n * f\|_{\infty,\mathbb{D}} < \infty.$$

Since $W_n * f$ is a polynomial, we can use the $\|\cdot\|_{\infty,\mathbb{D}}$-estimate of Theorem 4.9 to derive $B_{\infty,1}(\mathbb{D})$-functional calculus estimates. This follows the same lines as in **[Vit05a]**, however, using the improved constant dependence of our result in Theorem 4.9.

THEOREM 4.27. *Let* T *be a Tadmor–Ritt operator on a Banach space* X. *Then,*

$$\|f(T)\| \lesssim C(T)(\log(C(T) + 1))\|f\|_* \tag{4.59}$$

i.e., for all $f \in B_{\infty,1}(\mathbb{D})$, *where* $f(T)$ *is defined by* $\sum_{n=0}^\infty (W_n * f)(T)$.

PROOF. Since $W_n * f \in H^\infty[2^{n-1}, 2^{n+1}]$ for $n \geqslant 1$ and $W_0 * f \in H^\infty[0, 1]$, see Remark 4.11 for the definition of $H^\infty[m, n]$, we can apply Theorem 4.9 to derive

$$\|(W_n * f)(T)\| \leqslant aC(T) \left(2 \log C(T) + b + \log \frac{2^{n+1} + 1}{2^{n-1} + 1} \right) \|W_n * f\|_{\infty,\mathbb{D}},$$

for $n \geqslant 1$, with absolute constants $a, b > 0$. Clearly, $\frac{2^{n+1}+1}{2^{n-1}+1} \leqslant 5$. Analogously, $\|(W_0 * f)(T)\|$ can be estimated. Thus, $\sum_{n=0}^\infty \|(W_n * f)(T)\| \lesssim \|f\|_*$, and hence, $f(T)$ is well-defined with

$$\|f(T)\| \leqslant aC(T) (2 \log C(T) + b + \log 5) \|f\|_*.$$

□

In **[Vit05a**, Theorem 2.5] a similar $\|\cdot\|_{B_{\infty,1}(\mathbb{D})}$-estimate as in (4.59) is derived, but with a $C(T)$-dependence of $C(T)^5$.

We conclude this chapter by mentioning the, to us, most interesting open questions related to the presented results in this chapter.

Q1) *Is* $C(T)\log(C(T)+1)$*, up to a constant, the optimal bound for a Tadmor–Ritt operator* T*?*

Q2) *What is the optimal power-bound of a Tadmor–Ritt operator in a finite-dimensional space?*

CHAPTER 5

Discrete vs. continuous time problems

Abstract. In the previous chapters we have seen some analogy between functional calculus results for continuous and discrete-time. This chapter deals with the transformation from the continuous to the discrete setting via the Cayley transform. This leads to the prominent *Inverse Generator Problem* and the *Cayley Transform Problem* for C_0-semigroups. We show the equivalence of these two problems and the fact that we can even reduce these problems to the case where the semigroup is exponentially stable. Furthermore, we give an overview on existing results in the literature and state some open questions.

5.1. The Cayley transform

In this section we introduce a conformal mapping that takes the left-half-plane of the complex plane, $\mathbb{C}_-$, to the unit disc $\mathbb{D}$. We further show how images of sectors and half-planes under this mapping look like.

DEFINITION 5.1. The mapping

$$\tau : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\} : \quad z \mapsto \frac{1+z}{1-z}, \tag{5.1}$$

is called the *Cayley transform* from $\mathbb{C}_-$ to $\mathbb{D}$. Respectively, we call $\tau_-(z) = \tau(-z)$ the *Cayley transform* from $\mathbb{C}_+$ to $\mathbb{D}$.

Obviously, the Cayley transform is a special case of a *Möbius transform*,

$$z \mapsto \frac{az+b}{cz+d} \tag{5.2}$$

for $a, b, c, d \in \mathbb{C}$ with $ad - bc \neq 0$. Such transformation are well-studied in complex analysis, therefore, we state the following elementary properties without a proof.

LEMMA 5.2. *The Cayley transform* τ *has the following properties.*

(i) $\tau(z) = \frac{1+z}{1-z} = -1 + \frac{2}{1-z}$.

(ii) τ *is bijective and the inverse mapping* τ^{-1} *equals* $-\tau_-$.

(iii) τ *maps* $\mathbb{C}_-$ *onto* $\mathbb{D}$, $i\mathbb{R}$ *onto* $\mathbb{T}$ *and* $\mathbb{C}_+ \setminus \{1\}$ *onto* $\overline{\mathbb{D}}^c$. *Furthermore,* $\tau(\infty) = -1$, $\tau(0) = 1$ *and* $\tau(1) = \infty$.

(iv) The restriction $\tau|_{\mathbb{C}_-}$ *lies in* $H^\infty(\mathbb{C}_-)$ *and*

$$|\tau(iy)| = 1 \quad \forall y \in \mathbb{R},$$

thus, τ *is an inner function on* $\mathbb{C}_-$ *and* $\|\tau^n\|_{\infty,\mathbb{C}_-} = 1$ *for all* $n \in \mathbb{N}_0$.

From Lemma 5.2 it follows that subsets of $\mathbb{C}_-$ are mapped into the unit disc. For half-planes $\mathbb{C}_\omega = \{z \in \mathbb{C} : \operatorname{Re} z < \omega\}$ with $\omega < 0$, $\tau(\mathbb{C}_\omega)$ is a disc inside $\mathbb{D}$ that touches the point -1. On the other hand, the image of sectors, with angle less than π, cannot approach the points 1 and -1 tangentially. These results are summarized in the following lemma, see also Figures 5.1 and 5.2.

LEMMA 5.3. *Let* $\omega \leqslant 0$ *and* $\theta \in (0, \frac{\pi}{2}]$. *Then, the following holds for the Cayley transform* τ.

(i) τ *maps the half-plane* $\mathbb{C}_\omega$ *onto the ball* $B_{\frac{1}{1-\omega}}(\frac{\omega}{1-\omega})$.

(ii) τ *maps the sector* $\Xi_\theta = \mathbb{C} \setminus \overline{\Sigma_{\pi-\theta}}$ *onto* $B_{\frac{1}{\sin\theta}}(\frac{i}{\tan\theta}) \cap B_{\frac{1}{\sin\theta}}(\frac{-i}{\tan\theta})$.

(iii) $\tau(\Xi_\theta) \subset \operatorname{co}(\{1\} \cup \{-1\} \cup B_{\sin\theta}(0))$.

Here, $\operatorname{co}(X)$ *denotes the convex hull of* $X \subset \mathbb{C}$ *and* $\Sigma_{\pi-\theta} = \{z \neq 0 : |\arg z| < \pi - \theta\}$.

PROOF. It is well known that a Möbius transformations (5.2) maps a line in $\mathbb{C}$ to either a circle or a line. We will use this fact to prove *(i)* and *(ii)*.
Consider the line $\omega' + i\mathbb{R}$ for $\omega' \leqslant \omega$. Since $\tau(\infty) = -1$, $\tau(\omega') = \frac{1+\omega'}{1-\omega'} \in (-1, 1]$ and $\tau(\omega' + i) \notin \mathbb{R}$, we conclude that $\tau(\omega' + i\mathbb{R})$ is a circle going through the points $z_1 = -1, z_2 = \frac{1+\omega'}{1-\omega'}$. Because $\tau(\omega' + iy) = \overline{\tau(\omega' - iy)}$ for $y \in \mathbb{R}$, the circle $\tau(\omega' + i\mathbb{R})$ is symmetric w.r.t. the real axis. Therefore, the radius $r_{\omega'}$ and the centre $s_{\omega'}$ can be calculated by $2r_{\omega'} = |z_1 - z_2| = \frac{2}{1-\omega'}$ and $s_{\omega'} = \frac{z_1+z_2}{2} = \frac{\omega'}{1-\omega'}$. Since $\cup_{\omega' \leqslant \omega} \partial B_{r_{\omega'}}(s_{\omega'}) = B_{\frac{1}{1-\omega}}(\frac{\omega}{1-\omega})$, this proves *(i)*.
To show *(ii)*, consider the line $e^{i(\pi-\theta)}\mathbb{R}$. Similar as for *(i)*, it follows that $\tau(e^{i(\pi-\theta)}\mathbb{R})$ is a circle going through the points $\tau(\infty) = -1$, $\tau(0) = 1$. Moreover, by symmetry, the centre s of the circle has to lie on $i\mathbb{R}$. To derive s and the radius r, we observe that $\frac{d}{dt}\tau(te^{i(\pi-\theta)})(0) = 2e^{i(\pi-\theta)}$. Hence, the angle (oriented clockwise) between the real axis and the tangent to $\tau(e^{i(\pi-\theta)}\mathbb{R})$ at point 1 equals θ. From that and considering the triangle spanned by the points 0, 1 and s in the complex plane, we derive $r = \frac{1}{\sin\theta}$ and $s = \frac{-i}{\tan\theta}$. Analogously, $\tau(e^{-i(\pi-\theta)}\mathbb{R})$

equals the circle with radius $\frac{1}{\sin\theta}$ and centre $\frac{i}{\tan\theta}$. Moreover, it follows that the (rotated) half-planes $\Gamma_1 = e^{i(\frac{\pi}{2}-\theta)}\mathbb{C}_-$, $\Gamma_2 = e^{-i(\frac{\pi}{2}-\theta)}\mathbb{C}_-$ are mapped onto the balls $B_{\frac{1}{\sin\theta}}(\frac{-i}{\tan\theta})$, $B_{\frac{1}{\sin\theta}}(\frac{i}{\tan\theta})$, respectively. Since $\Gamma_1 \cap \Gamma_2 = \Sigma_\theta$, the assertion follows.

(iii) follows by noting that the angles spanned by the boundary of $\mathrm{co}\left(\{1\} \cup \{-1\} \cup B_{\sin\theta}(0)\right)$ and 1 and -1, respectively (considered from 'inside' the convex hull), both equal 2θ. See also Figure 5.2.

□

FIGURE 5.1. The Cayley transform $\tau(z) = \frac{1+z}{1-z}$ on half-planes.

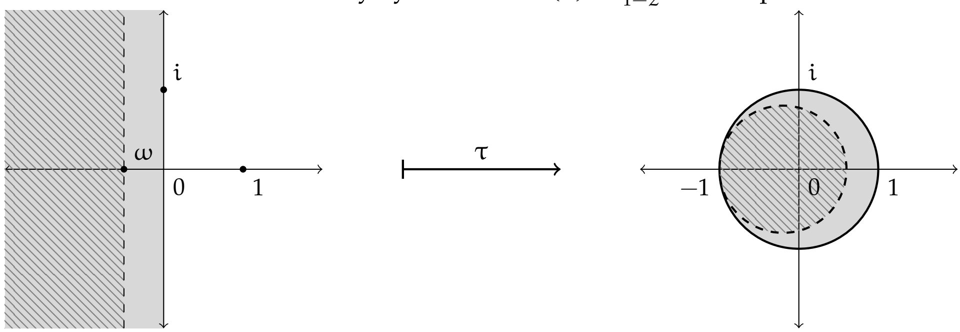
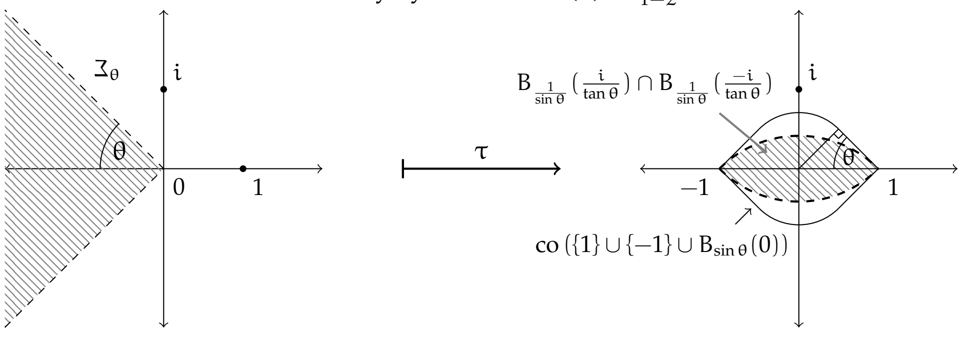


FIGURE 5.2. The Cayley transform $\tau(z) = \frac{1+z}{1-z}$ and sectors.



In Figure 5.2, note that

$$\mathrm{co}\big(\{1\} \cup \{-1\} \cup B_{\sin\theta}(0)\big) \subset \overline{\mathcal{B}_\theta} \cup \big(-\overline{\mathcal{B}_\theta}\big)\,, \tag{5.3}$$

where $\overline{\mathcal{B}_\theta} = \overline{\mathrm{co}\big(\{1\} \cup B_{\sin\theta}(0)\big)}$ was defined in Section 4.1.1, see also Figure 4.1.

The following result rests on the spectral mapping theorem for resolvents, which can be found, e.g., in [**Haa06a**, Chapter A.3].

PROPOSITION 5.4. *Let* $C : D(C) \subset X \to X$ *be a closed operator on the Banach space* X *and assume that* $1 \in \rho(C)$. *Then, the operator*

$$\mathcal{C}ay(C) := \tau(C) := (I + C)(I - C)^{-1} = -I + 2R(1, C) \tag{5.4}$$

is bounded on X. *It further holds that*

$$\sigma(\mathcal{C}ay(C)) = \tau(\tilde{\sigma}(C)), \quad \rho(\mathcal{C}ay(C)) = \tau(\tilde{\rho}(C)), \tag{5.5}$$

where $\tilde{\rho}(C)$ *denotes the extended resolvent set, i.e.*

$$\tilde{\rho}(C) = \begin{cases} \rho(C) \cup \{\infty\}, & \text{if } C \in \mathcal{B}(X), \\ \rho(C), & \text{if } C \notin \mathcal{B}(X), \end{cases} \tag{5.6}$$

and $\tilde{\sigma}(C) = (\mathbb{C} \cup \{\infty\}) \setminus \tilde{\rho}(C)$.

Proposition 5.4, together with Lemma 5.3, shows how the spectra of $\mathcal{C}ay(A)$ for A generating bounded, exponentially stable and bounded analytic and C_0-semigroups, respectively, look like.

In fact, the Cayley transform of the generator A of a bounded semigroup has spectrum in the closed unit disc. If the semigroup is exponentially stable, then the spectrum of $\mathcal{C}ay(A)$ is contained in a smaller disc inside $\overline{\mathbb{D}}$, touching the unit circle at -1. See Figure 5.1.

Similarly, the spectrum of $\mathcal{C}ay(A)$ is contained in the closure of the set $B_{\frac{1}{\sin\theta}}(\frac{i}{\tan\theta}) \cap B_{\frac{1}{\sin\theta}}(\frac{-i}{\tan\theta})$ (grey-lined in Figure 5.2), if A generates a bounded analytic semigroup (i.e., $-A$ is sectorial of angle less than $\frac{\pi}{2}$), see also Figure 5.2. Here, θ equals the sectorality angle of $-A$.

Combining these properties yields that if the analytic semigroup is also exponentially stable (on $[0, \infty)$), then the spectrum of $\mathcal{C}ay(A)$ lies in the intersection of the grey-lined regions on the right-hand side of Figures 5.1 and 5.2. More precisely, if $-A$ is sectorial of angle θ and $\omega < 0$ is the growth bound of the semigroup e^{tA}, then

$$\begin{aligned} \sigma(\mathcal{C}ay(A)) &\subset \overline{\left(B_{\frac{1}{\sin\theta}}\left(\tfrac{i}{\tan\theta}\right) \cap B_{\frac{1}{\sin\theta}}\left(\tfrac{-i}{\tan\theta}\right)\right)} \cap \overline{B_{\frac{1}{1-\omega}}\left(\tfrac{\omega}{1-\omega}\right)} \\ &\subset \overline{\operatorname{co}\left(\{1\} \cup \{-1\} \cup B_{\sin\theta}(0)\right)} \cap \overline{B_{\frac{1}{1-\omega}}\left(\tfrac{\omega}{1-\omega}\right)}. \end{aligned}$$

Since $B_{\frac{1}{1-\omega}}\left(\frac{\omega}{1-\omega}\right)$ is a disc inside $\overline{\mathbb{D}}$ touching $\mathbb{T}$ only at -1, it follows that there exists $\theta' \in (0, \frac{\pi}{2})$ such that $\sigma(\mathcal{C}ay(A)) \subset \overline{\operatorname{co}(\{-1\} \cup B_{\sin\theta'}(0))} = -\overline{\mathcal{B}_{\theta'}}$. This indicates that Tadmor–Ritt operators can be seen as the discrete analog of generators of exponentially stable, analytic semigroup generators.

5.2. The Cayley transform and the Inverse Generator Problem

Having introduced the Cayley transform of a closed operator in Section 5.1, we are now going to study whether the Cayley transform preserves stability in the sense as discussed in Section 1.1. As we have seen in the introduction of this thesis, such questions originate from the analysis of differential and difference equations in numerical analysis, see, e.g., [**vDKS93, SW97**].

To simplify notation for the rest of the Chapter, if A is the generator of a C_0-semigroup, the corresponding semigroup will be denoted by $e^{tA} = \left(e^{tA}\right)_{t\geqslant 0}$. Moreover, all considered semigroups will be strongly continuous, therefore, we will often use the word *semigroup* rather than C_0-*semigroup*. Let us begin with stating the two main questions of this chapter.

PROBLEM 5.5 (*The Cayley Transform Problem - Banach (Hilbert) space form*).
Is the Cayley transform $\mathcal{C}ay(A)$ power-bounded, i.e.,

$$\sup_{n\in\mathbb{N}} \|\mathcal{C}ay(A)^n\| < \infty,$$

when A is the generator of a bounded C_0-semigroup on a Banach (Hilbert) space X?

PROBLEM 5.6 (*The Inverse Generator Problem - Banach (Hilbert) space form*).
Is A^{-1} the generator of a bounded C_0-semigroup, when A generates a bounded C_0-semigroup on a Banach (Hilbert) space X and A^{-1} is supposed to exist as a densely defined operator?

REMARK 5.7 (to Problems 5.5 and 5.6).

- Note that for the generator A of a bounded semigroup, the operator $\mathcal{C}ay(A)$ is bounded since $1 \in \rho(A)$, see Proposition 5.4.
- Suppose that e^{tA} is even exponentially stable. This implies that $0 \in \rho(A)$, and hence, A^{-1} is a bounded operator. Therefore, A^{-1} generates the semigroup given by the power series

$$e^{tA^{-1}} = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^{-n}.$$

  Thus, for an exponentially stable semigroup e^{tA}, Problem 5.6 reduces to the question whether this power series is uniformly bounded in $t \geqslant 0$.
- As we have seen in the Introduction, Section 1.1, Problem 5.5 is nothing else than the question whether the *Crank–Nicolson scheme*, with stepsize $h = 2$, is stable, cf. (1.3). Therefore, in the literature, Problem 5.5 is often referred to as the (question of) *stability of the Crank–Nicolson scheme*.
  We further remark that $\mathcal{C}ay(A)$ is sometimes called the *cogenerator* of the semigroup e^{tA}, see, e.g., [**EZ08, GZB11**].

The Cayley Transform and the Inverse Generator Problem have been studied intensively in the last decades, see, e.g., [**ABD04, deL88, EZ08, Gom99, Gom04, GZ07, GZT07, GZB11, GZ06, vC11, PZ07, Zwa07**]. For overviews see also [**Haa06a**, Sections 9.2.3, 9.2.4] and the theses [**Bes12, Mub11**]. In 1988, de Laubenfels [**deL88**] started the study of the Inverse Generator Problem, already answering affirmatively the (comparably simple) case of analytic semigroups. In the following 15 years, the problem seemed to be a bit forgotten, until it got back attention by the works of Gomilko, Tomilov and Zwart [**Gom99, Gom04, GZT07, Zwa07**], who pointed out a relation to the Cayley Transform Problem. However, in 1993, Palenica [**Pal93**], and, independently, Crouzeix, Larsson, Piskarev and Thomée [**CLPT93**] had already proved that the Cayley transform of a sectorial operator on a Banach space is power-bounded, i.e. the answer to Problem 5.5 is 'yes' for generators of bounded analytic semigroups, see also Section 4.1.

Both problems actually split up in into a couple of sub-problems on the one hand for special spaces (Hilbert vs. Banach spaces) and on the other hand in types of semigroups (e.g., bounded, contractive, exponentially stable, analytic). It is known that the answers to Problems 5.5 and 5.6 depend on these very situations. For example, we have positive answers if the semigroup e^{tA} is assumed to be analytic; see [**Pal93**] and Chapter 4 for sectorially bounded analytic semigroups on Banach spaces and [**GZ06**] for bounded analytic semigroups on Hilbert spaces. Whereas there exist examples of contraction semigroups on general Banach spaces such that the Cayley transform of the generator is not power-bounded and such that A^{-1} is not generating a bounded strongly continuous semigroup, see Example 5.9 below.

At the end of this chapter we will list an overview of the (so-far known) answers to the different cases, see Table 1. The most prominent question is whether the problems (for general bounded semigroups) have positive answers for Hilbert spaces. This still remains open.

