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REMARKS ON THE CROUZEIX–PALENCIA PROOF THAT THE NUMERICAL RANGE IS A $(1+\sqrt{2})$-SPECTRAL SET*

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Abstract. Crouzeix and Palencia recently showed that the numerical range of a Hilbert-space operator is a $(1+\sqrt{2})$-spectral set for the operator. One of the principal ingredients of their proof can be formulated as an abstract functional-analysis lemma. We give a new short proof of the lemma and show that, in the context of this lemma, the constant $(1+\sqrt{2})$ is sharp.

Key words. numerical range, spectral set, Cauchy transform, Crouzeix's conjecture

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1. Introduction.

Let H be a complex Hilbert space and let T be a bounded linear operator on H. The *numerical range* $W(T)$ of T is defined by

$$W(T) := \{\langle Tx, x\rangle : x \in H, \|x\| = 1\}.$$

It is a bounded convex set and is compact if $\dim H < \infty$.

In the recent paper [5], Crouzeix and Palencia, improving earlier results of Delyon and Delyon [6] and Crouzeix [4], showed that $\overline{W(T)}$ is always a $(1+\sqrt{2})$-spectral set for T. This means that, for every function f holomorphic on an open set containing $\overline{W(T)}$, the operator norm of $f(T)$ satisfies

$$\|f(T)\| \le (1+\sqrt{2}) \sup_{z\in W(T)} |f(z)|. \tag{1.1}$$

Crouzeix [3] has conjectured that $(1+\sqrt{2})$ may be replaced by 2. Simple examples show that the constant 2 is best possible.

The point of departure in all three papers [4, 5, 6] is the same. Let us fix a smoothly bounded open convex set Ω containing $\overline{W(T)}$. Denote by $A(\Omega)$ the algebra of continuous functions f on $\overline{\Omega}$ that are holomorphic on Ω, and write $\|f\|_\Omega := \sup_\Omega |f|$. As remarked in [6], the containment $\overline{W(T)} \subset \Omega$ is reflected in the fact that the operator-valued measure on $\partial\Omega$

$$\frac{1}{2\pi i}(\zeta I - T)^{-1}\, d\zeta,$$

used in defining $f(T)$, has positive real part. This quickly leads to the estimate

$$\|f(T) + (C\overline{f})(T)^*\| \le 2\|f\|_\Omega \quad (f \in A(\Omega)), \tag{1.2}$$

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where $C\overline{f}$ denotes the Cauchy transform of $\overline{f}$, namely,

$$(C\overline{f})(z) := \frac{1}{2\pi i}\int_{\partial\Omega} \frac{\overline{f(\zeta)}}{\zeta - z}\, d\zeta \quad (z \in \Omega).$$

The problem is how to get from (1.2), which is an estimate on $\|f(T) + (C\overline{f})(T)^*\|$, to an estimate on $\|f(T)\|$ alone. In [4, 5, 6], this is achieved in three different ways.

In [6], the authors prove the invertibility of the map $f \mapsto f + \overline{(C\overline{f})}|_{\partial\Omega}$, considered as a self-map of the space of continuous functions on $\partial\Omega$. They further obtain an estimate for the norm of the inverse, which, together with (1.2), leads to the bound

$$\|f(T)\| \leq \Big(\Big(\frac{2\pi\,\mathrm{diam}^2(\Omega)}{\mathrm{area}(\Omega)}\Big)^3 + 3\Big)\|f\|_\Omega \quad (f \in A(\Omega)).$$

In [4], Crouzeix estimates $\|(C\overline{f})(T)\|$ directly, still under the assumption that $\overline{W(T)} \subset \Omega$, and then uses (1.2) and the triangle inequality to obtain the bound

$$\|f(T)\| \leq 11.08\|f\|_\Omega \quad (f \in A(\Omega)). \tag{1.3}$$

This bound is universal, in the sense that the constant 11.08 is independent of Ω. For certain sets Ω, however, the bound can be improved. This is discussed in detail in [1]. In particular, if Ω is a disk, then $\|f(T)\| \leq 2\|f\|_\Omega$, and this is easily seen to be sharp. (There are now several proofs of this last inequality, originally due to Okubo and Ando. A particularly short one, obtained as a simple consequence of (1.2), can be found in the recent preprint of Caldwell, Greenbaum, and Li [2].)