We remark that the Inverse Generator Problem is sometimes stated slightly differently in the literature, dropping the *boundedness* for the semigroup $e^{tA^{-1}}$ in the question. However, this is an equivalent problem (if the problem is considered "for all A and all spaces X"), see [**Zwa07**] or Theorem 5.14 below.

5.3. The equivalence of the Cayley Transform and the Inverse Generator Problem

To shorten notation, let us introduce the following sets.

DEFINITION 5.8. For a Banach space X, $\omega \in \mathbb{R}$ and $M \geqslant 1$ we set

$$\mathcal{G}_{M,\omega}(X) := \left\{A : A \text{ generates a } C_0\text{-semigroup on } X \text{ and } \|e^{tA}\| \leqslant Me^{t\omega}\ \forall t \geqslant 0\right\},$$

and $\mathcal{G}_M := \mathcal{G}_{M,0}$. Furthermore, let $\mathcal{G}_{bdd}(X)$ and $\mathcal{G}_{exp}(X)$ be the sets of bounded and exponentially stable semigroups, respectively. So,

$$\mathcal{G}_{bdd}(X) = \bigcup_{M \geqslant 1} \mathcal{G}_M(X) \quad \text{and} \quad \mathcal{G}_{exp}(X) = \bigcup_{M \geqslant 1, \omega < 0} \mathcal{G}_{M,\omega}(X).$$

If we omit the space X, like $A \in \mathcal{G}_M$, then we mean that there exists some Banach space X such that $A \in \mathcal{G}_M(X)$. Similar, we write $A \in \mathcal{G}$ if A is a semigroup generator on some Banach space.

EXAMPLE 5.9.
The following example of an exponentially stable semigroup shows that neither the Cayley transform of the generator needs to be power-bounded, nor does the inverse of the generator need to generate a bounded semigroup. The part on the Cayley transform is slightly adapted from [**Bes12**, Lemma 1.2] and the part on the inverse generator can already be found in [**Zwa07**, Example 3.5].

Consider $X = C_0[0,1)$, which denotes the Banach space of continuous functions f on $[0,1)$ with $\lim_{t\to 1^-} f(t) = 0$, equipped with the supremum norm. Define

$$(T(t)f)(s) = \begin{cases} f(t+s), & t+s < 1, \\ 0, & t+s \geqslant 1. \end{cases}$$

It can be shown that T is a strongly continuous semigroup which is exponentially stable as $T(t) = 0$ for $t \geqslant 1$. Therefore, $\mathcal{C}ay(A)$, see Proposition 5.4, and A^{-1} are bounded operators, where A denotes the generator of T. Hence, A^{-1} generates a strongly continuous semigroup (given by the power series $e^{tA^{-1}}$). Therefore, it remains to study

$$\sup_{n\in\mathbb{N}} \|\mathcal{C}ay(A)^n\| \quad \text{and} \quad \sup_{t\geqslant 0} \|e^{tA^{-1}}\|.$$

We are going to use the following semigroup version of a well-known identity for Laguerre polynomials.

$$(-1)^n \mathcal{C}ay(A)^n x = I - 2\int_0^\infty e^{-t} L_{n-1}^{(1)}(2t) T(t) x \, dt, \quad x \in X, n \in \mathbb{N}, \tag{5.7}$$

where L_n^α denotes the generalized Laguerre polynomials see, e.g., [**Gom04**] or [**BZ10**, Lemma 4.4].

Applying (5.7) to our semigroup T, we get for $f \in C_0[0,1)$,

$$\begin{aligned}\|f\| + \|\mathrm{Cay}(A)^n f\| &\geqslant 2 \left\| s \mapsto \int_0^{1-s} e^{-t} L_{n-1}^{(1)}(2t) f(t+s)\,dt \right\|_\infty \\ &\geqslant 2 \left| \int_0^1 e^{-t} L_{n-1}^{(1)}(2t) f(t)\,dt \right|.\end{aligned}$$

Now we choose f such that it approximates the sign of the smooth function

$$t \mapsto e^{-t} L_{n-1}^{(1)}(2t) \quad \text{on} \quad (0,1).$$

More precisely, for any $\varepsilon > 0$, we find $f \in C_0[0,1)$ such that

$$\left| \int_0^1 e^{-t} L_{n-1}^{(1)}(2t) f(t)\,dt \right| \geqslant \int_0^1 |e^{-t} L_{n-1}^{(1)}(2t)|\,dt + \varepsilon,$$

(for a similar argument see [**Zwa07**, Example 3.5]).
Finally, we use that for all $t > 0$, $|L_n^{(1)}(t)| \sim n^{\frac{1}{4}} t^{-\frac{3}{4}} \left|\cos\left(2\sqrt{nt} - \frac{3\pi}{4}\right)\right|$, see [**Sze67**], to conclude that

$$\int_0^1 |e^{-t} L_{n-1}^{(1)}(2t)|\,dt \gtrsim n^{\frac{1}{4}}.$$

This shows that the Cayley transform is not power-bounded.
Similarly, using the following identity for exponentially stable semigroups due to Zwart [**Zwa07**, Lem. 3.2],

$$e^{A^{-1}\tau} x = x - \int_0^\infty \tau h_{ac}(t\tau) e^{tA} x\,dt, \quad \tau > 0, \tag{5.8}$$

where $h_{ac} = \frac{1}{\sqrt{t}} J_1(2\sqrt{t})$ and J_1 denotes the *Bessel function of first kind*, it can be shown that $\|e^{tA^{-1}}\| \sim t^{\frac{1}{4}}$ for $t \geqslant 0$. Thus, $e^{tA^{-1}} \notin \mathcal{G}_{bdd}$.

Let us now study the relation between Problems 5.5 and 5.6. It has been known for a long time that there is a strong connection between these questions. For instance, the following results are known.

THEOREM 5.10. *Let A be an operator on a Hilbert space H. The following implications hold.*

(i) If $A \in \mathcal{G}_{bdd}(H)$, A injective and $A^{-1} \in \mathcal{G}_{bdd}(H)$, then $\mathrm{Cay}(A)$ is power-bounded and the powers-bound can be estimated by

$$\sup_{n\in\mathbb{N}} \|\mathrm{Cay}(A)^n\| \leqslant \frac{e}{2}\left(\frac{1}{4} + M_1^2 + M_2^2\right), \tag{5.9}$$

where $M_1 = \sup_{t\geqslant 0} \|e^{tA}\|$ and $M_2 = \sup_{t\geqslant 0} \|e^{tA^{-1}}\|$.

(ii) If $A \in \mathcal{G}_{exp}(H)$ and $\mathrm{Cay}(A)$ is power-bounded, then $A^{-1} \in \mathcal{G}_{bdd}(H)$.

PROOF. The first item was first proved by **[Gom04]**, see also [**ABD04**], [**Bes12**, Theorem 7.6], [**GZ06**, Theorem 4.4] for alternative proofs.
The second assertion can be found in [**GZB11**, Theorem 4.11]. □

In particular, Theorem 5.10 shows that if $A \in \mathcal{G}_{\exp}(H)$, then the operator A^{-1} (which is even bounded as $0 \in \rho(A)$) generates a bounded semigroup if and only if $Cay(A)$ is power-bounded. In Theorem 5.14 we will partially generalize this equivalence to general $A \in \mathcal{G}_{bdd}(H)$ ("partially" in the sense that the equivalence then holds "for all Hilbert spaces H").

Versions of the following lemma can be traced back to Gomilko [**Gom04, GZ07, GZT07**] and have often been used in the study of the relation between the Cayley Transform Problem and the Inverse Generator Problem, see, e.g., [**GZB11**]. For completeness, we include an elementary proof.

LEMMA 5.11 (Gomilko's trick). *Let A be such that $(0,\infty) \subset \rho(A)$ and assume that A is injective. Then, for $\lambda > 0$ and $A_\lambda := 2\lambda A - I$,*

$$-\lambda\left(\lambda - A^{-1}\right)^{-1} = (I + A_\lambda)(I - A_\lambda)^{-1},$$
$$= Cay(A_\lambda).$$

PROOF. Since $(0,\infty) \subset \rho(A)$ by assumption, $1 \in \rho(A_\lambda)$ for $\lambda > 0$. Thus, by Proposition 5.4, $Cay(A_\lambda)$ is well-defined as a bounded operator. By inserting the definition of $A_\lambda = 2\lambda A - I$ and noting that $(CD)^{-1} = D^{-1}C^{-1}$ for closed operators C and D, we derive

$$\begin{aligned}(I + A_\lambda)(I - A_\lambda)^{-1} &= \lambda A(I - \lambda A)^{-1} \\ &= \lambda\left[(I - \lambda A)A^{-1}\right]^{-1} \\ &= -\lambda\left(\lambda I - A^{-1}\right)^{-1}, \end{aligned} \tag{5.10}$$

where the last equality follows since for all $x \in R(A)$,

$$(I - \lambda A)A^{-1}x = (A^{-1} - \lambda I)x$$

and $D(A^{-1} - \lambda I) = R(A)$. □

The following two theorems show that if Problems 5.5 and 5.6 have positive answers for all $A \in \mathcal{G}_{M,\omega}$, then the power-bound of the Cayley transform and the bound of the inverse semigroup can be bounded by constants only depending on the constants M and ω.

THEOREM 5.12. *Let $M \geqslant 1$ and $\omega \leqslant 0$. The following assertions are equivalent.*

(i) For every Banach space X and every $A \in \mathcal{G}_{M,\omega}(X)$, it follows that the Cayley transform of A is power-bounded, i.e.,

$$\exists C_A > 0 \qquad \sup_{n\in\mathbb{N}} \|\mathrm{Cay}(A)^n\|_X \leqslant C_A. \tag{5.11}$$

(ii) There exists a constant $C_{M,\omega} > 0$ such that for every Banach space X and every $A \in \mathcal{G}_{M,\omega}(X)$, the powers of the Cayley transform of A are bounded by $C_{M,\omega}$, i.e.,

$$\sup_{n\in\mathbb{N}} \|\mathrm{Cay}(A)^n\|_X \leqslant C_{M,\omega}. \tag{5.12}$$

The equivalence remains true if one replaces "Banach space" by "Hilbert space" in both (i) and (ii).

PROOF. Clearly, we only have to show *(i)* $\Rightarrow$ *(ii)*.
Fix $M \geqslant 1$, $\omega \leqslant 0$. Let us assume that *(ii)* does not hold. Therefore, since by *(i)*, $\mathrm{Cay}(A)$ is power-bounded for every Hilbert space H and $A \in \mathcal{G}_{bdd}(H)$, we can assume that there exists a sequence $(A_m)_{m\in\mathbb{N}} \subset \mathcal{G}_{M,\omega}(X_m)$ such that $\sup_{m\in\mathbb{N}} \sup_{n\in\mathbb{N}} \|\mathrm{Cay}(A_m)^n\|_{X_m} = \infty$. Define

$$A := \operatorname*{diag}_{m\in\mathbb{N}}(A_m) = \begin{bmatrix} A_1 & & & \\ & A_2 & & \\ & & \ddots & \\ & & & \ddots \end{bmatrix},$$

with maximal domain in the Banach space

$$\bigoplus_{m\in\mathbb{N}} X_m = \left\{ x \in \prod_{m\in\mathbb{N}} X_m : \|x\|_{\oplus}^2 := \sum_{m\in\mathbb{N}} \|x_m\|_{X_m}^2 < \infty \right\},$$

equipped with the norm $\|\cdot\|_{\oplus}$. By its structure, it is easily seen that A generates a C_0-semigroup e^{tA} with $\|e^{tA}\| \leqslant Me^{t\omega}, t \geqslant 0$. Thus, $A \in \mathcal{G}_{M,\omega}(\bigoplus_m X_m)$, and since we assumed that *(i)* holds, the Cayley transform $\mathrm{Cay}(A)$ is power-bounded. On the other hand, since $\mathrm{Cay}(A) = \mathrm{diag}_m(\mathrm{Cay}(A_m))$, we have that

$$\sup_{n\in\mathbb{N}} \|\mathrm{Cay}(A)^n\|_{\mathcal{B}(\bigoplus_{m\in\mathbb{N}} X_m)} = \sup_{n\in\mathbb{N}} \sup_{m\in\mathbb{N}} \|\mathrm{Cay}(A_m)^n\|_{\mathcal{B}(X_m)} = \infty, \tag{5.13}$$

which contradicts (5.13).
The proof for Hilbert spaces is the same, noting that for Hilbert spaces X_m, the space $\bigoplus_m X_m$ is a Hilbert space with inner product

$$\langle x, y\rangle := \sum_{m\in\mathbb{N}} \langle x_m, y_m\rangle_{X_m}, \quad x, y \in \bigoplus_{m\in\mathbb{N}} X_m.$$

□

A similar result holds for the Inverse Generator Problem.

THEOREM 5.13. *Let $M \geqslant 1$ and $\omega \leqslant 0$. The following assertions are equivalent.*

(i) For every Banach space X and every $A \in \mathcal{G}_{M,\omega}(X)$ such that A^{-1} is a densely defined operator, it follows that $A^{-1} \in \mathcal{G}_{bdd}(X)$.

(ii) There exists a $K_{M,\omega} > 0$ such that the following holds. For every Banach space X and every $A \in \mathcal{G}_{M,\omega}(X)$ such that A^{-1} is a densely defined operator, it follows that $A^{-1} \in \mathcal{G}_{K_{M,\omega},0}(X)$, i.e., A^{-1} generates a semigroup bounded by $K_{M,\omega}$.

The equivalence remains true if one replaces "Banach space" by "Hilbert space" in both (i) and (ii).
Note that for $\omega < 0$, $A \in \mathcal{G}_{M,\omega}$ implies that $A^{-1} \in \mathcal{B}(X)$, see Remark 5.7.

PROOF. The proof is similar to the one of Theorem 5.12. The implication *(ii)* $\Rightarrow$ *(i)* is trivial.

To show that *(i)* implies *(ii)*, we fix $M \geqslant 1$, $\omega \leqslant 0$.
First, let us observe that if an operator B and some $K \geqslant 1$, the condition that $B \notin \mathcal{G}_{K,0}$ is equivalent to the alternative,

($\heartsuit$) $B \in \mathcal{G}$ and $\sup_{t\geqslant 0} \|e^{tB}\| > K$, or

($\clubsuit$) $B \notin \mathcal{G}$,

i.e., B is a generator but not in $\mathcal{G}_{K,0}$, or B is not a generator of a semigroup at all.
Assume that *(ii)* is not true. Hence, for every $n \in \mathbb{N}$ there exists exists an operator $A_m \in \mathcal{G}_{M,\omega}(X_m)$ for some Banach space X_m, such that A_m^{-1} exists as a densely defined operator and $A_m^{-1} \notin \mathcal{G}_{K,0}$. Since, we assumed that *(i)* holds, A_m^{-1} generates a bounded semigroup, i.e., $A_m^{-1} \in \mathcal{G}_{bdd}$. Hence, in particular, $A_m^{-1} \in \mathcal{G}$, and since $A_m^{-1} \notin \mathcal{G}_{K,0}$, it follows that A_m^{-1} has to be of type ($\heartsuit$) in the list above (with $B = A_m^{-1}$).
Therefore, we have a sequence $(A_m)_{m\in\mathbb{N}}$ of generators on Banach spaces X_m with the following properties. For $m \in \mathbb{N}$, $A_m \in \mathcal{G}_{M,\omega}(X_m)$, A_m^{-1} is a densely defined operator, $A_m^{-1} \in \mathcal{G}_{bdd}(X_m)$ and

$$\sup_{m\in\mathbb{N}} \sup_{t\geqslant 0} \|e^{tA_m^{-1}}\| = \infty. \tag{5.14}$$

Consider $A := \mathrm{diag}_m(A_m)$ defined on $\bigoplus_m X_m$, where $\bigoplus_m X_m$ is defined as in proof of Theorem 5.12. It is easy to see that A generates the semigroup $\mathrm{diag}_m(e^{tA_m})$, which implies that $A \in \mathcal{G}_{M,\omega}(\bigoplus_m X_m)$. Furthermore, it holds that

$$A^{-1} = \mathrm{diag}_m(A_m^{-1})$$

exists as a densely defined operator. By *(i)*, we conclude that $A^{-1} = \mathrm{diag}_m(A_m^{-1})$ has to generate a bounded semigroup too.

Next, we show that $e^{tA^{-1}}$ equals $\mathrm{diag}_m(e^{tA_m^{-1}})$ (at least on some dense subspace). Let $x = (x_m)_m \in \bigoplus_m X_m$ such that finitely many x_m are non-zero in X_m. Then, for $s > 0$,

$$\begin{aligned}\mathfrak{L}\left(e^{\cdot A^{-1}}x\right)(s) &= \left(sI - A^{-1}\right)^{-1}x \\ &= \left(\operatorname*{diag}_{m\in\mathbb{N}}(sI_{X_m} - A_m^{-1})\right)^{-1} x \\ &= \left((sI_{X_m} - A_m^{-1})^{-1}x_m\right)_{m\in\mathbb{N}} \\ &= \left(\mathfrak{L}\left(e^{\cdot A_m^{-1}}x_m\right)(s)\right)_{m\in\mathbb{N}} \\ &= \mathfrak{L}\left(\operatorname*{diag}_{m\in\mathbb{N}}\left(e^{\cdot A_m^{-1}}\right)x\right)(s),\end{aligned}$$

where in the last step we used that x_m is the non-zero for only finitely many $m \in \mathbb{N}$. Since the Laplace transform is injective and the semigroup trajectories are continuous, we derive that

$$e^{tA^{-1}}x = \operatorname*{diag}_{m\in\mathbb{N}}\left(e^{tA_m^{-1}}\right)x, \quad t \geqslant 0, x \in D,$$

where $D := \{(x_m) \in \bigoplus_m X_m : x_m = 0 \text{ for a.e. } m \in \mathbb{N}\}$ is a dense subspace of $\bigoplus_m X_m$. In particular, it follows that $\mathrm{diag}_m(e^{tA_m^{-1}})$ is a bounded operator on D[1]and

$$\left\|e^{tA^{-1}}\right\| = \left\|\mathrm{diag}_m(e^{tA_m^{-1}})\right\|,$$

Therefore,

$$\sup_{t\geqslant 0}\|e^{tA^{-1}}\| = \sup_{t\geqslant 0}\left\|\operatorname*{diag}_{m\in\mathbb{N}}(e^{tA_m^{-1}})\right\| = \sup_{t\geqslant 0}\sup_{m\in\mathbb{N}}\|e^{tA_m^{-1}}\|_{\mathcal{B}(X_m)} = \infty,$$

where the last step follows from (5.14). Hence, $e^{tA^{-1}} \notin \mathcal{G}_{bdd}$, and thus *(i)* cannot hold. This concludes the proof for the "Banach space" version of the Theorem.

The "Hilbert space" version of the Theorem follows completely analogously noting that $\bigoplus_m X_m$ becomes a Hilbert space if the X_m's are Hilbert spaces (see also proof of Theorem 5.12). □

For a closed operator A on X, the condition that A^{-1} exists as a densely defined operator is equivalent to the condition that $0 \in \sigma_c(A) \cup \rho(A)$, where

$$\sigma_c(A) = \{\lambda \in \sigma(A) : \lambda I - A \text{ is injective and } R(\lambda I - A) \text{ is dense in } X\},$$

denotes the *continuous spectrum* of A.

[1]Note that the norm of a bounded, densely defined operator B is defined by $\|B\| = \sup_{x\in D(B)} \frac{\|Bx\|}{\|x\|}$.

THEOREM 5.14. *Let* $M \geqslant 1$. *The following assertions are equivalent.*

(i) For every Hilbert space H*:*

$$A \in \mathcal{G}_M(H) \implies \sup_{n\in\mathbb{N}} \|\mathrm{Cay}(A)^n\| < \infty.$$

(ii) For every Hilbert space H*:*

$$A \in \mathcal{G}_M(H) \cap \mathcal{G}_{\exp}(H) \implies \sup_{n\in\mathbb{N}} \|\mathrm{Cay}(A)^n\| < \infty.$$

(iii) For every Hilbert space H*:*

$$A \in \mathcal{G}_{M,-1}(H) \implies \sup_{n\in\mathbb{N}} \|\mathrm{Cay}(A)^n\| < \infty.$$

(iv) For every Hilbert space H*:*

$$A \in \mathcal{G}_M(H) \text{ with } 0 \in \sigma_c(A) \cup \rho(A) \implies A^{-1} \in \mathcal{G}_{bdd}(H).$$

(v) For every Hilbert space H*:*

$$A \in \mathcal{G}_M(H) \cap \mathcal{G}_{\exp}(H) \implies A^{-1} \in \mathcal{G}_{bdd}(H).$$

(vi) For every Hilbert space:

$$A \in \mathcal{G}_M(H) \text{ with } 0 \in \sigma_c(A) \cup \rho(A) \implies A^{-1} \text{ generates } C_0\text{-semigroup}.$$

If "Hilbert space" gets replaced by "Banach space" in the above assertions, then none of them hold.

PROOF. Note that the implications *(i)* ⇒ *(ii)* ⇒ *(iii)* and *(iv)* ⇒ *(v)* are trivial (for both the "Banach space" and the "Hilbert space"-version). The equivalence of *(iv)* and *(vi)* follows by [**Zwa07**, Theorem 2.2], where it was only shown for the "Hilbert space"-version. However, the proof is completely analogous for Banach spaces.

Let us now give an overview of the different proof steps.

For the "Hilbert space"-version, we show the implications *(iii)* ⇒ *(iv)*, *(iv)* ⇒ *(i)* and *(v)* ⇒ *(ii)*, which concludes the proof of the equivalence.

Then, we will observe that for the "Banach space"-version, the chain of implications,

$$\text{(i)} \Rightarrow \text{(ii)} \Rightarrow \text{(iii)} \Rightarrow \text{(iv)} \Rightarrow \text{(v)}, \tag{5.15}$$

still holds. Since, by Example 5.9, we have a counterexample to *(v)*, we conclude that none of the assertions can hold.

(iii) ⇒ *(iv)* : Let $A \in \mathcal{G}_M(H)$, for a Hilbert space H, such that $0 \in \sigma_c(A) \cup \rho(A)$. Thus, A^{-1} exists as a densely defined operator. For $\lambda > 0$, define $A_\lambda = 2\lambda A - I$. A_λ generates a semigroup satisfying

$$\|e^{A_\lambda t}\| \leqslant M e^{-t}, \qquad \lambda, t > 0. \tag{5.16}$$

Thus, $A_\lambda \in \mathcal{G}_{M,-1}(H)$ for all $\lambda > 0$. By the assumption of *(iii)* and Theorem 5.12, there exists a constant $K_{M,-1}$ such that

$$\|\mathcal{C}ay(A_\lambda)^n\| \leqslant K_{M,-1}, \qquad \forall n \in \mathbb{N}, \lambda > 0.$$

Using Lemma 5.11, we get that $\mathcal{C}ay(A_\lambda) = -\lambda\left(\lambda I - A^{-1}\right)^{-1}$. Hence,

$$\|\left(\lambda - A^{-1}\right)^{-n}\| \leqslant \frac{K_{M,-1}}{\lambda^n}, \qquad \forall n \in \mathbb{N}. \tag{5.17}$$

Since, by assumption, A^{-1} is densely defined and, as the inverse of the closed operator A, closed, we conclude by the Hille-Yosida Theorem [**EN00**, Theorem II.3.8] that A^{-1} generates a bounded semigroup. This shows *(iv)* for the "Hilbert space version".

(iv) ⇒ *(i)*: In the following lines, we will see that this implication is a consequence of Theorem 5.13 and Theorem 5.10 *(i)*.
In fact, let H be a Hilbert and $A \in \mathcal{G}_M(H)$ for some $M \geqslant 1$. Clearly, for any $\varepsilon > 0$, $A - \varepsilon I$ is also in $\mathcal{G}_M(H)$ and $0 \in \rho(A - \varepsilon I)$. By the assumption of *(iv)* and Theorem 5.13, it follows that there exists a constant $C_M > 0$ such that $(A - \varepsilon I)^{-1} \in \mathcal{G}_{C_M,0} = \mathcal{G}_{C_M}$ for all $\varepsilon > 0$. To sum up, we have seen that for all $\varepsilon > 0$,

$$(A - \varepsilon I) \in \mathcal{G}_M, 0 \in \rho(A - \varepsilon I) \text{ and } (A - \varepsilon I)^{-1} \in \mathcal{G}_{C_M}.$$

Therefore, by Theorem 5.10 *(i)*, we conclude that $\mathcal{C}ay(A - \varepsilon I)$ is power-bounded and that

$$\|\mathcal{C}ay(A - \varepsilon I)^n\| \leqslant \frac{e}{2}\left(\frac{1}{4} + M^2 + C_M^2\right), \quad \forall n \in \mathbb{N}. \tag{5.18}$$

By Proposition 5.4, $\mathcal{C}ay(A - \varepsilon I) = -I + 2R(1, A - \varepsilon I) = -I + 2R(1 + \varepsilon, A)$, and since the mapping $\lambda \mapsto R(\lambda, A)$ is analytic on $\rho(A)$, it follows that $\mathcal{C}ay(A - \varepsilon I) \to \mathcal{C}ay(A)$ in $\mathcal{B}(H)$ as $\varepsilon \to 0^+$. Hence, by (5.18),

$$\|\mathcal{C}ay(A)^n\| \leqslant \frac{e}{2}\left(\frac{1}{4} + M^2 + C_M^2\right), \quad \forall n \in \mathbb{N}.$$

Therefore, the Cayley transform of A is power-bounded.
Implication *(v)* ⇒ *(ii)* follows directly from Theorem 5.10 *(i)*.

Using the "Banach space"-version of Theorem 5.12, it is easy to see that the proof of *(iii)* ⇒ *(iv)* also holds for the "Banach space"-version of the assertions. Therefore, (5.15) and the proof is finished. □

REMARK 5.15.

(i) An open question is whether for $0 \in \sigma_c(A) \cup \rho(A)$, the implication

$$A, A^{-1} \in \mathcal{G}_{bdd}(X) \implies \mathcal{C}ay(A) \text{ is power-bounded}$$

remains true on general Banach spaces X (the Hilbert space is covered by Theorem 5.10).

(ii) For generators of exponentially stable semigroups on Hilbert spaces, our Theorem 5.14 yields that

$$\big(\forall A \in \mathcal{G}_{\exp} : \ A^{-1} \in \mathcal{G}_{bdd}\big) \iff \left(\forall A \in \mathcal{G}_{\exp} : \ \sup_{n \in \mathbb{N}} \|\mathrm{Cay}(A)^n\| < \infty\right), \qquad (5.19)$$

whereas, Theorem 5.10 yields that for all Hilbert spaces H,

$$\forall A \in \mathcal{G}_{\exp}(H) : (A^{-1} \in \mathcal{G}_{bdd}(H) \iff \sup_{n \in \mathbb{N}} \|\mathrm{Cay}(A)^n\| < \infty). \qquad (5.20)$$

By comparing (5.19) and (5.20), we see that Theorem 5.14 only yields a weaker assertion than the stronger ("pointwise") result in Theorem 5.10.
In fact, the novelty of Theorem 5.14 is rather that it proves the general equivalence of the Cayley Transform and the Inverse Generator Problem for *bounded* semigroups on Hilbert spaces. More importantly, it shows that both problems for *bounded* semigroups can be reduced to the case of exponentially stable semigroups.

It is well-known that for a contraction semigroup on a Hilbert space, the Cayley transform is power-bounded. In fact, by the Lumer-Phillips theorem, it follows by a little exercise that for a semigroup e^{tA} on a Hilbert space with $1 \in \rho(A)$,

$$\|\mathrm{Cay}(A)\| \leqslant 1 \iff \sup_{t \geqslant 0} \|e^{tA}\| \leqslant 1 \overset{\text{Def.}}{\iff} A \in \mathcal{G}_1, \qquad (5.21)$$

where the last equivalence follows by definition. See, e.g., [**Bes12**, Theorem 1.9] for a proof, however, the result is much older. We refer to, e.g., [**SNF70, EZ08, Fac14, Fac15, KW10**] for related results. Clearly, (5.21) implies that the Cayley transform of a contraction semigroup generator on a Hilbert space is power-bounded. Therefore, by Theorem 5.14, the Inverse Generator Problem for contraction semigroups can be solved in the most general setting.

THEOREM 5.16. *If* A *generates a contraction semigroup on a Hilbert space* H*, i.e.* $A \in \mathcal{G}_1(H)$ *and* $0 \in \rho(A) \cup \sigma_c(A)$*, then,* A^{-1} *generates a bounded semigroup. However, there exists a Banach space* X *and* $A \in \mathcal{G}_1(X)$ *with* $0 \in \rho(B)$ *such that* $e^{tA^{-1}} \notin \mathcal{G}_{bdd}$.

PROOF. By (5.21), it follows that $\mathrm{Cay}(A)$ is power-bounded for every $A \in \mathcal{G}_1(H)$ and every Hilbert space H. Therefore, by Theorem 5.14 (Implication *(i)* ⇒ *(iv)*), the first assertion follows.
The second assertion follows by Example 5.9. □

However, we want to emphasize that there is also a very simple, direct proof of Theorem 5.16, for which our main result, Theorem 5.14, is not needed. In fact, by the *Lumer–Phillips* theorem, see **[EN00]**, it holds that A generates a contraction semigroup on a Hilbert space if and only if

(a) the range of $I - A$ equals H, and

(b) $\mathrm{Re}\langle Ax, x\rangle \leqslant 0$ for all $x \in D(A)$.

If A generates a contraction semigroup on H and A^{-1} exists as densely defined operator, then it follows by (b) that $\langle y, A^{-1}y\rangle \leqslant 0$ for all $y \in D(A^{-1}) = R(A)$. Obviously, $(I - A^{-1})y = (A - I)A^{-1}y$ for $y \in D(A^{-1}) = R(A)$. Since A^{-1} maps $D(A^{-1})$ to $D(A)$, we conclude by (a) that the range of $(I - A^{-1})$ equals H. Therefore, A^{-1} generates a contraction semigroup by Lumer–Phillips.
We observe that the assumption of $D(A^{-1})$ being dense was actually not needed in the argument above, and moreover, we even derived that $e^{tA^{-1}}$ generates a contraction semigroup.

5.4. Notes

In the following we want to discuss some known results concerning Problems 5.5 and 5.6.

We start with the already mentioned fact that the answers to both problems are 'yes', if the semigroup T is bounded analytic, which is equivalent to $-A$ being densely defined and sectorial of angle less than $\frac{\pi}{2}$. Note that this property is stronger than the property that T is analytic on some sector Σ_δ (T is not necessarily bounded on Σ_δ) and bounded on $[0, \infty)$.

THEOREM 5.17. *Let* A *generate an analytic semigroup* $e^{\cdot A} : \Sigma_\theta \to \mathcal{B}(X)$ *for some sector* Σ_θ, $\theta \in (0, \pi)$, *and a Banach space* X. *If either*

(a) $z \mapsto e^{zA}$ *is bounded on* Σ_θ, *or*

(b) $t \mapsto e^{tA}$ *is bounded on* $[0, \infty)$ *and* X *is a Hilbert space,*

then the answer to Problem 5.5 is 'yes'. If in addition $0 \in \rho(A) \cup \sigma_c(A)$, *then also Problem 5.6 has an affirmative answer.*

PROOF. If (a) holds, then $-A$ is is densely defined and sectorial of angle $\omega < \frac{\pi}{2}$, see Section 3.1. By Palencia **[Pal93]**, the Cayley transform of a sectorial operator is power-bounded.
If $0 \in \rho(A) \cup \sigma_c(A)$, i.e., the inverse A^{-1} exists as a densely defined operator, then it is well-known that $-A^{-1}$ is also sectorial of angle ω, see, e.g., **[Haa06a]**, and, thus, generates a bounded analytic semigroup.
The assertions for (b) are due to Guo and Zwart **[GZ06]**, see also **[GZ07]**. □

REMARK 5.18. In the proof of Theorem 5.17, we used Palencia's result that the Cayley transform of a sectorial operator is power-bounded. Essentially, we have seen this result in Chapter 4. In fact, by using the results in Section 5.1, it is not hard to see that the Cayley transforms provides a one-to-one correspondence between sectorial operators $-A$ with $0 \in \rho(A)$ and Tadmor-Ritt operators. By Corollary 4.10, Tadmor-Ritt operators are power-bounded. Thus, it follows that $Cay(A)$ is power-bounded for sectorial $-A$ with $0 \in \rho(A)$.
We remark that the proof of Palencia's result for sectorial operators $-A$ with $0 \notin \rho(A)$ can analogously be derived as in Chapter 4, by adapting the arguments to operators S $(= Cay(A))$ of the form

$$\sigma(S) \subset \overline{\mathbb{D}} \text{ and } \quad \|R(z,S)\| \leqslant M(S)(|z+1|^{-1} + |z-1|^{-1}), \quad |z| > 1. \tag{5.22}$$

The spectrum $\sigma(S)$ then lies in the closure of

$$\mathrm{co}\big(\{1\} \cup \{-1\} \cup B_{\sin\theta}(0)\big)$$

for some $\theta \in (0, \frac{\pi}{2})$, see Figures 5.1 and 5.2 (right-hand side).

As we have seen at the end of Section 1.1, certain functional calculus estimates imply that the Cayley transform is power-bounded. For a definition of a bounded H^∞-calculus for semigroup generators, we refer to Section 1.2 and Chapter 2.

THEOREM 5.19. *If $A \in \mathcal{G}_{\exp}$ has a bounded $H^\infty(\mathbb{C}_-)$-calculus, then Problems 5.5 and 5.6 have affirmative answers.*

PROOF. This follows directly from the estimate

$$\|f(A)\| \lesssim \|f\|_{\infty,\mathbb{C}_-}, \quad f \in H^\infty(\mathbb{C}_-),$$

applied to $f_n(z) = \tau(z)^n = \left(\frac{1+z}{1-z}\right)^n$ and $f_t(z) = e^{\frac{t}{z}}$, which both have $\|\cdot\|_{\infty,\mathbb{C}_-}$-norm less or equal to 1, and the fact that $f_t(A) = e^{tA^{-1}}$, where the right-hand side is defined by the power series of the exponential function (this last identity holds since $f_t(A) = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^{-n}$, which follows by the Convergence lemma, Lemma 2.45). □

REMARK 5.20. As mentioned already in Chapter 3, even bounded analytic semigroup generators do not have a bounded H^∞-calculus in general (not even on Hilbert spaces), **[MY90]**. Hence, in the view of Theorem 5.17, it seems that the assumption of a bounded $H^\infty(\mathbb{C}_-)$-calculus in Theorem 5.19 can be weakened.
Obviously, the Cayley Transform Problem for $A \in \mathcal{G}_{bdd}$ is equivalent to the statement

$$\|f(A)\| \lesssim \|f\|_{\infty,\mathbb{C}_-}, \text{ for all } f = \tau^n, n \in \mathbb{N}. \tag{5.23}$$

Therefore, we can see the Cayley Transform Problem as the question whether the $H^\infty(\mathbb{C}_-)$-calculus restricted to the set $\{\tau^n : n \in \mathbb{N}\}$ is *bounded*. Let us investigate this in the following.

We recall that *finite Blaschke products* (for $\mathbb{C}_-$) are functions of the form

$$B(z) = e^{i\phi} \prod_{i=1}^{n} b_{\lambda_i}(z), \quad \text{with } b_\lambda(z) = \frac{\lambda + z}{\overline{\lambda} - z},$$

where $\lambda_i > 0$ and $\phi \in [0, 2\pi)$. We observe that $B \in H^\infty(\mathbb{C}_-)$ with $\|B\|_\infty \leqslant 1$. Blaschke products play an important role in Hardy space theory. We refer to **[Gar07, Dur70, Nik02a]** for more information.

Clearly, τ^n is a finite *Blaschke product* for any n ($\lambda_i = 1$). Therefore, a bounded *Blaschke-product-calculus*, i.e.,

$$\|B(A)\|_{\infty,\mathbb{C}_-} \leqslant C \text{ for a constant } C > 0 \text{ and all Blaschke products } B,$$

implies that the Cayley transform $Cay(A)$ is power-bounded[2].

However, using Schauder multipliers, see Section 3.3.1, we can construct an example of a generator such that the Blaschke-product-calculus is unbounded. In fact, using the notation and theory of Section 3.3.1, one can choose a Schauder basis (ψ_n) of some Banach space X such that there exists a sequence $\mu_n \in \ell^\infty(\mathbb{N}, \mathbb{C})$ with unbounded multiplier $\mathcal{M}_{\mu_n}$. By **[Str88]**, there exists a Blaschke B product such that $B(2^n) = \mu_n$ for all $n \in \mathbb{N}$. Let $A = \mathcal{M}_{2^{-n}}$ w.r.t. (ψ_n). Then, A generates a bounded analytic semigroup, but $B(A) = \mathcal{M}_{\mu_n}$ is unbounded.

Moreover, Kriegler and Weis **[KW10]** pointed out that, by the Convergence lemma (see Theorem 2.45), a bounded Blaschke-product-calculus always implies that the $H^\infty(\mathbb{C}_-)$-calculus is bounded. They actually showed it for generators of bounded analytic semigroups (with dense domain and range), however, the argument is the same for generators of exponentially stable semigroups.

We conclude with Table 1, which gives an overview on the answers for the different sub cases of the Cayley Transform and the Inverse Generator Problem, and a list of some related open questions.

TABLE 1. Answers to the Cayley Transform / Inverse Generator Problem

	Hilbert space	Banach space
A has bounded $H^\infty(\mathbb{C}_-)$-calculus	✓ [Thm.5.19]	✓ [Thm.5.19]
A generates bounded analytic semigroup	✓ [Thm.5.17]	✓[Thm.5.17]
A generates contraction semigroup	✓ [Thm.5.16]	✗ [Ex.5.9]
A generates exponentially stable semigroup	???	✗ [Ex.5.9]
A generates bounded semigroup	???	✗ [Ex.5.9]

[2] A Blaschke product is an example of an *inner function*, which is a function f in $H^\infty(\mathbb{C}_-)$ such that $|f(i\omega)| = 1$ for a.e. $\omega \in \mathbb{R}$. The name of this Ph.D. project was *Semigroups with an Inner Function Calculus*.

Q1: *Is* $\mathrm{Cay}(A)$ *power-bounded, if we assume that* A *and* A^{-1} *are generators of bounded semigroups on a Banach space* X*?*
For Hilbert spaces, the answer is 'yes', see Theorem 5.10.

Q2: *What are suffcient additional assumptions in 5.4, if the answer to 5.4 is 'no'?*

Q3: *What is the answer to Problems 5.5 and 5.6 for Hilbert spaces?*
As we showed in Theorem 5.14, it suffices to consider A's generating exponentially stable semigroups.

Part II

On certain norm estimates for cosine families

CHAPTER 6

Zero-two laws for cosine families

Abstract. We show that for $(C(t))_{t\geqslant 0}$ being a strongly continuous cosine family on a Banach space, the estimate $\limsup_{t\to 0^+}\|C(t)-I\|<2$ implies that $C(t)$ converges to I in the operator norm (Section 6.2). This implication has become known as the *zero-two law*. We further prove that the stronger assumption of $\sup_{t\geqslant 0}\|C(t)-I\|<2$ yields that $C(t)=I$ for all $t\geqslant 0$ (Section 6.3). For discrete cosine families the assumption $\sup_{n\in\mathbb{N}}\|C(n)-I\|\leqslant r<\frac{3}{2}$ yields that $C(n)=I$ for all $n\in\mathbb{N}$. For $r\geqslant\frac{3}{2}$ this assertion does no longer hold. [1]
More general and using different techniques, we show that, for $(C(t))_{t\in\mathbb{R}}$ being a cosine family on a unital Banach algebra, the estimate $\limsup_{t\to\infty^+}\|C(t)-I\|<2$ implies that $C(t)=I$ for all $t\in\mathbb{R}$ (Section 6.4). We also state the corresponding result for discrete cosine families and for semigroups.
In the last part (Section 6.5) we consider scaled versions of above laws. We show that from the estimate $\sup_{t\geqslant 0}\|C(t)-\cos(at)I\|<1$ we can conclude that $C(t)$ equals $\cos(at)I$. Here $(C(t))_{t\geqslant 0}$ is again a strongly continuous cosine family on a Banach space. [2]

6.1. Introduction

Let $(T(t))_{t\geqslant 0}$ denote a strongly continuous semigroup on the Banach space X with infinitesimal generator A. It is well-known that the inequality

$$\limsup_{t\to 0^+}\|T(t)-I\|<1, \tag{6.1}$$

implies that the generator A is a bounded operator, see e.g., [**Sta05**, Remark 3.1.4]. Or equivalently, that the semigroup is uniformly continuous (at 0), i.e.,

$$\limsup_{t\to 0^+}\|T(t)-I\|=0. \tag{6.2}$$

---

[1]Sections 6.1, 6.2 and 6.3 are adapted from the article
F.L. SCHWENNINGER, H. ZWART, *Zero-two law for cosine families, Journal of Evolution Equations, to appear 2015.*
[2]Section 6.5 is adapted from the article
F.L. SCHWENNINGER, H. ZWART, *Less than one implies zero, submitted*, available at arXiv: 1310.6202.

This implication has become known as *zero-one* law for semigroups. Surprisingly, the same law holds for general semigroups on *semi-normed algebras*, i.e., (6.1) implies (6.2), see e.g., [**Est04**]. For a nice overview and related results, we refer the reader to [**CEP15a**].

In this chapter we study similar laws for *cosine families*. Therefore, we first recall the definition of a cosine family. For more information (about strongly continuous cosine families), we refer to [**ABHN11**] and [**Fat69**]. In the following a *normed unital algebra* $\mathcal{A}$ refers to a normed linear space (not equal to $\{0\}$) over $\mathbb{C}$ with a mapping $\bullet : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ such that

- ◇ $\bullet$ is bilinear and associative,
- ◇ there exists a *unity element* $I \in \mathcal{A}$, i.e., $a \bullet I = I \bullet a = a \ \forall a \in \mathcal{A}$ and $\|I\| = 1$,
- ◇ for all $a, b \in \mathcal{A}$, we have that $\|a \bullet b\| \leqslant \|a\| \cdot \|b\|$.