Crouzeix's proof of (1.3) is technical and requires a lot of work. In the Crouzeix–Palencia article [5], the passage from (1.2) to a bound for $\|f(T)\|$ is quite different and much simpler. It is effected using an abstract functional-analysis argument, which, for convenience, we summarize in the form of a lemma.

Lemma 1.1. *Let T be a bounded Hilbert-space operator and let Ω be a bounded open set containing the spectrum of T. Suppose that, for each $f \in A(\Omega)$, there exists $g \in A(\Omega)$ such that*

$$\|g\|_\Omega \leq \|f\|_\Omega \quad \textit{and} \quad \|f(T) + g(T)^*\| \leq 2\|f\|_\Omega. \tag{1.4}$$

Then

$$\|f(T)\| \leq (1 + \sqrt{2})\|f\|_\Omega \quad (f \in A(\Omega)). \tag{1.5}$$

To prove (1.1), this lemma is applied with Ω a smoothly bounded open convex set containing $\overline{W(T)}$ and with $g := C\overline{f}$. The second inequality in (1.4) is then just (1.2). The first inequality is a fundamental property of the Cauchy transform, namely, that $f \mapsto C\overline{f}$ is a contraction of $A(\Omega)$ into itself when Ω is convex (see, e.g., [5, Lemma 2.1]). Thus (1.5) holds, and the main result (1.1) then follows upon "shrinking" Ω down to $W(T)$.

It is quite striking that neither the numerical range nor the Cauchy transform appears explicitly in Lemma 1.1. They enter merely through the inequalities (1.4).

Our purpose in this note is to give a very short proof of Lemma 1.1 (even simpler than the argument given in [5]) and to show that, in the context of this lemma, the constant $(1 + \sqrt{2})$ is sharp. We conclude the article with a brief discussion of how these remarks relate to Crouzeix's conjecture.

2. Short proof of Lemma 1.1. Let K denote the norm of the continuous homomorphism $f \mapsto f(T) : A(\Omega) \to B(H)$. Our goal is to show that $K \leq 1+\sqrt{2}$.

Let $f \in A(\Omega)$ with $\|f\|_\Omega \leq 1$. By (1.4), there exists $g \in A(\Omega)$ such that $\|g\|_\Omega \leq 1$ and $\|f(T)+g(T)^*\| \leq 2$. We then have

$$f(T)f(T)^*f(T)f(T)^* = f(T)\big(f(T)+g(T)^*\big)^* f(T)f(T)^* - (fgf)(T)f(T)^*,$$

and, since $fgf \in A(\Omega)$, it therefore follows that

$$\|f(T)\|^4 \leq \|f(T)+g(T)^*\|\|f(T)\|^3 + \|(fgf)(T)\|\|f(T)\| \leq 2K^3+K^2.$$

Taking the supremum over all f with $\|f\|_\Omega \leq 1$, we deduce that $K^4 \leq 2K^3+K^2$, whence $K \leq 1+\sqrt{2}$, as desired.

3. Sharpness of Lemma 1.1. We exhibit a pair (T,Ω), satisfying all the hypotheses of Lemma 1.1, such that equality holds in (1.5) for a particular (nonzero) choice of $f \in A(\Omega)$. This shows that, in the context of Lemma 1.1, the constant $(1+\sqrt{2})$ is sharp.