We will always write ab for $a \bullet b$. If $\bullet$ is commutative, $\mathcal{A}$ is called commutative and if the normed space is complete, $\mathcal{A}$ is called a *unital Banach algebra*. We refer to [**Rud91**] for details about normed algebras.

DEFINITION 6.1. Let $\mathcal{A}$ be a unital normed algebra. Then, a family $C = (C(t))_{t\in\mathbb{R}} \subset \mathcal{A}$ is called a *cosine family* in $\mathcal{A}$ if following two conditions hold.

(i) $C(0) = I$, and

(ii) for all $t, s \in \mathbb{R}$, *d'Alembert's functional equation* holds, i.e.,

$$2C(t)C(s) = C(t+s) + C(t-s). \tag{6.3}$$

If $\mathcal{A} = \mathcal{B}(X)$ for some Banach space X and if for all $x \in X$ and all $t \in \mathbb{R}$ we have that

$$\lim_{h\to 0} C(t+h)x = C(t)x,$$

then C is called a *strongly continuous cosine family* on X.

Similar as for strongly continuous semigroups we can define the infinitesimal generator.

DEFINITION 6.2. Let C be a strongly continuous cosine family on the Banach space X, then the *infinitesimal generator* A is defined as

$$Ax = \lim_{t\to 0} \frac{2(C(t)x - x)}{t^2},$$

with its domain consisting of those $x \in X$ for which this limit exists.

It can be shown that the infinitesimal generator of a strongly continuous cosine family is a closed, densely defined operator, see e.g., [**ABHN11**].

One of the main goals of this chapter is to study the *zero-two law* for strongly continuous cosine families on Banach spaces, i.e., whether

$$\limsup_{t\to 0^+} \|C(t)-I\| < 2 \quad \text{implies that} \quad \limsup_{t\to 0^+} \|C(t)-I\| = 0. \tag{6.4}$$

This implication holds if the Banach space is UMD, see [**Fac13**, Corollary 4.2], hence, in particular for Hilbert spaces. On the other hand the $0-3/2$ *law*, i.e.

$$\limsup_{t\to 0^+} \|C(t)-I\| < \frac{3}{2} \quad \text{implies that} \quad \limsup_{t\to 0^+} \|C(t)-I\| = 0,$$

holds for cosine families on general Banach spaces as was proved by W. Arendt in [**Are12**, Theorem 1.1 in Three Line Proofs]. The result even holds without assuming that the cosine family is strongly continuous. In the same work, Arendt poses the question whether the zero-two law holds for cosine families, [**Are12**, Question 1.2 in Three Line Proofs]. In the following theorem we answer this question positively for strongly continuous cosine families. For its proof we refer to Section 6.2.

THEOREM 6.3. *Let* $(C(t))_{t\geqslant 0}$ *be a strongly continuous cosine family on the Banach space* X. *Then*

$$\limsup_{t\to 0^+} \|C(t)-I\| < 2, \tag{6.5}$$

implies that $\lim_{t\to 0^+} \|C(t)-I\| = 0$.

By taking $X = \ell^2$ and

$$C(t) = \begin{pmatrix} \cos(t) & 0 & \cdots & & \\ 0 & \cos(2t) & 0 & \cdots & \\ \vdots & & \ddots & & \end{pmatrix},$$

it is easy to see that this result is optimal.
Very recently, in the preprints [**Cho15a**], [**Est15b**], W. Chojnacki and, independently J. Esterle showed that Theorem 6.3 can be extended to the case of cosine families in unital Banach algebras (in [**Cho15a**] even to normed unital algebras).

The zero-one law for semigroups and the zero-two law for cosine families tells something about the behaviour near $t = 0$. Instead of studying the behaviour around zero, we could study the behaviour on the whole time axis. A result dating back to the sixties is the following. For a semigroup the assumption

$$\sup_{t\geqslant 0} \|T(t)-I\| < 1, \tag{6.6}$$

implies that $T(t) = I$ for all $t \geqslant 0$, see e.g., Wallen [**Wal67**] and Hirschfeld [**Hir68**]. The corresponding result for cosine families, i.e.,

$$\sup_{t\in\mathbb{R}} \|C(t)-I\| < 2 \quad \text{implies that} \quad C(t) = I \tag{6.7}$$

has been been open until now. We prove (6.7) for strongly continuous cosine families on Banach spaces, see Theorem 6.8. This result is strongly motivated by recent work of A. Bobrowski and W. Chojnacki. In [**BC13**, Theorem 4], they showed that for $a \in \mathbb{R}$,

$$\sup_{t \geqslant 0} \|C(t) - \cos(at)I\| < \frac{1}{2}, \tag{6.8}$$

implies that $C(t) = \cos(at)I$ for all $t \geqslant 0$. They used this to conclude that scalar cosine families are isolated points in the (metric) space of bounded strongly continuous cosine families on a fixed Banach space X, with the metric

$$d(C_1, C_2) = \sup_{t \in \mathbb{R}} \|C_1(t) - C_2(t)\|.$$

Hence, (6.7) shows that (6.8) can be improved for $a = 0$. It is easy to see that the number 2 in (6.7) is optimal. Furthermore, we will show that also for $a \neq 0$, the constant $\frac{1}{2}$ in (6.8) can be improved. Precisely, in Theorem 6.22 we prove that for a strongly continuous cosine family $(C(t))_{t \in \mathbb{R}}$, and

$$\sup_{t \in \mathbb{R}} \|C(t) - \cos(at)I\| < r \quad \text{implies that} \quad C(t) = \cos(at)I \tag{6.9}$$

for $r = 1$. Very recently, A. Bobrowski, W. Chojnacki and A. Gregosiewicz [**BCG15**], and independently J. Esterle [**Est15a**] showed that the implication in (6.9) even holds for $r \leqslant \frac{8}{3\sqrt{3}}$ and general cosine families (of elements in a normed unital algebra, or unital Banach algebra respectively). This constant r is optimal, as $\sup_{t \in \mathbb{R}} |\cos(3t) - \cos(t)| = \frac{8}{3\sqrt{3}}$.

The lay-out of this chapter is as follows. In Section 6.2 we prove the zero-two law for strongly continuous cosine families, i.e., Theorem 6.3 is proved.
In Section 6.3, we prove the implication in (6.7). Furthermore, we study the corresponding discrete version, and show that there the constant 2 has to be replaced by $\frac{3}{2}$. Finally, we give an elementary alternative proof for strongly continuous semigroups. The content of Sections 6.2 and 6.3 have, with minor adaptations, been published in the article [**SZ15c**].
In Section 6.4, we consider general cosine families of elements in an unital Banach algebra and generalize (6.7) to

$$\limsup_{t \to \infty} \|C(t) - I\| < 2 \quad \text{implies that} \quad C(t) = I.$$

The used techniques based on a result by J. Esterle [**Est15b**] are different to the ones in Section 6.3. We also show a discrete and a semigroup version.
Finally, in Section 6.5 we give a proof of (6.9) for $r = 1$. The content of this section has, up to minor changes, been submitted for publication in the article [**SZ15b**].

6.2. The zero-two law at the origin

In this section we prove that for a strongly continuous cosine family C on the Banach space X, Theorem 6.3 holds; i.e.,

$$\limsup_{t\to 0^+} \|C(t) - I\| < 2 \quad \text{implies that} \quad \limsup_{t\to 0^+} \|C(t) - I\| = 0.$$

For the proof of Theorem 6.3, the following well-known estimates, which can be found in [**Fat69**, Lemma 5.5 and 5.6], are needed.

LEMMA 6.4. *Let* C *be a strongly continuous cosine family with generator* A. *Then, there exists* $\omega \geqslant 0$ *and* $M \geqslant 1$ *such that*

$$\|C(t)\| \leqslant Me^{\omega t} \qquad \forall t \geqslant 0. \tag{6.10}$$

Furthermore, for $\operatorname{Re}\lambda > \omega$ *we have* $\lambda^2 \in \rho(A)$ *and*

$$\|\lambda^2 R(\lambda^2, A)\| \leqslant M \cdot \frac{|\lambda|}{\operatorname{Re}\lambda - \omega}. \tag{6.11}$$

Hence the above lemma shows that the spectrum of A must lie within the parabola $\{s \in \mathbb{C} \mid s = \lambda^2 \text{ with } \operatorname{Re}\lambda = \omega\}$. To study the spectral properties of the points within this parabola, we use the following lemma.

LEMMA 6.5. *Let* C *be a strongly continuous cosine family on the Banach space* X *and let* A *be its generator. Then, for* $\lambda \in \mathbb{C}$ *and* $s \in \mathbb{R}$ *the following assertions hold.*

(i) $S(\lambda, s)$ *defined by*

$$S(\lambda, s)x = \int_0^s \sinh(\lambda(s-t))C(t)x\, dt, \quad x \in X, \tag{6.12}$$

is a linear and bounded operator on X *and its norm satisfies*

$$\|S(\lambda, s)\| \leqslant \sup_{t\in[0,|s|]} \|C(t)\| \cdot \frac{\sinh(|s| \operatorname{Re}\lambda)}{\operatorname{Re}\lambda}. \tag{6.13}$$

(ii) For $x \in X$ *we have* $S(\lambda, s)x \in D(A)$,

$$(\lambda^2 I - A)S(\lambda, s)x = \lambda(\cosh(\lambda s)I - C(s))x. \tag{6.14}$$

Furthermore, $S(\lambda, s)A \subset AS(\lambda, s)$.

(iii) The bounded operators $S(\lambda, s)$ *and* $C(s)x - \cosh(\lambda s)I$ *commute.*

(iv) If $\lambda \neq 0$ *and* $\cosh(\lambda s) \in \rho(C(s))$, *then* $\lambda^2 \in \rho(A)$ *and*

$$\begin{aligned} \|R(\lambda^2, A)\| &\leqslant \frac{1}{|\lambda|} \cdot \|S(\lambda, s)\| \cdot \|R(\cosh(\lambda s), C(s))\| \\ &\leqslant \sup_{t\in[0,|s|]} \|C(t)\| \cdot \frac{2|s|e^{|s \operatorname{Re}\lambda|}}{|\lambda|} \cdot \|R(\cosh(\lambda s), C(s))\|. \end{aligned} \tag{6.15}$$

PROOF. We begin by showing item *(i)*. Since the cosine family is strongly continuous, the integral in (6.12) is well-defined. Hence $S(\lambda, s)$ is well defined and linear. For the estimate (6.13) we consider

$$\begin{aligned} \|S(\lambda, s)x\| &\leqslant \sup_{t\in[0,|s|]} \|C(t)\| \cdot \|x\| \cdot \int_0^{|s|} |\sinh(\lambda t)|\, dt \\ &= \sup_{t\in[0,|s|]} \|C(t)\| \cdot \|x\| \cdot \frac{1}{2}\int_0^{|s|} |e^{\lambda t} - e^{-\lambda t}|\, dt \\ &\leqslant \sup_{t\in[0,|s|]} \|C(t)\| \cdot \|x\| \cdot \frac{e^{|s|\operatorname{Re}\lambda} - e^{-|s|\operatorname{Re}\lambda}}{2\operatorname{Re}\lambda}. \end{aligned}$$

By definition, the last fraction equals $\frac{\sinh(|s|\operatorname{Re}\lambda)}{\operatorname{Re}\lambda}$, and so the inequality (6.13) is shown.

Item (ii). See [**Nag74**, Lemma 4].

Item (iii). This is clear, since $C(t)$ and $C(s)$ commute for $s, t \in \mathbb{R}$.

Item (iv). We define the bounded operator

$$B = \frac{1}{\lambda} S(\lambda, s) R(\cosh(\lambda s), C(s)).$$

By item *(ii)*, we see that $(\lambda^2 I - A)B = I$. By item *(iii)*, we get that

$$B = \frac{1}{\lambda} R(\cosh(\lambda s), C(s)) S(\lambda, s).$$

Thus, again by *(ii)*., $B(\lambda^2 I - A)x = x$ for $x \in D(A)$. Hence, $\lambda^2 \in \rho(A)$ and the first inequality of (6.15) follows. By using the power series of the exponential function, it is easy to see that $\frac{\sinh(|s|\operatorname{Re}\lambda)}{\operatorname{Re}\lambda} \leqslant 2|s|e^{|s\operatorname{Re}\lambda|}$. Combining this with (6.13) gives the second inequality in (6.15). □

With the use of the above lemma we show that the spectrum of A is contained in the intersection of a ball and a parabola, provided that (6.5) holds, i.e., provided $\limsup_{t\to 0^+} \|C(t) - I\| < 2$.

LEMMA 6.6. *Let C be a strongly continuous cosine family on the Banach space X with generator A. Assume that there exists $c > 0$ such that*

$$\limsup_{t\to 0^+} \|C(t) - I\| < c < 2. \tag{6.16}$$

Then, there exists $M_c, r_c > 0$ and $\phi_c \in (0, \frac{\pi}{2})$ such that

$$\mathcal{R}_c := \left\{\lambda^2 \mid \lambda \in \mathbb{C}, |\lambda| > r_c, |\arg(\lambda)| \in \left(\phi_c, \frac{\pi}{2}\right]\right\} \subset \rho(A), \tag{6.17}$$

and

$$\forall \mu \in \mathcal{R}_c \qquad \|\mu R(\mu, A)\| \leqslant M_c. \tag{6.18}$$

PROOF. First, we note that by (6.16) we have the existence of $t_0 > 0$ such that $\|C(t) - I\| < c$ for all $t \in [0, t_0)$, and by symmetry, for all $t \in (-t_0, t_0)$. Using the assumption, we find that $\frac{1}{2}\|C(t) - I\| < \frac{c}{2} < 1$, and hence $I + \frac{1}{2}(C(t) - I) = \frac{1}{2}(C(t) + I)$ is invertible with $\|(C(t) + I)^{-1}\| < \frac{1}{2-c}$ for all $t \in (-t_0, t_0)$. In other words, $-1 \in \rho(C(t))$. By standard spectral theory, it follows that the open ball centered at -1 with radius $\|R(-1, C(t))\|^{-1}$, i.e., $B_{\|R(-1,C(t))\|^{-1}}(-1)$, is included in $\rho(C(t))$. Therefore,

$$B_{\frac{2-c}{2}}(-1) \subset B_{\frac{1}{2\|R(-1,C(t))\|}}(-1) \subset \rho(C(t)) \quad \forall t \in (-t_0, t_0), \tag{6.19}$$

and by the analyticity of the resolvent, we have for $\mu \in B_{\frac{2-c}{2}}(-1)$ and $t \in (-t_0, t_0)$ that

$$\begin{aligned} \|R(\mu, C(t))\| &= \left\| \sum_{n=0}^{\infty} (\mu + 1)^n R(-1, C(t))^{n+1} \right\| \\ &\leqslant 2\|R(-1, C(t))\| < \frac{2}{2-c}. \end{aligned} \tag{6.20}$$

Since $\cosh(t)$ is entire and $\cosh(i\pi) = -1$, there exists $\tilde{r} > 0$ such that

$$\cosh(B_{\tilde{r}}(i\pi)) \subset B_{\frac{2-c}{2}}(-1). \tag{6.21}$$

Let $\lambda \in \mathbb{C}$ be such that $|\arg(\lambda)| \leqslant \frac{\pi}{2}$. We search for $s \in \mathbb{R}$ such that $\lambda s \in B_{\tilde{r}}(i\pi)$. Let $s_\lambda = \frac{\pi \sin(\arg(\lambda))}{|\lambda|}$ be the unique element on the line $\{\lambda s : s \in \mathbb{R}\}$ which is closest to $i\pi$. We have that $|i\pi - \lambda s_\lambda| = \pi \cos(\arg(\lambda))$. Now, choose $\phi_c \in (0, \frac{\pi}{2})$ large enough such that $\pi \cos(\phi_c) < \tilde{r}$ and choose $r_c > 0$ such that $\frac{\pi}{r_c} < t_0$. Then, for all $\lambda^2 \in \mathcal{R}_c$, we have that $\lambda s_\lambda \in B_{\tilde{r}}(i\pi)$ with $s_\lambda \in (-t_0, t_0)$. By (6.21), $\cosh(\lambda s_\lambda) \in B_{\frac{2-c}{2}}(-1)$. Thus,

$$\cosh(\lambda s_\lambda) \in \rho(C(s_\lambda)), \quad \text{and} \quad \|R(\cosh(\lambda s_\lambda), C(s_\lambda))\| \leqslant \frac{2}{2-c}, \tag{6.22}$$

by (6.19) and (6.20). Therefore, Lemma 6.5 *(iv)* implies that $\lambda^2 \in \rho(A)$ and

$$\begin{aligned} \|R(\lambda^2, A)\| &\leqslant \sup_{t \in [0, |s_\lambda|]} \|C(t)\| \cdot \frac{2|s_\lambda| e^{|s_\lambda \operatorname{Re} \lambda|}}{|\lambda|} \cdot \|R(\cosh(\lambda s), C(s_\lambda))\| \\ &\leqslant \sup_{t \in [0, t_0]} \|C(t)\| \cdot \frac{2\pi e^{\pi}}{|\lambda|^2} \cdot \frac{2}{2-c} \leqslant \frac{M_c}{|\lambda|^2} \end{aligned}$$

for some M_c only depending on $\sup_{t \in [0,t_0]} \|C(t)\|$ and c. $\square$

Combining the results from Lemmas 6.4 and 6.6 enables us to prove Theorem 6.3. As for semigroups we can prove a slightly more general result.

THEOREM 6.7 (Zero-two law for cosine families). *Let* C *be a strongly continuous cosine family on the Banach space* X. *Denote by* A *its infinitesimal generator. Then the following assertions are equivalent*

(i) $\limsup_{t\to 0^+} \|C(t) - I\| < 2$;

(ii) $\limsup_{t\to 0^+} \|C(t) - I\| = 0$;

(iii) A *is a bounded operator.*

PROOF. Trivially, the second item implies the first one. If the assertion in item *(iii)* holds, then the corresponding cosine family is given by

$$C(t) = \sum_{n=0}^{\infty} A^n \frac{(-1)^n t^{2n}}{(2n)!}.$$

From this, the property in item *(ii)* is easy to show. Hence it remains to show that item *(i)* implies item *(iii)*.

Let c be the constant from equation (6.16), and let $r_c > 0, \phi_c \in [0, \frac{\pi}{2})$ be the constants from Lemma 6.6. By Lemma 6.4, we have that there exists $\omega' > \omega \geqslant 0$ such that

$$\sup_{\lambda \in R_{\omega'} \cap S_{\phi_c}} \|\lambda^2 R(\lambda^2, A)\| < \infty, \tag{6.23}$$

where $R_{\omega'} = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geqslant \omega'\}$ and $S_{\phi_c} = \{\mu \in \mathbb{C} : |\arg \mu| \leqslant \phi_c\}$. Now, let λ such that $|\lambda| > r_c$ and $|\arg(\lambda)| \in (\phi_c, \frac{\pi}{2}]$. Thus $\lambda^2 \in \mathcal{R}_c$, see (6.17), and so by Lemma 6.6,

$$\sup_{\lambda^2 \in \mathcal{R}_c} \|\lambda^2 R(\lambda^2, A)\| < \infty. \tag{6.24}$$

Let $f(z) = z^2$. It is easy to see that the closure of $\mathbb{C}\backslash(\mathcal{R}_c \cup f(R_{\omega'} \cap S_{\phi_c}))$ is compact. Thus, (6.23) and (6.24) yield that there exists an $R > 0$ such that the spectrum $\sigma(A)$ lies within the open ball $B_R(0)$ and

$$\sup_{|\mu| > R} \|\mu R(\mu, A)\| < \infty. \tag{6.25}$$

Hence we have that $\mu \mapsto R(\mu, A)$ has a removable singularity at ∞. Since A is closed, this implies that A is a bounded operator, [**Kat95**, Theorem I.6.13], and therefore item *(iii)* is shown. □

6.3. Similar laws on $\mathbb{R}$ and $\mathbb{N}$

In the previous section we showed that uniform estimates in a neighbourhood of zero imply additional properties. In this section we study estimates which hold on $\mathbb{R}$, $(0, \infty)$, $\mathbb{Z}$, or $\mathbb{N}$. For $\mathbb{R}$ and $(0, \infty)$ we show that by applying a scaling trick, the results can be obtained from the already proved laws. The main theorem of this section is the following.

THEOREM 6.8. *The following assertions hold*

(i) For a semigroup T *we have that (6.6) implies that* $T(t) = I$ *for all* $t \geqslant 0$.

(ii) If the strongly continuous cosine family C *on the Banach space* X *satisfies*

$$\sup_{t \geqslant 0} \|C(t) - I\| = r < 2 \tag{6.26}$$

then $C(t) = I$ *for all* $t \in \mathbb{R}$.

PROOF. Since the proof of the two items is very similar, we concentrate on the second one.

For the Banach space X we define $\ell^2(\mathbb{N}; X)$ as

$$\ell^2(\mathbb{N}; X) = \{(x_n)_{n \in \mathbb{N}} \mid x_n \in X, \sum_{n \in \mathbb{N}} \|x_n\|^2 < \infty\}. \tag{6.27}$$

With the norm

$$\|(x_n)\| = \sqrt{\sum_{n \in \mathbb{N}} \|x_n\|^2},$$

this is a Banach space. On this extended Banach space we define $C_{ext}(t)$, $t \in \mathbb{R}$ as

$$C_{ext}(t)(x_n) = (C(nt)x_n). \tag{6.28}$$

Hence it is a diagonal operator with scaled versions of C on the diagonal. By a standard argument and (6.26) it follows that the cosine family C_{ext} is strongly continuous. Now we estimate the distance from this cosine family to the identity on $\ell^2(\mathbb{N}; X)$ for $t \in (0, 1]$.

$$\begin{aligned} \|C_{ext}(t) - I\|^2 &= \sup_{\|(x_n)\|=1} \|C_{ext}(t)(x_n) - (x_n)\|^2 \\ &= \sup_{\|(x_n)\|=1} \sum_{n \in \mathbb{N}} \|C(nt)x_n - x_n\|^2 \\ &\leqslant \sup_{\|(x_n)\|=1} \sum_{n \in \mathbb{N}} r^2 \|x_n\|^2 = r^2, \end{aligned}$$

where we have used (6.26). In particular, this implies that

$$\limsup_{t \to 0^+} \|C_{ext}(t) - I\| < 2.$$

By Theorem 6.7, we conclude that the infinitesimal generator of C_{ext} is bounded. Since $C_{ext}(t)$ is a diagonal operator, it is easy to see that its infinitesimal generator A_{ext} is diagonal as well. Furthermore, the n'th diagonal element equals nA. Since n runs to infinity, A_{ext} can only be bounded if $A = 0$. This immediately implies that $C(t) = I$ for all $t \in \mathbb{R}$. □

From the above proof it is clear that if Theorem 6.7 would hold for non-strongly continuous cosine families, then the strong continuity assumption can be removed from item 2 in the above theorem as well.

We emphasize that for semigroups no continuity assumption was needed. As mentioned in the introduction, this can also be proved using operator algebraic result going back to Wallen **[Wal67]**. In Subsection 6.3.2, we present an (also simple) alternative proof. However, first we study the analog of Theorem 6.8 for discrete cosine families.

6.3.1. Discrete cosine families.

DEFINITION 6.9 (Discrete cosine family). Let $\mathcal{A}$ be a unital normed algebra (with unity I). A family $C = (C(n))_{n\in\mathbb{Z}} \subset \mathcal{A}$ is called a *discrete cosine family (or cosine sequence) in* $\mathcal{A}$ when $C(0) = I$ and (6.3) holds for all $t, s \in \mathbb{Z}$.

THEOREM 6.10. *If a discrete cosine family* C *in a unital normed algebra* $\mathcal{A}$ *satisfies*

$$\sup_{n\in\mathbb{N}} \|C(n) - I\| = r < \frac{3}{2}, \tag{6.29}$$

then $C(n) = I$ *for all* n.
Furthermore, there exists a discrete cosine family such that $C(n) \neq I$ *for all* $n \in \mathbb{N}$ *and*

$$\sup_{n\in\mathbb{N}} \|C(n) - I\| = \frac{3}{2}.$$

PROOF. We closely follow the proof in **[Are12]**. Using equation (6.3) we find for $n \in \mathbb{Z}$ that

$$2\,(C(n) - I)^2 = C(2n) - I - 4(C(n) - I).$$

Hence

$$4(C(n) - I) = C(2n) - I - 2\,(C(n) - I)^2\,.$$

Taking norms, we find

$$4\|C(n) - I\| \leqslant \|C(2n) - I\| + 2\|C(n) - I\|^2. \tag{6.30}$$

Let $L := \sup_{n\in\mathbb{N}} \|C(n) - I\|$, then (6.30) implies that

$$4L \leqslant L + 2L^2$$

In other words, $L = 0$ or $L \geqslant 3/2$. By assumption, the latter does not hold, and therefore, $L = 0$, or equivalently $C(n) = I, n \geqslant 0$. This proves the first part of the theorem. To show that the constant $3/2$ is sharp, we consider the following scalar discrete cosine family on $X = \mathbb{C}$,

$$C(n) = \cos\left(\frac{2\pi}{3}n\right), \qquad n \in \mathbb{Z}.$$

It is easy to see that this family only takes the values 1 and $-\frac{1}{2}$, and thus

$$\sup_{n\in\mathbb{N}} \|C(n) - I\| = \sup_{n\in\mathbb{N}} \left|\cos\left(\frac{2\pi}{3}n\right) - 1\right| = \frac{3}{2}. \tag{6.31}$$

Hence we conclude that $\frac{3}{2}$ is the best possible constant in (6.29). □

6.3.2. An elementary proof for semigroups. We now give an elementary proof of the following known result.

THEOREM 6.11. *Let* T *be a strongly continuous semigroup on the Banach space* X*, and let* A *denote its infinitesimal generator. If*

$$r := \sup_{t\geqslant 0} \|T(t) - I\| < 1, \tag{6.32}$$

then $T(t) = I$ *for all* $t \geqslant 0$.

PROOF. In general it holds that

$$T(t)x - x = A\int_0^t T(s)x\,ds, \qquad t > 0, x \in X. \tag{6.33}$$

For $t > 0$ let B_t denote the bounded operator $x \mapsto B_t x := \int_0^t T(s)x ds$. For $x \in X$,

$$\|x - t^{-1}B_t x\| = \frac{1}{t}\left\|\int_0^t x - T(s)x\,ds\right\| \leqslant \frac{1}{t}\int_0^t \|x - T(s)x\|ds \leqslant r\|x\|.$$

Thus, since $r < 1$, it follows that $t^{-1}B_t$ is boundedly invertible for all $t > 0$ and

$$\|tB_t^{-1}\| \leqslant \frac{1}{1-r} \Leftrightarrow \|B_t^{-1}\| \leqslant \frac{1}{t(1-r)}. \tag{6.34}$$

By (6.33) and (6.32), we have that $\|AB_t\| \leqslant 1$. Thus,

$$\|A\| \leqslant \|B_t^{-1}\| \overset{(6.34)}{\leqslant} \frac{1}{t(1-r)} \qquad \forall t > 0, \tag{6.35}$$

hence, $A = 0$ which concludes the proof. □

6.4. The zero-two law at ∞

In Theorems 6.7 and 6.8 we have proved laws of the form

$$\limsup_{t\to 0} \|C(t) - I\| < 2 \implies \lim_{t\to 0} \|C(t) - I\| = 0, \tag{limsup-law}$$

$$\sup_{t\in\mathbb{R}} \|C(t) - I\| < 2 \implies C(t) = I\ \forall t \in \mathbb{R}, \tag{sup-law}$$

where $(C(t))_{t\in\mathbb{R}}$ is a strongly continuous cosine family on a Banach space X. Following these results published in **[SZ15c]**, Chojnacki **[Cho15a]** and Esterle **[Est15b]** have generalized them to the situation of a cosine family of elements in a general unital Banach algebra (In **[Cho15a]** even normed unital algebras are considered).

In this section we consider cosine families in a unital Banach algebra $\mathcal{A}$ (with unity element I) satisfying

$$\limsup_{t\to\infty} \|C(t) - I\| < 2. \tag{6.36}$$

Clearly, this condition is weaker than the premise in the *sup-law*, however, we show that it still allows for the same conclusion. In fact, in Theorem 6.16 we prove that

$$\limsup_{t\to\infty} \|C(t) - I\| < 2 \implies C(t) = I\ \forall t \in \mathbb{R}. \qquad \text{(limsup-}\infty\text{-law)}$$

The proof (of (*limsup-∞-law*)) uses techniques by J. Esterle developed recently in **[Est15b]**. Finally we state the corresponding result for semigroups.

6.4.1. A $\limsup_{t\to\infty}$- law. In the following, for a normed unital algebra $\mathcal{A}$, let I always denote the unity element. Let us recall the notions *spectrum* and *resolvent* for elements in $\mathcal{A}$. An element $a \in \mathcal{A}$ is called *invertible* if there exists a $b \in \mathcal{A}$ such that $ab = ba = I$. Furthermore,

$$\rho(a) = \{\lambda \in \mathbb{C} : (\lambda I - a) \text{ is invertible}\}, \quad \sigma(a) = \mathbb{C} \setminus \rho(a).$$

LEMMA 6.12. *Let $(C(t))_{t\in\mathbb{R}}$ be a cosine family in a unital Banach algebra. If*

$$\limsup_{t\to\infty} \|C(t) - I\| = 0,$$

then $C(t) = I$ for all $t \in \mathbb{R}$.

PROOF. From the assumption follows that $\lim_{t\to\infty} C(t) = I$. By d'Alembert's defining identity for cosine families,

$$C(t+s) + C(t-s) = 2C(t)C(s), \tag{6.37}$$

for all $s, t \in \mathbb{R}$. Thus, letting $t \to \infty$, we derive $2I = 2C(s)$ for all $s \in \mathbb{R}$. □

The following lemma is a slight extension of Esterle's Lemma 2.1 in **[Est15b]**, as we also allow for $t_0 = \infty$. The proof is analogous the case $t_0 = 0$.

LEMMA 6.13. *Let* $(c(t))_{t\in\mathbb{R}}$ *be a complex-valued cosine family and* $t_0 \in \{0,\infty\}$. *Then, we have one of the following situations.*

(i) $\limsup_{t\to t_0} |c(t)-1| = \infty$,

(ii) $\limsup_{t\to t_0} |c(t)-1| = 2$,

(iii) $\limsup_{t\to t_0} |c(t)-1| = 0$.

Moreover, in case (iii), it follows that

$$c(t) = \begin{cases} 1 & \text{if} \quad t_0 = \infty, \\ \cos(\alpha t) & \text{if} \quad t_0 = 0, \end{cases} \tag{6.38}$$

for some $\alpha \geqslant 0$.

PROOF. As mentioned the proof is analogous to the one in [**Est15b**, Lemma 2.1]. In case (iii) and $t_0 = \infty$, it follows by Lemma 6.12 that $c(t) = 1$ for all $t \in \mathbb{R}$. ☐

For a unital Banach algebra $\mathcal{A}$, which in additional is *commutative*, i.e., $ab = ba$ for $a, b \in \mathcal{A}$, define the *characters* (or *complex homomorphisms*)

$$\Delta_{\mathcal{A}} = \{\chi : \mathcal{A} \to \mathbb{C}, \chi \text{ is linear and multiplicative}, \chi \neq 0\}, \tag{6.39}$$

where $\chi \neq 0$ means that χ is not the zero functional. It is easy to see that $\chi(I) = 1$ for every $\chi \in \Delta_{\mathcal{A}}$. Moreover, one can show that $|\chi(a)| \leqslant \|a\|$ for $a \in \mathcal{A}$ and $\chi \in \Delta_{\mathcal{A}}$. We equip $\Delta_{\mathcal{A}}$ with the initial topology induced by the set of mappings $\{\hat{a} : a \in \mathcal{A}\}$, where

$$\hat{a} : \Delta_{\mathcal{A}} \to \mathbb{C}, \chi \mapsto \chi(a). \tag{6.40}$$

The mapping $a \mapsto \hat{a}$ is the well-known *Gelfand transform* and therefore, the topology on $\Delta_{\mathcal{A}}$ is called *Gelfand topology*. It is well known that $\Delta_{\mathcal{A}}$ is compact and that the Gelfand transform is a continuous algebra homomorphism from $\mathcal{A}$ to $C(\Delta_{\mathcal{A}})$, the space of continuous functions on $\Delta_{\mathcal{A}}$ (equipped with the supremum norm). Furthermore, for the *spectral radius* $r(a) = \sup_{\lambda\in\sigma(a)} |\lambda|$ of an element $a \in \mathcal{A}$ we have that

$$\forall a \in \mathcal{A} : \quad r(a) = \|\hat{a}\|_\infty = \sup_{\chi\in\Delta_{\mathcal{A}}} |\chi(a)|. \tag{6.41}$$

For detailed information about Banach algebras, the Gelfand transform and the space of characters we refer to e.g., [**Rud91**, Chapter 11].

For a subset S of a (not necessarily commutative) Banach algebra $\mathcal{A}$, let $\overline{\text{alg}}(S)$ denote the closure of the smallest subalgebra of $\mathcal{A}$ containing S, which is called the *Banach algebra generated by* S. It can be shown that if S is commutative, i.e., $ab = ba$ for $a, b \in S$, then $\overline{\text{alg}}(S)$ is commutative. Hence, $\overline{\text{alg}(S)}$ is a commutative Banach algebra.

PROPOSITION 6.14. *Let* $(C(t))_{t\in\mathbb{R}}$ *be a cosine family in the unital Banach algebra. If* $\limsup_{t\to\infty} r(C(t)-I) < 2$, *then* $r(C(t)-I) = 0$ *for all* $t \in \mathbb{R}$.

PROOF. W.l.o.g. we can assume that the considered Banach algebra $\mathcal{A}$ is commutative, otherwise consider the Banach algebra generated by the cosine family, $\mathcal{A} = \overline{\text{alg}}\left((C(t))_{t\in\mathbb{R}}\right)$, which is commutative. Let $\Delta_{\mathcal{A}}$ denote the characters on $\mathcal{A}$, see (6.39). By (6.41), we have that for all $t \in \mathbb{R}$,

$$r(C(t)-I) = \sup_{\chi\in\Delta_{\mathcal{A}}} |\chi(C(t)-I)| = \sup_{\chi\in\Delta_{\mathcal{A}}} |\chi(C(t))-1|. \tag{6.42}$$

Thus, by the assumption we get that $\limsup_{t\to\infty} |\chi(C(t))-1| < 2$ for all $\chi \in \Delta_{\mathcal{A}}$. Since C is a cosine family and $\chi \in \Delta_{\mathcal{A}}$ a linear, multiplicative functional, it follows directly that $(\chi(C(t)))_{t\in\mathbb{R}}$ is a complex-valued cosine family. Now Lemma 6.13 implies that $\chi(C(t)) = 1$ for all $t \in \mathbb{R}$ and $\chi \in \hat{A}$. Using this in (6.42), we deduce that $r(C(t)-I) = 0$ for all $t \in \mathbb{R}$. □

As pointed by Esterle [**Est15b**], if $\mathcal{A}$ is commutative, the following *square root* can be defined. For $a \in \mathcal{A}$ with $\|a\| \leqslant 1$ set

$$\sqrt{I-a} := \sum_{n=0}^{\infty} (-1)^n \alpha_n a^n, \tag{6.43}$$

where $\alpha_0 = 1$, $\alpha_n = \frac{1}{n!}\frac{1}{2}(\frac{1}{2}-1)...(\frac{1}{2}-n+1) = (-1)^{n-1}\frac{1}{n2^{n-1}}\binom{2(n-1)}{n-1}$, $n > 0$. We remark $(-1)^n\alpha_n$ are the Taylor coefficients of the function $z \to \sqrt{1-z}$ at the origin (with convergence radius equal to 1). Then $\left(\sqrt{I-a}\right)^2 = I-a$ and since $(-1)^{n-1}\alpha_n > 0$ for $n \geqslant 1$,

$$\left\|I-\sqrt{I-a}\right\| \leqslant \sum_{n=1}^{\infty} |\alpha_n|\|a\|^n = \sum_{n=1}^{\infty} (-1)^{n-1}\alpha_n\|a\|^n = 1-\sqrt{1-\|a\|}. \tag{6.44}$$

For details see [**Est15b**]. Using (6.43) applied to the Banach algebra generated by the cosine family, the following result can be proved.

LEMMA 6.15 (Esterle, [**Est15b**]). *Let* $(C(t))_{t\in\mathbb{R}}$ *be a cosine family in a unital Banach algebra and* $s \in \mathbb{R}$. *If* $\|C(2s)-I\| \leqslant 2$ *and* $r(C(s)-I) < 1$, *then,*

$$C(s) = \sqrt{I-\frac{I-C(2s)}{2}},$$

where the square root is defined as described above.

With the above preparatory results, the *limsup-∞-law* is now easy to show. The proof is analogous to the one in [**Est15b**, Theorem 3.2], which in turn can be seen as an elegant refinement of the technique used in the *three-lines-proof* in [**Are12**].

THEOREM 6.16. *Let* $(C(t))_{t\in\mathbb{R}}$ *be a cosine family in a unital Banach algebra. Then,* $\limsup_{t\to\infty} \|C(t) - I\| < 2$ *implies that* $C(t) = I$ *for all* $t \in \mathbb{R}$.

PROOF. By Proposition 6.14, we have that $r(C(t)-I) = 0$ for $t \in \mathbb{R}$. Furthermore, there exists s_0 such that $\|C(s)-I\| < 2$ for $s > s_0$. Thus, we can apply Lemma 6.15 and Eq. (6.44) so that for all $s > s_0$,

$$\|I - C(s)\| \leqslant 1 - \sqrt{1 - \left\|\frac{I - C(2s)}{2}\right\|} \leqslant 1.$$

With $S := \limsup_{s\to\infty} \|C(s) - I\|$, this yields that

$$S \leqslant 1 - \sqrt{1 - \frac{S}{2}} \leqslant 1. \tag{6.45}$$

Therefore, $1 - \frac{S}{2} \leqslant (1 - S)^2$ and hence, $\frac{3}{2}S \leqslant S^2$. Thus, either $S = 0$ or $S \geqslant \frac{3}{2}$. Since $S \leqslant 1$ by (6.45), this implies that $S = 0$. Finally, Lemma 6.12 yields the assertion. □

REMARK 6.17. After discussing Theorem 6.16 with J. Esterle, he pointed out that the following alternative in the proof can be used. Instead of applying Lemma 6.15, we could use Theorem 2.3 in **[Est15a]** which asserts that

$$r(C(t) - I) = 0\ \forall t \in \mathbb{R} \quad \Longrightarrow \quad C(t) = I\ \forall t \in \mathbb{R}, \tag{6.46}$$

for a bounded cosine family C.

REMARK 6.18. It is clear that Theorem 6.16 generalizes the *sup-law*. We remark that the known proofs of the *sup-law*, see Theorem 6.8 and **[Cho15a]**, which use a diagonalization argument and the *limsup-law*, cannot be generalized to the assertion of Theorem 6.16.

6.4.2. A discrete lim sup**-law.** For discrete cosine families, or *cosine sequences* $(C(n))_{n\in\mathbb{Z}}$, the following is a slight generalization of Theorem 6.10.

THEOREM 6.19. *Let* $(C(n))_{n\in\mathbb{Z}}$ *be a discrete cosine family in a unital Banach algebra. Then,*

$$\limsup_{n\to\infty} \|C(n) - I\| < \frac{3}{2} \quad \Longrightarrow \quad C(n) = I\ \forall n \in \mathbb{Z}.$$

There exists a discrete cosine family such that $C(n) \neq I$ *for all* $n \in \mathbb{N}$ *and*

$$\limsup_{n\to\infty} \|C(n) - I\| = \frac{3}{2}.$$

PROOF. The proof is completely analogous to the one for Theorem 6.10, with $L := \limsup_{n\to\infty} \|C(n) - I\|$ after (6.30). □

6.4.3. The corresponding semigroup result. Let us finally state the corresponding result for (discrete) semigroups in a normed unital algebra. This is a corollary of a well-known result by Wallen [**Wal67**].

THEOREM 6.20. *Let* $(T_n)_{n\in\mathbb{N}}$ *be a semigroup in a normed unital algebra. Then,*

$$\limsup_{n\to\infty} \|T_n - I\| < 1 \implies T_n = I \ \forall n \in \mathbb{N}. \tag{6.47}$$

PROOF. If $\limsup_{n\to\infty} \|T_n - I\| < 1$, then $\liminf_{n\in\mathbb{N}} \frac{1}{n}\sum_{j=1}^{n} \|T_j - I\| < 1$. By Wallen [**Wal67**], the assertion follows. ☐

REMARK 6.21. Clearly, Theorem 6.20 implies that for a semigroup T on $[0,\infty)$, we have that

$$\limsup_{t\to\infty} \|T(t) - I\| < 1 \implies T(t) = I \ \forall t \geqslant 0. \tag{6.48}$$

6.5. Less than one implies zero

6.5.1. Scaled zero-r laws. Turning again to the semigroup case, it is easy to see that the *zero-one law*

$$\sup_{t\geqslant 0} \|T(t) - I\| < 1 \implies T(t) = I \ \forall t \geqslant 0,$$

implies the scaled version

$$\sup_{t\geqslant 0} \|T(t) - e^{\lambda t} I\| < 1 \implies T(t) = e^{\lambda t} I \ \forall t \geqslant 0, \tag{6.49}$$

for $\operatorname{Re}\lambda \geqslant 0$. Note that (6.49) is not true for $\operatorname{Re}\lambda < 0$, as can be seen by the example $T(t) = e^{2\lambda t}$. In the following we investigate a similar question for cosine families $(C(t))_{t\geqslant 0}$. As mentioned in Section 6.1, Bobrowski and Chojnacki showed in [**BC13**, Theorem 4] that

$$\sup_{t\geqslant 0} \|C(t) - \cos(at)I\| < \frac{1}{2}, \tag{6.50}$$

implies $C(t) = \cos(at)I$ for all $t \geqslant 0$. The purpose of this section is to extend Bobrowski and Chojnacki's result by showing that the number $\frac{1}{2}$ in (6.50) may be replaced by 1. More precisely, we prove the following.