Let

$$T := \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix},$$

and let $\Omega := \Omega_0 \cup \Omega_1$, where Ω_j is the open disk with center j and radius 1/4. Clearly Ω contains the spectrum of T. Also, if $f \in A(\Omega)$, then

$$f(T) = \begin{pmatrix} f(1) & f(1)-f(0) \\ 0 & f(0) \end{pmatrix}. \tag{3.1}$$

Indeed, this is clear if $f(z)=z^n$, since $T^n = T$ for all $n \geq 1$. The identity for general f follows by the linearity and continuity of the map $f \mapsto f(T)$. (It could also be deduced directly from the definition of $f(T)$ as a Cauchy integral.)

Given $f \in A(\Omega)$, define $g \in A(\Omega)$ by

$$g(z) := \begin{cases} -\overline{f(0)}, & z \in \Omega_0, \\ -\overline{f(1)}, & z \in \Omega_1. \end{cases}$$

Clearly $\|g\|_\Omega \leq \|f\|_\Omega$. Also

$$\begin{aligned} \|f(T)+g(T)^*\| &= \left\| \begin{pmatrix} f(1) & f(1)-f(0) \\ 0 & f(0) \end{pmatrix} + \begin{pmatrix} -f(1) & 0 \\ -(f(1)-f(0)) & -f(0) \end{pmatrix} \right\| \\ &= |f(1)-f(0)| \left\| \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\| \leq 2\|f\|_\Omega. \end{aligned}$$

Thus (1.4) holds.

Finally, let $h \in A(\Omega)$ be given by

$$h(z) := \begin{cases} -1, & z \in \Omega_0, \\ 1, & z \in \Omega_1. \end{cases}$$

Then $\|h\|_\Omega = 1$ and

$$\|h(T)\| = \left\| \begin{pmatrix} h(1) & h(1)-h(0) \\ 0 & h(0) \end{pmatrix} \right\| = \left\| \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} \right\| = 1+\sqrt{2}.$$

Thus equality holds in (1.5) when $f = h$.

4. Crouzeix's conjecture. As mentioned in the introduction, Crouzeix has conjectured that $\overline{W(T)}$ is always a 2-spectral set for T. What does the example in section 3 tell us about the conjecture?

First of all, the example is *not* a counterexample to the conjecture, because in the example $W(T) \not\subset \Omega$. Indeed, the eigenvalues $0, 1$ of T belong to different components Ω_0, Ω_1 of Ω, whereas the numerical range $W(T)$ is a convex set containing 0 and 1. In fact, Crouzeix's conjecture is known to be true for 2×2 matrices (see [3]).

What the example *does* tell us is that it is not possible to improve upon the constant $(1 + \sqrt{2})$ merely by adjusting the proof of Lemma 1.1. In this sense, the grouping of terms in the proof presented in section 2 is already optimal.

If one is to approach the conjecture along the same lines as in Lemma 1.1, then more information is needed about the choice of g. Here is one possibility. Recall that, in the application of Lemma 1.1, $g = C\overline{f}$, the Cauchy transform of $\overline{f}$. The map $f \mapsto C\overline{f}$ is both antilinear and unital (i.e., it maps 1 to 1). In the example in section 3, the map $f \mapsto g$ is also antilinear, but it is *not* unital; on the contrary it sends $1 \mapsto -1$. This suggests the following question.

Question 4.1. *Let T be a bounded Hilbert-space operator and let Ω be a bounded open set containing the spectrum of T. Suppose that there exists a unital antilinear map $\alpha : A(\Omega) \to A(\Omega)$ such that, for all $f \in A(\Omega)$,*

$$\|\alpha(f)\|_\Omega \leq \|f\|_\Omega \quad \textit{and} \quad \|f(T) + (\alpha(f))(T)^*\| \leq 2\|f\|_\Omega.$$

Does it follow that

$$\|f(T)\| \leq 2\|f\|_\Omega \quad (f \in A(\Omega))?$$

An affirmative answer to this question would prove the Crouzeix conjecture.

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