THEOREM 6.22. *Let* $(C(t))_{t\geqslant 0}$ *be a strongly continuous cosine family on the Banach space* X *and let* $a \geqslant 0$. *If the following inequality holds for* $r = 1$,

$$\sup_{t\geqslant 0} \|C(t) - \cos(at)I\| < r, \tag{6.51}$$

then $C(t) = \cos(at)I$.

Clearly, the case $a = 0$ in Theorem 6.22 is not interesting, since then $r \leqslant 1$ can be weakend to $r \leqslant 2$ as we have seen in Theorems 6.8 and 6.16.
After the first draft[3] of Theorem 6.22 had appeared, Chojnacki **[Cho15b]** generalized the result to the situation of cosine families on normed algebras indexed by general abelian groups. Later, Bobrowski, Chojnacki and Gregosiewicz **[BCG15]** and, independently, Esterle **[Est15a]** extended Theorem 6.22 to $r < \frac{8}{3\sqrt{3}} \approx 1.54$. This is optimal as can be seen by choosing $C(t) = \cos(3at)I$ and the fact that $\sup_{t\geqslant 0} |\cos(3t) - \cos(t)| = \frac{8}{3\sqrt{3}}$. Again, their results do not require the strong continuity assumption and hold for cosine families on general normed algebras with a unity element.

The outline of this section is as follows. First we show some technical lemmata we need later, see Section 6.5.2.
In Section 6.5.3 we prove Theorem 6.22 for $a \neq 0$ using elementary techniques, which seem to be less involved than the technique used in **[BC13]** (which lead to the worse constant $r = \frac{1}{2}$, see (6.50)).

6.5.2. Some technical lemmata.

LEMMA 6.23. *If* $a, b \geqslant 0$ *and* $a \neq b$, *then* $\sup_{t\geqslant 0} |\cos(at) - \cos(bt)| > 1$.

PROOF. If $a = 0$, the assertion is clear as $\cos(\pi) = -1$. Hence, let $a, b > 0$. By scaling, it suffices to prove that

$$\forall a \in (0, 1)\ \exists s \geqslant 0 : \quad |\cos(as) - \cos(s)| > 1.$$

Since $\cos(2k\pi) = 1$ for $k \in \mathbb{Z}$ and $\cos(as) < 0$ for $t \in \frac{\pi}{a}(\frac{1}{2} + 2m, \frac{3}{2} + 2m)$, $m \in \mathbb{Z}$, we are done if we find $(k, m) \in \mathbb{Z} \times \mathbb{Z}$ such that

$$k \in \frac{1}{a}\left(\frac{1}{4} + m, \frac{3}{4} + m\right).$$

This is equivalent to $ka - m \in (\frac{1}{4}, \frac{3}{4})$. It is easy to check that for $a \in (2^{-n-1}, 2^{-n}] \cup [1 - 2^{-n-1}, 1 - 2^{-n})$ we can choose $k = 2^{n-1}$ and $m = \lfloor ka \rfloor$. ☐

The following lemma gives the Fourier series of odd powers of the cosine. We omit the proof as it can be checked by the reader easily.

LEMMA 6.24. *Let* $n \in \mathbb{N}$. *Then, for all* $t \in \mathbb{R}$,

$$\cos(t)^{2n+1} = \sum_{k=0}^{n} a_{2k+1,2n+1} \cos\left((2k + 1)t\right),$$

where $a_{2k+1,2n+1} = 2^{-2n}\binom{2n+1}{n-k}$.

[3] F. Schwenninger, H. Zwart, *Less than one implies zero*, `http://arxiv.org/abs/1310.6202v1.pdf`.

LEMMA 6.25. *For any* $n \in \mathbb{N}$ *and* $a_{1,2n+1}$ *chosen as in Lemma 6.24 holds that*

- $b_n := \lim_{q\to 0^+} q \cdot \int_0^\infty e^{-qt}|\cos(t)^n|\,dt$ *exists and* $b_n \geqslant b_{n+1}$,
- $a_{1,2n+1} = 2b_{2n+2}$,
- $\lim_{n\to\infty} \frac{a_{1,2n+1}}{2b_{2n+1}} = 1$.

PROOF. Because $t \mapsto |\cos(t)^n|$ is π-periodic,

$$q\int_0^\infty e^{-qt}|\cos(t)^n|\,dt = \frac{q\int_0^\pi e^{-qt}|\cos(t)^n|dt}{1-e^{-q\pi}},$$

which goes to $\frac{1}{\pi}\int_0^\pi |\cos(t)^n|dt$ as $q \to 0^+$. Furthermore,

$$2_{b_{2n+2}} = \frac{2}{\pi}\int_0^\pi |\cos(t)^{2n+2}|dt = \frac{1}{\pi}\int_0^{2\pi} \cos(t)^{2n+1}\cos(t)dt$$

equals $a_{1,2n+1}$ by the Fourier series of $\cos(t)^{2n+1}$, see Lemma 6.24.
By the same lemma we have that for $n \geqslant 1$

$$\frac{a_{1,2n-1}}{a_{1,2n+1}} = \frac{2^{-2n+2}\binom{2n-1}{n}}{2^{-2n}\binom{2n+1}{n}} = \frac{(2n+1)2n}{4(n+1)n},$$

which goes to 1 as $n \to \infty$. This implies that $\frac{a_{1,2n+1}}{2b_{2n+1}}$ goes to 1 as

$$a_{1,2n+1} = 2b_{2n+2} \leqslant 2b_{2n+1} \leqslant 2b_{2n} = a_{1,2n-1}, \quad n \in \mathbb{N}.$$

□

6.5.3. Proof of Theorem 6.22. In the following, let $(C(t))_{t\geqslant 0}$ always be a strongly continuous cosine family on the Banach space X with infinitesimal generator of A which has domain $D(A)$.

Assume that for some $r > 0$,

$$\sup_{t\geqslant 0} \|C(t) - \cos(at)I\| = r. \tag{6.52}$$

The following lemma shows that if (6.52) holds, the spectrum of A is a singleton. This will be essential in proving Theorem 6.22.

LEMMA 6.26. *If* $(C(t))_{t\in\mathbb{R}}$ *is a strongly continuous cosine family such that (6.52) holds for* $r < 1$ *and* $a \geqslant 0$*, then the spectrum of the generator* A *satisfies* $\sigma(A) \subseteq \{-a^2\}$.

PROOF. The case $r = 0$ is trivial, thus let $r > 0$. From (6.52) it follows in particular that the cosine family $(C(t))_{t\geqslant 0}$ is bounded. Using Lemma 5.4 from **[Fat69]** we conclude that for every $s \in \mathbb{C}$ with positive real part s^2 lies in the resolvent set of A, i.e., $s^2 \in \rho(A)$. Thus the spectrum of A lies in $\mathbb{R}_- \cup \{0\}$.

To determine the spectrum, we use Lemma 6.5. By *(ii)* of the Lemma, for $\lambda \in \mathbb{C}$, $s \in \mathbb{R}$ and $x \in D(A)$ there holds

$$\frac{1}{\lambda} S(\lambda, s)(\lambda^2 I - A)x \, dt = (\cosh(\lambda s) I - C(s))x,$$

where the operator $S(\lambda, s)$ is defined in (6.12) and bounded (by Lemma 6.5 *(i)*). By this and the definition of the approximate point spectrum,

$$\sigma_{ap}(A) = \left\{ \lambda \in \mathbb{C} \mid \exists (x_n)_{n\in\mathbb{N}} \subset D(A), \|x_n\| = 1, \lim_{n\to\infty} \|(A - \lambda I)x_n\| = 0 \right\},$$

it follows that if $\lambda^2 \in \sigma_{ap}(A)$, then $\cosh(\lambda s) \in \sigma_{ap}(C(s))$. Hence,

$$\cosh\left(s\sqrt{\sigma_{ap}(A)} \right) \subset \sigma_{ap}(C(s)), \qquad \forall s \in \mathbb{R}. \tag{6.53}$$

Since $\sigma(A) \subset \mathbb{R}_- \cup \{0\}$, the boundary of the spectrum equals $\sigma(A)$. Combining this with the fact that the boundary of the spectrum is contained in the approximate point spectrum [**EN00**, Prop.VI.1.10], we see that $\sigma(A) = \sigma_{ap}(A)$. Let $-\lambda^2 \in \sigma(A)$ for $\lambda \geqslant 0$. Then, by (6.53),

$$\cosh(\pm si\lambda) = \cos(s\lambda) \in \sigma_{ap}(C(s)), \qquad \forall s \in \mathbb{R}.$$

If $\lambda \neq a$, we can find $s_0 > 0$ such that $|\cos(s_0\lambda) - \cos(as_0)| \geqslant 1$, see Lemma 6.23. Since $\cos(s_0\lambda) \in \sigma_{ap}(C(s_0))$, we find a sequence $(x_n)_{n\in\mathbb{N}} \subset X$ such that $\|x_n\| = 1$ and $\lim_{n\to\infty} \|(C(s_0) - \cos(s_0\lambda)I)x_n\| = 0$. Therefore,

$$\|(C(s_0) - \cos(as_0)I)x_n\| \geqslant |\cos(s_0\lambda) - \cos(as_0)| - \|\left(C(s_0) - \cos(s_0\lambda)I\right)x_n\|.$$

Thus $\|C(s_0) - \cos(as_0)I\| \geqslant 1$. This contradicts assumption (6.52) as $r < 1$. $\square$

If $a > 0$ in (6.52), we may apply scaling on t. Hence in that situation, we can take, without loss of generality, $a = 1$, thus

$$\sup_{t\geqslant 0} \|C(t) - \cos(t)I\| = r. \tag{6.54}$$

For the rest of the section assume that (6.54) holds with $r < 1$. Hence, we know that the norm of the difference

$$e(t) := C(t) - \cos(t)I$$

is uniformly below one, i.e., $\sup_{t\geqslant 0} \|e(t)\| < 1$, and we want to show that it equals zero. The idea is to work on the following inequality (which holds by the uniform boundedness of e).

$$\left\| \int_0^\infty h_n(q,t)e(t)dt \right\| \leqslant r \int_0^\infty |h_n(q,t)|dt, \tag{6.55}$$

with $h_n(q,t) = -2qe^{-qt}\cos(t)^{2n+1}$, $n \in \mathbb{N}$, where $q > 0$ is an auxiliary variable to be dealt with later.

Since $(C(t))_{t\geqslant 0}$ is bounded, it is well-known (see e.g., [**Fat69**, Lemma 5.4]) that for s with $\mathrm{Re}(s)>0$, $s^2\in\rho(A)$ and we can define $E(s)$ as the Laplace transform of $e(t)$,

$$E(s) := \int_0^\infty e^{-st}e(t)\,dt = s(s^2I-A)^{-1} - \frac{s}{s^2+1}I \tag{6.56}$$

To calculate the left-hand side of (6.55) we need the following result (together with Lemma 6.24).

PROPOSITION 6.27. *For* $h_n(q,t) = -2qe^{-qt}\cos(t)^{2n+1}$ *and* $q>0$ *we have*

$$\int_0^\infty h_n(q,t)e(t)dt = a_{1,2n+1}\left[g(q)I + qB(A,q)\right] + G(A,q),$$

where a_n *as in Lemma 6.24,* $g(q) = \frac{2q^2+4}{(q^2+4)}$,

$$B(A,q) = R_{(q+i)^2}\,2q\left[A-(q^2+1)I\right]R_{(q-i)^2},$$

where $R_\lambda = R(\lambda, A)$ *and* $G(A,q)$ *is such that* $\lim_{q\to 0^+} G(A,q)=0$ *in the operator norm.*

PROOF. By Lemma 6.24, we have that

$$\begin{aligned}\int_0^\infty h_n(q,t)e(t)dt &= -\sum_{k=0}^n a_{2k+1,2n+1}\,2q\int_0^\infty e^{-qt}\cos\left((2k+1)t\right)e(t)\,dt\\ &= -\sum_{k=0}^n a_{2k+1,2n+1}q\left[E(q+(2k+1)i)+E(q-(2k+1)i)\right].\end{aligned}$$

Let us first consider the term in the sum corresponding to $k=0$. By (6.56),

$$E(q\pm i) = (q\pm i)R_{(q\pm i)^2} - \frac{q\pm i}{q(q\pm 2i)}I. \tag{6.57}$$

Hence,

$$\begin{aligned}E(q+i)+E(q-i) &= -\frac{2q^2+4}{q(q^2+4)}I + (q+i)R_{(q+i)^2} + (q-i)R_{(q-i)^2}\\ &= -\frac{g(q)}{q}I + R_{(q+i)^2}\left[(q+i)((q-i)^2I-A)+\right.\\ &\quad\left. + ((q-i)^2I-A)(q-i)\right]R_{(q-i)^2}\\ &= -\frac{g(q)}{q}I + R_{(q+i)^2}2q\left[q^2I+I-A\right]R_{(q-i)^2}\\ &= -\frac{g(q)}{q}I - B(A,q).\end{aligned}$$

Thus, it remains to show that $\lim_{q\to 0^+} G(A,q)=0$, with

$$G(A,q) := -\sum_{k=1}^n a_{2k+1,2n+1}q\left[E(q+(2k+1)i)-E(q-(2k+1)i)\right].$$

Let $d_k = (2k+1)i$. By (6.56),

$$E(q \pm (2k+1)i) = (q \pm d_k)R_{(q\pm d_k)^2} - \frac{q \pm d_k}{(q \pm d_k)^2 + 1} I.$$

For $k \neq 0$, $d_k^2 \in \rho(A)$ by Lemma 6.26. Thus, by continuity of $\lambda \mapsto R_\lambda$,

$$\lim_{q \to 0^+} E(q \pm (2k+1)i) = \pm d_k R_{d_k^2} \pm \frac{d_k}{d_k^2 + 1} I,$$

for $k \neq 0$, hence, $\lim_{q\to 0^+} G(A, q) = 0$. Therefore, the assertion follows. □

Now we are ready to prove Theorem 6.22.

PROOF (Proof of Theorem 6.22). Let $r = 1 - 2\varepsilon$ for some $\varepsilon > 0$. Since $\lim_{n\to\infty} \frac{2b_{2n+1}}{a_{1,2n+1}} = 1$ by Lemma 6.25, we can choose $n \in \mathbb{N}$ such that

$$r\frac{2b_{2n+1}}{a_{1,2n+1}} < 1 - \varepsilon. \tag{6.58}$$

Let us abbreviate $a_{1,2n+1}$ by a_{2n+1}. By (6.55) and Proposition 6.27, we have that

$$\|a_{2n+1}\,[g(q)I + qB(A, q)] + G(A, q)\| \leqslant 2rq \int_0^\infty e^{-qt} |\cos(t)^{2n+1}| dt,$$

hence, dividing by $g(q)a_{2n+1}$,

$$\left\| I + \frac{1}{g(q)} \left(qB(A, q) + \frac{1}{a_{2n+1}} G(A, q) \right) \right\| \leqslant \frac{2rq}{g(q)a_{2n+1}} \int_0^\infty e^{-qt} |\cos(t)^{2n+1}| dt,$$

For $q \to 0^+$, $g(q) \to 1^+$, $G(A, q) \to 0$ by Proposition 6.27 and by Lemma 6.25, $q \int_0^\infty e^{-qt} |\cos(t)^{2n+1}| dt \to b_{2n+1}$. Thus, there exists $q_0 > 0$ (depending only on ε and n) such that

$$\left\| I + \frac{q}{g(q)} B(A, q) \right\| \leqslant r\frac{2b_{2n+1}}{a_{2n+1}} + \varepsilon =: \delta, \qquad \forall q \in (0, q_0),$$

Since $\delta < 1$ by (6.58), $B(A, q)$ is invertible for $q \in (0, q_0)$. Moreover,

$$\|B(A, q)^{-1}\| \leqslant \frac{q}{g(q)} \cdot \frac{1}{1 - \delta}.$$

Since

$$B(A, q)^{-1}x_1 = \frac{1}{2} \left((q - i)^2 I - A\right)\; q^{-1} \left[A - (q^2 + 1)I\right]^{-1} \left((q + i)^2 I - A\right) x_1,$$

for $x_1 \in D(A)$, which is dense in X, we conclude that

$$\|((q - i)^2 - A)R(q^2 + 1, A)((q + i)^2 - A)x_1\| \leqslant \frac{q^2}{g(q)} \cdot \frac{2}{1 - \delta} \cdot \|x_1\|.$$

As $q \to 0^+$, the right-hand-side goes to 0, whereas the left hand side tends to $\|(I+A)\,[I - A]^{-1}\,(I+A)x_1\|$ as $1 \in \rho(A)$. Since $-1 \in \rho(A)$, we derive $(I+A)x_1 = 0$. Therefore, $A = -I$ since $D(A)$ is dense in X. □

REMARK 6.28.

(i) As mentioned before, by the recent results of Bobrowski, Chojnacki and Gregosiewicz [**BCG15**] and, independently, Esterle [**Est15a**], we know that the implication of our 'Less than one implies zero' law, Theorem 6.22, even holds if the constant 1 in the premise gets replaced by $\frac{8}{3\sqrt{3}}$. Their proof techniques, however, differ completely from the ones we use for Theorem 6.22. It is an interesting question whether our proof can be extended to the more general result.

(ii) Recently, Esterle [**Est15c**] has studied the discrete-version of Theorem 6.22, proving that for every $a \in \mathbb{R}$ there exists $k(a) \in \left[\frac{\sqrt{5}}{2}, \frac{8}{3\sqrt{3}}\right]$ such that for every discrete cosine family C in a unital Banach algebra it holds that

$$\sup_{n\in\mathbb{N}} \|C(n) - \cos(an)I\| < k(a) \quad \Longrightarrow \quad C(n) = \cos(an)\ \forall n \in \mathbb{Z}.$$

He also proves that for a cosine family $(C(g))_{g\in G}$ in a unital Banach algebra and a scalar cosine family $(c(g))_{g\in G}$, where G is an abelian group, the assertion

$$\sup_{g\in G} \|C(g) - c(g)I\| < \frac{\sqrt{5}}{2} \quad \Longrightarrow \quad C(g) = c(g)\ \forall n \in \mathbb{Z}$$

holds. Moreover, the number $\frac{\sqrt{5}}{2}$ is shown to be optimal.

APPENDIX A

Maximum principles for operator-valued functions

For an open set $\Omega \subset \mathbb{C}$, $H(\Omega)$ denotes the (complex-valued) holomorphic functions on Ω.

THEOREM A.1 (Phragmén-Lindelöf principle - general version, **[PL08]**). *Let* $f \in H(\Omega)$, *where* $\Omega \subset \mathbb{C}$ *is an open, connected set and denote by* Γ *the boundary of in the extended complex plane* Ω *in* $\mathbb{C} \cup \{\infty\}$. *Let* $E \subset \Gamma$ *and assume that there exists* $M > 0$ *such that*

$$\limsup_{z \to \zeta, z \in \Omega} |f(z)| \leqslant M, \qquad \zeta \in \Gamma \setminus E.$$

Further assume the existence of a function $w \in H(\Omega)$ *with* $0 < |w| \leqslant 1$ *on* Ω *and for all* $\sigma > 0$,

$$\limsup_{z \to \zeta, z \in \Omega} |w(z)^{\sigma} f(z)| \leqslant M, \quad \zeta \in E.$$

Then,

$$|f(z)| \leqslant M, \quad z \in \Omega.$$

Recall that $\partial\Omega$ denotes the boundary of Ω in $\mathbb{C}$. Setting $E = \{\infty\} \cap \Gamma$ and $\omega(z) = 1$ in Theorem A.1 yields the following maximum principle for H^{∞}-functions.

COROLLARY A.2 (Phragmén-Lindelöf principle for H^{∞}-functions). *Let* $\Omega \subset \mathbb{C}$ *be open and connected. For a scalar-valued, bounded analytic function* f *on* Ω *we have*

$$\sup_{z \in \Omega} |f(z)| = \sup_{\zeta \in (\partial\Omega \setminus \{\infty\})} \widetilde{f}(\zeta),$$

where $\widetilde{f}(\zeta) = \limsup_{z \to \zeta, z \in \Omega} |f(z)|$.
If f *is continuous at* $\zeta \in \partial\Omega$, *then* $\widetilde{f}(\zeta) = |f(\zeta)|$.

THEOREM A.3 (Phragmén-Lindelöf principle for operator-valued H^∞-functions). *Let $\Omega \subset \mathbb{C}$ be a connected, open set and X a Banach space. For a function bounded and analytic function $F : \Omega \to \mathcal{B}(X)$, it holds that*

$$\sup_{z\in\Omega} \|F(z)\| = \sup_{\zeta\in\partial\Omega\setminus\{\infty\}} \widetilde{F}(\zeta),$$

where $\widetilde{F}(\zeta) = \limsup_{z\to\zeta, z\in\Omega} \|F(z)\|$.
If F is continuous at $\zeta \in \partial\Omega$, then $\widetilde{F}(\zeta) = \|F(\zeta)\|$.

PROOF. Since for $\zeta \in \partial\Omega$, $\widetilde{F}(\zeta) \leqslant \sup_{z\in\Omega} \|F(z)\|$, it follows that

$$\sup_{z\in\Omega} \|F(z)\| \geqslant \sup_{\zeta\in\Omega\setminus\{\infty\}} \widetilde{F}(\zeta).$$

To prove the converse inequality, let $x \in X$, $y \in X'$ and define $F_{x,y}(z) = \langle y, F(z)x\rangle$. Since F is analytic if and only if $F_{x,y}$ is analytic for all x, y, we can apply Corollary A.2 to derive that $\sup_{z\in\Omega} |F_{x,y}(z)| = \sup_{\zeta\in\partial\Omega\setminus\{\infty\}} \widetilde{F}_{x,y}(\zeta)$. Hence,

$$\begin{aligned}
\sup_{z\in\Omega} \|F(z)\| &= \sup_{z\in\Omega} \sup_{\|x\|=\|y\|=1} |F_{x,y}(z)| \\
&= \sup_{\|x\|=\|y\|=1} \sup_{\zeta\in\partial\Omega\setminus\{\infty\}} \widetilde{F}_{x,y}(\zeta) \\
&= \sup_{\zeta\in\partial\Omega\setminus\{\infty\}} \sup_{\|x\|=\|y\|=1} \limsup_{z\to\zeta, z\in\Omega} |\langle y, F(z)x\rangle| \\
&\leqslant \sup_{\zeta\in\partial\Omega\setminus\{\infty\}} \sup_{\|x\|=\|y\|=1} \limsup_{z\to\zeta, z\in\Omega} \|F(z)\| \\
&\leqslant \sup_{\zeta\in\partial\Omega\setminus\{\infty\}} \limsup_{z\to\zeta, z\in\Omega} \|F(z)\| \\
&= \sup_{\zeta\in\partial\Omega\setminus\{\infty\}} \widetilde{F}(\zeta).
\end{aligned}$$

□

In Corollary A.2 and Theorem A.3 we have seen that for general domains Ω, the supremum norm of a bounded, analytic function on Ω is 'attained' at the boundary. For $\Omega \in \{\mathbb{D}, \mathbb{C}_-, \mathbb{C}_+\}$, by Hardy space theory, one can even define a 'boundary function' $f^* \in L^\infty(\partial\Omega)$ such that $\|f\|_{\infty,\Omega} = \|f^*\|_{L^\infty}$. Hence, H^∞ isometrically embedded in $L^\infty(\partial\Omega)$. The operator-valued analog is not true in general, but at least for Hilbert spaces X an boundary function $F^* \in L^\infty(\partial\Omega, \mathcal{B}(X))$ can be defined (where the boundary function is defined in the strong operator topology). The case for separable Hilbert spaces is well-known, see, e.g., **[CZ95, Tho97]**, the non-separable case was proved by Mikkola, see **[Mik08]**, which answered a problem stated by Thomas **[Tho97]**.

Bibliography

[AS64] M. Abramowitz and I. A. Stegun. *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, volume 55 of *National Bureau of Standards Applied Mathematics Series*. U.S. Government Printing Office, Washington, D.C., 1964.

[AHvN13] J. Abreu, B. Haak, and J. van Neerven. The stochastic Weiss conjecture for bounded analytic semigroups. *J. Lond. Math. Soc. (2)*, 88(1):181–201, 2013.

[ADM96] D. Albrecht, X. Duong, and A. McIntosh. Operator theory and harmonic analysis. In *Instructional Workshop on Analysis and Geometry, Part III (Canberra, 1995)*, volume 34 of *Proc. Centre Math. Appl. Austral. Nat. Univ.*, pages 77–136. Austral. Nat. Univ., Canberra, 1996.

[Are12] W. Arendt. A $0 - 3/2$ - Law for Cosine Functions. *Ulmer Seminare, Funktionalanalysis und Evolutionsgleichungen*, 17:349–350, 2012.

[ABHN11] W. Arendt, C. J. K. Batty, M. Hieber, and F. Neubrander. *Vector-valued Laplace transforms and Cauchy problems*, volume 96 of *Monographs in Mathematics*. Birkhäuser/Springer Basel AG, Basel, second edition, 2011.

[AB02] W. Arendt and S. Bu. The operator-valued Marcinkiewicz multiplier theorem and maximal regularity. *Math. Z.*, 240(2):311–343, 2002.

[ABD04] T. Y. Azizov, A. I. Barsukov, and A. Dijksma. Decompositions of a Krein space in regular subspaces invariant under a uniformly bounded C_0-semigroup of bicontractions. *J. Funct. Anal.*, 211(2):324–354, 2004.

[BC91] J.-B. Baillon and P. Clément. Examples of unbounded imaginary powers of operators. *J. Funct. Anal.*, 100(2):419–434, 1991.

[Bak03] N. Y. Bakaev. Constant size control in stability estimates under some resolvent conditions. *Num. methods and Prog.*, 4:348–357, 2003.

[BE04] A. Bátkai and K.-J. Engel. Abstract wave equations with generalized Wentzell boundary conditions. *J. Differential Equations*, 207(1):1–20, 2004.

[BHM13] C. Batty, M. Haase, and J. Mubeen. The holomorphic functional calculus approach to operator semigroups. *Acta Sci. Math. (Szeged)*, 79:289–323, 2013.

[BN99] N.-E. Benamara and N. Nikolski. Resolvent tests for similarity to a normal operator. *Proc. London Math. Soc. (3)*, 78(3):585–626, 1999.

[BG94] E. Berkson and T. A. Gillespie. Spectral decompositions and harmonic analysis on UMD spaces. *Studia Math.*, 112(1):13–49, 1994.

[Bes12] N. Besseling. *Stability analysis in continuous and discrete time*. PhD thesis, University of Twente, 2012.

[BZ10] N. Besseling and H. Zwart. Stability analysis in continuous and discrete time, using the Cayley transform. *Integral Equations Operator Theory*, 68(4):487–502, 2010.

[Blu01a] S. Blunck. Analyticity and discrete maximal regularity on L_p-spaces. *J. Funct. Anal.*, 183(1):211–230, 2001.

[Blu01b] S. Blunck. Maximal regularity of discrete and continuous time evolution equations. *Studia Math.*, 146(2):157–176, 2001.

[Boa54] R. P. Boas, Jr. *Entire functions*. Academic Press Inc., New York, 1954.

[BC13] A. Bobrowski and W. Chojnacki. Isolated points of some sets of bounded cosine families, bounded semigroups, and bounded groups on a Banach space. *Studia Math.*, 217(3):219–241, 2013.

[BCG15] A. Bobrowski, W. Chojnacki, and A. Gregosiewicz. On close-to-scalar one-parameter cosine families. *J. Math. Anal. Appl.*, 429(1):383–394, 2015.

[BDEM10] H. Bounit, A. Driouich, and O. El-Mennaoui. A direct approach to the Weiss conjecture for bounded analytic semigroups. *Czechoslovak Math. J.*, 60(135)(2):527–539, 2010.

[CEP15a] I. Chalendar, J. Esterle, and J. R. Partington. Dichotomy results for norm estimates in operator semigroups. In W. Arendt, R. Chill, and Y. Tomilov, editors, *Semigroups meet complex analysis, Harmonic Analysis and Mathematical Physics*, volume 250 of *Operator Theory: Advances and Applications*. Birkhäuser, 2015. Available at arXiv: 1502.05251.

[CEP15b] I. Chalendar, J. Esterle, and J. R. Partington. Lower estimates near the origin for functional calculus on operator semigroups. *J. Funct. Anal.*, 269(6):1620–1635, 2015.

[Cho15a] W. Chojnacki. Around Schwenninger and Zwart's zero-two law for cosine families. Submitted, 2015.

[Cho15b] W. Chojnacki. On cosine families close to scalar cosine families. *J. Aust. Math. Soc.*, 2015. Preprint available at arXiv:1411.0854, DOI:10.1017/S1446788715000038.

[CDMY96] M. Cowling, I. Doust, A. McIntosh, and A. Yagi. Banach space operators with a bounded H^∞ functional calculus. *J. Austral. Math. Soc. Ser. A*, 60(1):51–89, 1996.

[CN47] J. Crank and P. Nicolson. A practical method for numerical evaluation of solutions of partial differential equations of the heat-conduction type. *Proc. Cambridge Philos. Soc.*, 43:50–67, 1947.

[CLPT93] M. Crouzeix, S. Larsson, S. Piskarëv, and V. Thomée. The stability of rational approximations of analytic semigroups. *BIT*, 33(1):74–84, 1993.

[CZ95] R. F. Curtain and H. Zwart. *An introduction to infinite-dimensional linear systems theory*, volume 21 of *Texts in Applied Mathematics*. Springer-Verlag, New York, 1995.

[deL88] R. deLaubenfels. Inverses of generators. *Proc. Amer. Math. Soc.*, 104(2):443–448, 1988.

[deL95] R. deLaubenfels. Automatic extensions of functional calculi. *Studia Math.*, 114(3):237–259, 1995.

[DS88] N. Dunford and J. T. Schwartz. *Linear operators. Part III*. Wiley Classics Library. John Wiley & Sons, Inc., New York, 1988. Spectral operators, With the assistance of William G. Bade and Robert G. Bartle, Reprint of the 1971 original, A Wiley-Interscience Publication.

[Dur70] P. L. Duren. *Theory of* H^p *spaces*. Pure and Applied Mathematics, Vol. 38. Academic Press, New York, 1970.

[ER13] M. Egert and J. Rozendaal. Convergence of subdiagonal Padé approximations of C_0-semigroups. *J. Evol. Equ.*, 13(4):875–895, 2013.

[EZ06] T. Eisner and H. Zwart. Continuous-time Kreiss resolvent condition on infinite-dimensional spaces. *Math. Comp.*, 75(256):1971–1985, 2006.

[EZ08] T. Eisner and H. Zwart. The growth of a C_0-semigroup characterised by its cogenerator. *J. Evol. Equ.*, 8(4):749–764, 2008.

[EFR02] O. El-Fallah and T. Ransford. Extremal growth of powers of operators satisfying resolvent conditions of Kreiss-Ritt type. *J. Funct. Anal.*, 196(1):135–154, 2002.

[EN00] K.-J. Engel and R. Nagel. *One-parameter semigroups for linear evolution equations*, volume 194 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2000.

[Est04] J. Esterle. Zero-one and zero-two laws for the behavior of semigroups near the origin. In *Banach algebras and their applications*, volume 363 of *Contemp. Math.*, pages 69–79. Amer. Math. Soc., Providence, RI, 2004.

[Est15a] J. Esterle. Bounded cosine functions close to continuous scalar bounded cosine functions. *J. Int. Eq. and Op. Th.*, 2015. To appear. Available at arXiv:1502.00150, `https://hal.archives-ouvertes.fr/hal-01111839`.

[Est15b] J. Esterle. A short proof of the zero-two law for cosine families. *Archiv der Mathematik*, 2015. To appear, DOI: 10.1007/s00013-015-0811-0.

[Est15c] J. Esterle. The zero-$\frac{\sqrt{5}}{2}$ law for cosine families. Submitted, available at arXiv:1505.06064, 2015.

[Fac13] S. Fackler. Regularity of semigroups via the asymptotic behaviour at zero. *Semigroup Forum*, 87(1):1–17, 2013.

[Fac14] S. Fackler. *Regularity Properties of Sectorial Operators: Extrapolation, Counterexamples and Generic Classes*. PhD thesis, Universität Ulm, 2014.

[Fac15] S. Fackler. Regularity Properties of Sectorial Operators: Counterexamples and Open Problems. In W. Arendt, R. Chill, and Y. Tomilov, editors, *Semigroups meet complex analysis, Harmonic Analysis and Mathematical Physics*, volume 250 of *Operator Theory: Advances and Applications*. Birkhäuser, 2015. Preprint available at `http://arxiv.org/abs/1407.1142`.

[Fat69] H. O. Fattorini. Ordinary differential equations in linear topological spaces. I. *J. Differential Equations*, 5:72–105, 1969.

[Fog64] S. R. Foguel. A counterexample to a problem of Sz.-Nagy. *Proc. Amer. Math. Soc.*, 15:788–790, 1964.

[Fré06] M. Fréchet. Sur quelques points du calcul fonctionnel. *Rendiconti del Circolo Matematico di Palermo*, 22(1):1–72, 1906.

[GMY11] J. E. Galé, P. J. Miana, and D. V. Yakubovich. H^∞-functional calculus and models of Nagy-Foiaş type for sectorial operators. *Math. Ann.*, 351(3):733–760, 2011.

[Gar07] J. B. Garnett. *Bounded Analytic Functions*, volume 236 of *Graduate Texts in Mathematics*. Springer, New York, first edition, 2007.

[Gau60] W. Gautschi. Some elementary inequalities relating to the gamma and incomplete gamma function. *J. Math. and Phys.*, 38:77–81, 1959/60.

[Gol85] J. A. Goldstein. *Semigroups of linear operators and applications*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 1985.

[GZ07] A. Gomilko and H. Zwart. The Cayley transform of the generator of a bounded C_0-semigroup. *Semigroup Forum*, 74(1):140–148, 2007.

[Gom99] A. M. Gomilko. On conditions for the generating operator of a uniformly bounded C_0-semigroup of operators. *Funktsional. Anal. i Prilozhen.*, 33(4):66–69, 1999.

[Gom04] A. M. Gomilko. The Cayley transform of the generator of a uniformly bounded C_0-semigroup of operators. *Ukraïn. Mat. Zh.*, 56(8):1018–1029, 2004.

[GZT07] A. M. Gomilko, K. Zvart, and Y. Tomilov. On the inverse operator of the generator of a C_0-semigroup. *Mat. Sb.*, 198(8):35–50, 2007.

[GZB11] A. M. Gomilko, H. Zwart, and N. Besseling. Growth of semigroups in discrete and continuous time. *Studia Math.*, 206(3):273–292, 2011.

[GZ06] B. Z. Guo and H. Zwart. On the relation between stability of continuous- and discrete-time evolution equations via the Cayley transform. *Integral Equations Operator Theory*, 54(3):349–383, 2006.

[GG71] V. I. Gurariĭ and N. I. Gurariĭ. Bases in uniformly convex and uniformly smooth Banach spaces. *Izv. Akad. Nauk SSSR Ser. Mat.*, 35:210–215, 1971.

[Haa05] B. H. Haak. *Kontrolltheorie in Banachräumen und quadratische Abschätzungen*. PhD thesis, Universität Karlsruhe, 2005.

[Haa12] B. H. Haak. The Weiss conjecture and weak norms. *J. Evol. Equ.*, 12(4):855–861, 2012.

[Haa05] M. Haase. A general framework for holomorphic functional calculi. *Proc. Edinb. Math. Soc. (2)*, 48(2):423–444, 2005.

[Haa06a] M. Haase. *The Functional Calculus for Sectorial Operators*, volume 169 of *Operator Theory: Advances and Applications*. Birkhäuser Verlag, Basel, 2006.

[Haa06b] M. Haase. Semigroup theory via functional calculus. Preprint available at: `http://fa.its.tudelft.nl/~haase/files/semi.pdf`, 2006.

[Haa11] M. Haase. Transference principles for semigroups and a theorem of Peller. *J. Funct. Anal.*, 261(10):2959–2998, 2011.

[HR13] M. Haase and J. Rozendaal. Functional calculus for semigroup generators via transference. *J. Funct. Anal.*, 265(12):3345–3368, 2013.

[Hal64] P. R. Halmos. On Foguel's answer to Nagy's question. *Proc. Amer. Math. Soc.*, 15:791–793, 1964.

[HP57] E. Hille and R. S. Phillips. *Functional analysis and semi-groups*. American Mathematical Society Colloquium Publications, vol. 31. American Mathematical Society, Providence, R. I., 1957. rev. ed.

[Hir68] R. A. Hirschfeld. On semi-groups in Banach algebras close to the identity. *Proc. Japan Acad.*, 44:755, 1968.

[JP01] B. Jacob and J. R. Partington. The Weiss conjecture on admissibility of observation operators for contraction semigroups. *Integral Equations Operator Theory*, 40(2):231–243, 2001.

[JPP02] B. Jacob, J. R. Partington, and S. Pott. Admissible and weakly admissible observation operators for the right shift semigroup. *Proc. Edinb. Math. Soc. (2)*, 45(2):353–362, 2002.

[JPP09] B. Jacob, J. R. Partington, and S. Pott. Zero-class admissibility of observation operators. *Systems Control Lett.*, 58(6):406–412, 2009.

[KW01] N. Kalton and L. Weis. The H^∞-Functional Calculus and Square Function Estimates. Unpublished, available at arXiv: 1411.0472, 2001.

[KP08] N. J. Kalton and P. Portal. Remarks on ℓ_1 and ℓ_∞-maximal regularity for power-bounded operators. *J. Aust. Math. Soc.*, 84(3):345–365, 2008.

[KW01] N. J. Kalton and L. Weis. The H^∞-calculus and sums of closed operators. *Math. Ann.*, 321(2):319–345, 2001.

[Kat95] T. Kato. *Perturbation Theory for Linear Operators*. Classics in Mathematics. Springer-Verlag, Berlin, 1995. Reprint of the 1980 edition.

[Kre62] H.-O. Kreiss. Über die Stabilitätsdefinition für Differenzengleichungen die partielle Differentialgleichungen approximieren. *Nordisk Tidskr. Informations-Behandling*, 2:153–181, 1962.

[KW10] C. Kriegler and L. Weis. Contractivity of the H^∞-calculus and Blaschke products. In *Operator algebras, operator theory and applications*, volume 195 of *Oper. Theory Adv. Appl.*, pages 231–244. Birkhäuser Verlag, Basel, 2010.

[KW04] P. C. Kunstmann and L. Weis. Maximal L_p-regularity for parabolic equations, Fourier multiplier theorems and H^∞-functional calculus. In *Functional analytic methods for evolution equations*, volume 1855 of *Lecture Notes in Math.*, pages 65–311. Springer, Berlin, 2004.

[LM13] F. Lancien and C. L. Merdy. On functional calculus properties of ritt operators. Preprint, available at arXiv: 1301.4875, 2013.

[Lap75] G. I. Laptev. Conditions for the uniform well-posedness of the Cauchy problem for systems of equations. *Dokl. Akad. Nauk SSSR*, 220:281–284, 1975.

[LM98] C. Le Merdy. The similarity problem for bounded analytic semigroups on Hilbert space. *Semigroup Forum*, 56(2):205–224, 1998.

[LM03] C. Le Merdy. The Weiss conjecture for bounded analytic semigroups. *J. London Math. Soc. (2)*, 67(3):715–738, 2003.

[LM14a] C. Le Merdy. α-admissibility for Ritt operators. *Complex Anal. Oper. Theory*, 8(3):665–681, 2014.

[LM14b] C. Le Merdy. H^∞ functional calculus and square function estimates for Ritt operators. *Rev. Mat. Iberoam.*, 30(4):1149–1190, 2014.

[LT84] R. J. LeVeque and L. N. Trefethen. On the resolvent condition in the Kreiss matrix theorem. *BIT*, 24(4):584–591, 1984.

[Lyu99] Y. Lyubich. Spectral localization, power boundedness and invariant subspaces under Ritt's type condition. *Studia Math.*, 134(2):153–167, 1999.

[MS65] C. A. McCarthy and J. Schwartz. On the norm of a finite Boolean algebra of projections, and applications to theorems of Kreiss and Morton. *Comm. Pure Appl. Math.*, 18:191–201, 1965.

[McI86] A. McIntosh. Operators which have an H_∞ functional calculus. In *Miniconference on operator theory and partial differential equations (North Ryde, 1986)*, volume 14 of *Proc. Centre Math. Anal. Austral. Nat. Univ.*, pages 210–231. Austral. Nat. Univ., Canberra, 1986.

[MY90] A. McIntosh and A. Yagi. Operators of type ω without a bounded H_∞ functional calculus. In *Miniconference on Operators in Analysis (Sydney, 1989)*, volume 24 of *Proc. Centre Math. Anal. Austral. Nat. Univ.*, pages 159–172. Austral. Nat. Univ., Canberra, 1990.

[Mik08] K. M. Mikkola. Fourier multipliers for L^2 functions with values in nonseparable Hilbert spaces and operator-valued H^p boundary functions. *Ann. Acad. Sci. Fenn. Math.*, 33(1):121–130, 2008.

[Mub11] J. Mubeen. *The bounded $\mathcal{H}^\infty$-calculus for sectorial, strip-type and half-plane operators*. PhD thesis, University of Oxford, 2011.

[Nag74] B. Nagy. On cosine operator functions in Banach spaces. *Acta Sci. Math. (Szeged)*, 36:281–289, 1974.

[NZ99] B. Nagy and J. Zemánek. A resolvent condition implying power boundedness. *Studia Math.*, 134(2):143–151, 1999.

[Nev93] O. Nevanlinna. *Convergence of iterations for linear equations*. Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 1993.

[Nik13] N. Nikolski. Sublinear dimension growth in the Kreiss matrix theorem. *Algebra i Analiz*, 25(3):3–51, 2013.

[Nik02a] N. K. Nikolski. *Operators, functions, and systems: an easy reading. Vol. 1*, volume 92 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2002.

[Nik02b] N. K. Nikolski. *Operators, functions, and systems: an easy reading. Vol. 2*, volume 93 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2002.

[Pal93] C. Palencia. A stability result for sectorial operators in Banach spaces. *SIAM J. Numer. Anal.*, 30(5):1373–1384, 1993.

[Paz83] A. Pazy. *Semigroups of linear operators and applications to partial differential equations*, volume 44 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1983.

[PL08] E. Phragmén and E. Lindelöf. Sur une extension d'un principe classique de l'analyse et sur quelques propriétés des fonctions monogènes dans le voisinage d'un point singulier. *Acta Math.*, 31(1):381–406, 1908.

[PZ07] S. Piskarev and H. Zwart. Crank-Nicolson scheme for abstract linear systems. *Numer. Funct. Anal. Optim.*, 28(5-6):717–736, 2007.

[PS25] G. Pólya and G. Szegö. *Aufgaben und Lehrsätze aus der Analysis*, volume Band I. Springer, Berlin, 1925.

[Rie16] M. Riesz. Über einen Satz des Herrn Serge Bernstein. *Acta Math.*, 40(1):337–347, 1916.

[Rit53] R. K. Ritt. A condition that $\lim_{n\to\infty} n^{-1}T^n = 0$. *Proc. Amer. Math. Soc.*, 4:898–899, 1953.

[RR97] M. Rosenblum and J. Rovnyak. *Hardy classes and operator theory*. Dover Publications, Inc., Mineola, NY, 1997. Corrected reprint of the 1985 original.

[Rud91] W. Rudin. *Functional analysis*. International Series in Pure and Applied Mathematics. McGraw-Hill, Inc., New York, second edition, 1991.

[Sch15a] F. L. Schwenninger. Functional calculus for Tadmor-Ritt operators. Submitted, available at arXiv: 1507.00049, 2015.

[Sch15b] F. L. Schwenninger. On measuring unboundedness of the H^∞-calculus for generators of analytic semigroups. *Submitted*, 2015.

[SZ12] F. L. Schwenninger and H. Zwart. Weakly admissible $\mathcal{H}^-_\infty$-calculus on reflexive Banach spaces. *Indag. Math. (N.S.)*, 23(4):796–815, 2012.

[SZ14] F. L. Schwenninger and H. Zwart. Generators with a closure relation. *Oper. Matrices*, 8(1):157–165, 2014.

[SZ15a] F. L. Schwenninger and H. Zwart. Functional calculus for c_0-semigroups using infinite-dimensional systems theory. In W. Arendt, R. Chill, and Y. Tomilov, editors, *Semigroups meet* *complex analysis, Harmonic Analysis and Mathematical Physics*, volume 250 of *Operator Theory: Advances and Applications*. Birkhäuser, 2015.

[SZ15b] F. L. Schwenninger and H. Zwart. Less than one implies zero. Submitted, available at arXiv:1310.6202, 2015.

[SZ15c] F. L. Schwenninger and H. Zwart. Zero-two law for cosine families. *J. Evol. Equ.*, 2015. To appear, DOI:10.1007/s00028-015-0272-8.

[Sin70] I. Singer. *Bases in Banach spaces. I*. Springer-Verlag, New York, 1970. Die Grundlehren der mathematischen Wissenschaften, Band 154.

[Spi91] M. N. Spijker. On a conjecture by LeVeque and Trefethen related to the Kreiss matrix theorem. *BIT*, 31(3):551–555, 1991.

[STW03] M. N. Spijker, S. Tracogna, and B. D. Welfert. About the sharpness of the stability estimates in the Kreiss matrix theorem. *Math. Comp.*, 72(242):697–713 (electronic), 2003.

[Sta05] O. Staffans. *Well-posed linear systems*, volume 103 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2005.

[Sto90] M. H. Stone. *Linear transformations in Hilbert space*, volume 15 of *American Mathematical Society Colloquium Publications*. American Mathematical Society, Providence, RI, 1990. Reprint of the 1932 original.

[Str88] A. Stray. Minimal interpolation by Blaschke products. II. *Bull. London Math. Soc.*, 20(4):329–332, 1988.

[SW97] J. C. Strikwerda and B. A. Wade. A survey of the Kreiss matrix theorem for power bounded families of matrices and its extensions. In *Linear operators (Warsaw, 1994)*, volume 38 of *Banach Center Publ.*, pages 339–360. Polish Acad. Sci., Warsaw, 1997.

[SNF70] B. Sz.-Nagy and C. Foias. *Harmonic analysis of operators on Hilbert space*. Translated from the French and revised. North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., Inc., New York; Akadémiai Kiadó, Budapest, 1970.

[Sze67] G. Szegő. *Orthogonal polynomials*. American Mathematical Society, Providence, R.I., third edition, 1967. American Mathematical Society Colloquium Publications, Vol. 23.

[Tad81] E. Tadmor. The equivalence of L_2-stability, the resolvent condition, and strict H-stability. *Linear Algebra Appl.*, 41:151–159, 1981.

[Tad86] E. Tadmor. The resolvent condition and uniform power boundedness. *Linear Algebra Appl.*, 80:250–252, 1986.

[Tad14] E. Tadmor. On Tadmor's paper "The resolvent condition and uniform boundedness", November 2014. Private Communication.

[Tho97] E. G. F. Thomas. Vector-valued integration with applications to the operator-valued H^∞ space. *IMA J. Math. Control Inform.*, 14(2):109–136, 1997. Distributed parameter systems: analysis, synthesis and applications, Part 2.

[TE05] L. N. Trefethen and M. Embree. *Spectra and pseudospectra*. Princeton University Press, Princeton, NJ, 2005.

[vC11] J. A. van Casteren. On the Crank-Nicolson scheme once again. *J. Evol. Equ.*, 11(2):457–476, 2011.

[vDKS93] J. L. M. van Dorsselaer, J. F. B. M. Kraaijevanger, and M. N. Spijker. Linear stability analysis in the numerical solution of initial value problems. In *Acta numerica, 1993*, Acta Numer., pages 199–237. Cambridge Univ. Press, Cambridge, 1993.

[Vit04a] P. Vitse. Functional calculus under the Tadmor-Ritt condition, and free interpolation by polynomials of a given degree. *J. Funct. Anal.*, 210(1):43–72, 2004.

[Vit04b] P. Vitse. The Riesz turndown collar theorem giving an asymptotic estimate of the powers of an operator under the Ritt condition. *Rend. Circ. Mat. Palermo (2)*, 53(2):283–312, 2004.

[Vit05a] P. Vitse. A band limited and Besov class functional calculus for Tadmor-Ritt operators. *Arch. Math. (Basel)*, 85(4):374–385, 2005.

[Vit05b] P. Vitse. A Besov class functional calculus for bounded holomorphic semigroups. *J. Funct. Anal.*, 228(2):245–269, 2005.

[vN96] J. von Neumann. *Mathematical foundations of quantum mechanics*. Princeton Landmarks in Mathematics. Princeton University Press, Princeton, NJ, 1996.

[Wal67] L. J. Wallen. On the magnitude of $x^n - 1$ in a normed algebra. *Proc. Amer. Math. Soc.*, 18:956, 1967.

[Wei89] G. Weiss. Admissible observation operators for linear semigroups. *Israel J. Math.*, 65(1):17–43, 1989.

[Wei91] G. Weiss. Two conjectures on the admissibility of control operators. In *Estimation and control of distributed parameter systems (Vorau, 1990)*, volume 100 of *Internat. Ser. Numer. Math.*, pages 367–378. Birkhäuser, Basel, 1991.

[Wyn10] A. Wynn. α-admissibility of observation operators in discrete and continuous time. *Complex Anal. Oper. Theory*, 4(1):109–131, 2010.

[Zwa05] H. Zwart. Sufficient conditions for admissibility. *Systems Control Lett.*, 54(10):973–979, 2005.

[Zwa07] H. Zwart. Is A^{-1} an infinitesimal generator? In *Perspectives in operator theory*, volume 75 of *Banach Center Publ.*, pages 303–313. Polish Acad. Sci., Warsaw, 2007.

[Zwa12] H. Zwart. Toeplitz operators and $\mathcal{H}^\infty$-calculus. *J. Funct. Anal.*, 263(1):167 – 182, 2012.

[ZJS03] H. Zwart, B. Jacob, and O. Staffans. Weak admissibility does not imply admissibility for analytic semigroups. *Systems Control Lett.*, 48(3-4):341–350, 2003.

Nomenclature

$\mathfrak{D}^{B}_{g,y}$ A (scalar) output mapping defined by $g \in H^\infty(\mathbb{C}_-)$, p. 23
$\mathcal{H}^2(Y)$ Y-valued Hardy space on $\mathbb{C}_+$, p. 19
$\mathcal{H}^2_\perp(Y)$ Y-valued Hardy space on $\mathbb{C}_-$, p. 19
$\mathfrak{L}$ (two-sided) Laplace transform, p. 20
$\mathcal{A}$ normed unital algebra, p. 130
$\mathcal{B}_\theta$ The interior of the convex hull of $\{\{1\}, B_{\sin\theta}(0)\}$, p. 87
$\mathcal{F}$ The Fourier transform, p. 20
$\mathcal{G}_{bdd}(X)$ The set of bounded a C_0-semigroups on X, p. 113
$\mathcal{G}_{exp}(X)$ The set of exponentially stable C_0-semigroups on X, p. 113
$\mathcal{G}_{M,\omega}(X)$ The set of semigroup generators A on X such that $\|e^{tA}\| \leqslant Me^{t\omega}$, p. 113
$\mathcal{G}_M$ $\mathcal{G}_{M,0}$, p. 113
Π_Y orthogonal projection from $L^2(i\mathbb{R})$ onto $\mathcal{H}^2$, p. 20
$\rho(a), \sigma(a)$ spectrum, resolvent of element a in a unital algebra, p. 140
$\sigma_c(A)$ The continuous spectrum of a closed operator A, p. 118
τ The Cayley transform $\tau(z) = \frac{1+z}{1-z}$, p. 107
$B_r(z_0)$ open ball with radius r and centre z_0, p. 10
$C(T)$ The Tadmor–Ritt constant of an Tadmor–Ritt operator, p. 83
C_Λ Lambda extension of an operator $C \in \mathcal{B}(D(A), X)$, p. 29
$Cay(A)$ The Cayley transform $(I + A)(I - A)^{-1}$ of A, p. 110
$H^\infty(\Omega)$ Banach algebra of bounded holomorphic functions on Ω, p. 10
$H^\infty[\varepsilon, \sigma]$ space of $H^\infty(\mathbb{C}_+)$ with Fourier spectrum in $[\varepsilon, \sigma]$, p. 57
$KR(X)$ The set of Kreiss operators on X, p. 83
$M(A, \delta)$ The sectoriality constant for A, p. 49
$m_\phi, \kappa_\phi, ub_\phi$ basis constants for the Schauder basis ϕ, p. 63
M_g Toeplitz operator on $L^2(\mathbb{R}_+, H)$, p. 21
$S(\theta)(X)$ The set of θ-sectorial operators on X, p. 58
$TR(X)$ The set of Tadmor–Ritt operators on X, p. 83
X_1 (Banach) space $D(A)$ equipped with graph norm for closed operator A, p. 12
$Ei(x)$ Exponential integral function, p. 54
M_- Borel measures supported in $(-\infty, 0]$ with bounded variation, p. 21

Summary

This thesis presents various results within the field of operator theory that are formulated in estimates for functional calculi. *Functional calculus* is the general concept of defining operators of the form $f(A)$, where f is a scalar function and A is an operator A, typically on a Banach space. For instance, an example is given by e^A. Norm estimates for $f(A)$ emerge in applications for example when studying stability of numerical schemes. This work is split into two parts.

The first part essentially deals with the H^∞*-functional calculus* for generators A of strongly continuous semigroups. Here, the functions f are bounded and analytic on a (suitable) half-plane in the complex plane.
The interest in this operator-functions pair comes, for instance, from the study of maximal regularity of p.d.e.'s and numerical analysis. Within this part, we further distinguish the following topics.
First, an alternative approach to the classical definition of the calculus for general strongly continuous semigroups is presented, motivated by notions from linear systems theory. The operator $f(A)$ is constructed via an output mapping of a linear system associated with f. As a consequence, sufficient conditions for a bounded H^∞-calculus are given, i.e., conditions which guarantee that $f(A)$ is a bounded operator for any f in the considered class.
Second, we restrict to generators of analytic semigroups and allow for functions f that are bounded and analytic on sectors, which is known as the classical H^∞-calculus for sectorial operators. We show that the possible unboundedness of the calculus can be measured by the norm of $\|f(A)T(t)\|$ for small t, where T denotes the semigroup generated by A. Whereas the general asymptotical behavior is $\mathcal{O}(|\log t|)$, the occurrence of square function estimates reduces the blow-up.
Third, we consider *Tadmor–Ritt* operators, the discrete-time analog of analytic semigroups. Since we are dealing with bounded operators, the functions f are bounded and analytic on bounded domains in this case. The provided functional calculus estimates imply in particular a new, simpler proof for the power-boundedness of Tadmor–Ritt operators, and also yield the best-known bound. Furthermore, the influence of discrete square function estimate is described.
Fourth, the relation between the above-mentioned results for analytic and Tadmor–Ritt operators is investigated in more generality. The interplay between the solution

of continuous-time systems, given through the semigroup, and solutions of discrete-time systems, given through the powers of an operator, is by nature of major importance when studying the stability for numerical methods. Here, *stability* can be phrased as the question wether the approximate solution is bounded on the infinite time axis given that the exact solution is bounded. In the view of semigroup theory, the Cayley transform of the generator yields a discrete-time system from a continuous-time system. The question of stability then refers to whether the Cayley transform is power-bounded if the semigroup is assumed to be bounded. Whereas the answer is in general 'no' for semigroups on Banach spaces, the case of semigroups on Hilbert spaces has become an enigmatic open problem in the last decades. Using a well-known link with the *Inverse Generator Problem*, we prove that for the Hilbert space case, it suffices to consider exponentially stable semigroups to find the answer to the question. Moreover, it is shown the *Cayley Transform Problem* and the *Inverse Generator Problem* are equivalent in a general sense.

The second part of the thesis is on *zero-two laws* for cosine families. *Cosine families* can be seen as the analog to semigroups for second-order Cauchy problems. For a semigroup T it is well-known that if the transient behavior of the semigroup at 0 is less than 1 in the operator norm, i.e., $\limsup_{t\to 0^+} \|T(t) - I\| < 1$, then the semigroup equals the identity I. This has become known as a *zero-one law*. We prove the corresponding *zero-two law* for strongly continuous cosine families on Banach spaces which has been open so far. Furthermore, related laws are provided as well as generalizations to cosine families on Banach algebras.

Samenvatting

Functional calculus is een deelgebied van operator theorie met als centrale thema het definiëren van een nieuwe operator ,$f(A)$, waarbij f een scalaire functie en A een operator op een Banachruimte is. Een voorbeeld hiervan wordt onder andere gegeven door e^A. Norm afschattingen van zulke operatoren $f(A)$ vinden toepassingen in, onder andere, numerieke wiskunde. Het voor u liggende proefschrift kan gezien worden als een collectie van dit soort afschattingen.
Het onderzoek is gesplitst in twee delen. Het centrale thema in het eerste deel is H^∞-*calculus* voor generatoren van *sterk continue halfgroepen*, waarbij de functies f begrensd en analytisch op een (passend) half-vlak van het complexe vlak zijn. De theorie van dit operator-functie paar heeft belangrijke toepassingen voor vragen betreffende maximale regulariteit van partiëlle differentiaal vergelijkingen. Verder worden in het eerste deel de volgende situaties onderscheiden.
Ten eerste wordt een alternatieve constructie voor de calculus aangetoond, gemotiveerd door ideeën uit de systeemtheorie. De operator $f(A)$ wordt ingevoerd als de uitgang van een lineair systeem geassocieerd met f. Als een gevolg worden voldoende condities aangetoond die een begrensde calculus impliceren, dat is, condities zodanig $f(A)$ een begrensde operator levert voor elke f in de toegelaten klasse.
Ten tweede beschouwen we generatoren A van analytische halfgroepen T en functies f die analytisch en begrensd op een sector zijn. In dit proefschrift wordt bewezen dat deze calculus niet willekeurig onbegrensd kan zijn. Echter de 'onbegrensdheid' kan door het gedrag van de norm van $f(A)T(t)$ voor kleine positieve t beschreven worden. In het algemeen is deze afhankelijkheid logarithmisch in t, maar onder bepaalde voorwaarden, *kwadratische afschattingen*, verbetert het gedrag.
Het derde onderdeel behandelt *Tadmor–Ritt* operatoren welke als discrete-tijd tegenhangers van analytische halfgroepen interpreteert kunnen worden. Omdat deze operatoren begrensd zijn, zijn de beschouwde functies f analytisch en begrensd op een begrensd gebied. We bewijzen norm afschattingen voor de bijhorende calculus. Deze impliceren ook de uniforme begrensdheid van machten van Tadmor–Ritt operatoren, en leveren tevens de best bekende grens.
Ten vierde wordt de relatie tussen discrete- en continue-tijd systemen onderzocht. De focus ligt hier op de stabiliteit van een bepaalde numerieke methode. Om precies te zijn betekent stabiliteit dat de numerieke approximatie begrensd is op de gehele tijd-as als de exacte oplossing begrensd is deze tijd-as. In onze situatie is de exacte

oplossing gegeven door de halfgroep en de discrete-tijd oplossing door machten van een begrensde operator. Deze operator is de *Cayley getransformeerde* van A en de numerieke methode is het *Crank–Nicolson* schema. Daardoor is onze stabiliteitsvraag equivalent met de vraag of de machten van de Cayley getransformeerde uniform begrensd zijn als wordt aangenomen dat de halfgroep uniform begrensd is. Voor Banachruimtes is het bekend dat het antwoord negatief is – er bestaan voorbeelden van begrensde halfgroepen zodanig dat de machten van de bijhorende Cayley getransformeerde niet uniform begrensd zijn. Echter het antwoord is onbekend voor Hilbertruimtes. In dit proefschrift wordt aangetoond dat de vraag gereduceerd kan worden tot het geval van exponentieel stabiele halfgroepen. Dit is een sterkere eis dan uniforme begrensdheid. Verder wordt bewezen dat deze vraag over stabiliteit equivalent is met het *Inverse Generator Probleem*.

Het tweede deel gaat over *cosinus families*. Dat zijn operator-waardige functies op de reëele as die aan soortgelijke regels voldoen als de cosinus. Zoals halfgroepen natuurlijk verbonden zijn met eerste orde Cauchy problemen, zo zijn cosinus families natuurlijk verbonden met tweede orde Cauchy problemen. Voor een halfgroep T is het algemeen bekend dat als het transitief gedrag van de halfgroep bij nul kleiner is dan 1, d.w.z. $\limsup_{t\to 0^+} \|T(t) - I\| < 1$, dan is T gelijk aan de identiteit I. Hier wordt bewezen dat de correspondeerde regel, de zogenaamde *zero-two law*, ook voor cosinus families geldt. Dit is was onbekend tot nu toe. Voorts worden soortgelijke regels bewezen voor t gaande naar oneindig, en voor discrete tijd.

List of publications

[Sch15a] F.L.S. *On functional calculus estimates for Tadmor–Ritt operators*.
Submitted to Journal Mathematical Analysis and Applications, 2015.

[SZ15b] F.L.S., HANS ZWART. *Less than one implies zero*.
Submitted to Studia Mathematica, 2015.

[Sch15b] F.L.S. *On measuring unboundedness of the* H^∞*-calculus for generators of analytic semigroups*.
Submitted to Journal of Functional Analysis, 2015.

[SZ15a] F.L.S., HANS ZWART. *Functional calculus for* C_0*-semigroups using infinite-dimensional systems theory*.
To appear in W. Arendt, R. Chill and Y. Tomilov (Eds.): *Operator Semigroups meet Complex Analysis, Harmonic Analysis and Mathematical Physics*, Operator Theory: Advances and Applications, 240, Birkhäuser Verlag, 2015.

[SZ15c] F.L.S., HANS ZWART. *Zero-two law for cosine families*.
To appear in Journal of Evolution Equations, 2015.

[SZ14] F.L.S., HANS ZWART. *Generators with a closure relation*.
Operators and Matrices, 8(1): 157–165, 2014.

[SZ12] F.L.S., HANS ZWART. *Weakly admissible* $\mathcal{H}_\infty^-$*-calculus on reflexive Banach spaces*.
Indagationes Mathematicae, 23(4): 796-815, 2012.