On input-to-state-stability and integral input-to-state-stability for parabolic boundary control systems

Birgit Jacob[1] Robert Nabiullin[1] Jonathan Partington[2] Felix Schwenninger[3]

Abstract— **This work contributes to the recently intensified study of input-to-state stability for infinite-dimensional systems. The focus is laid on the relation between input-to-state stability and integral input-to-state stability for linear systems with a possibly unbounded control operator. The main result is that for parabolic diagonal systems both notions coincide, even in the setting of inputs in L^∞, and a simple criterion is derived.**

I. Introduction

The concept of *input-to-state stability*, introduced by E. Sontag in 1989 [Son89], is a well-studied stability notion of control systems with respect to external inputs. For a survey on input-to-state stability for finite-dimensional systems we refer the reader to [Son08]. A variant of classic input-to-state stability is the notion of *integral input-to-state stability*, see e.g., [Son98]. We note that for linear, finite-dimensional systems input-to-state stability and integral input-to-state stability are equivalent and hence, the interest in different types of input-to-state stability lies in the study of nonlinear systems then.

For infinite-dimensional systems, input-to-state stability and integral input-to-state stability have been less studied, but more intensively in the recent past, see [DM13a], [DM13b], [JLR08], [Log13], [Mir16], [MI14], [MI15], [MW15], [KK16]. See also [MW16] for a study on the failure of equivalences in infinite-dimensions, which are known to hold true for finite-dimensional systems. In contrast to finite dimensions, even the case of linear systems is still not fully understood. This contribution aims to shed more light on the latter situation. In most of the references mentioned above, general nonlinear systems are studied, however, in such a way that the special case of linear equations is only covered when bounded control operators are considered. Concerning applications, this is a major restriction, see e.g. [TW09]. Moreover, if the system is linear and the control operator is bounded, then is easy to see that input-to-state stability and integral input-to-state stability are equivalent. Therefore, the focus of this paper is to allow for unbounded control operators and to address the question how these stability concepts are related. For linear, infinite-dimensional systems, the notion of admissibility, [Wei89], [Sal84], has proved to be very useful for the study of unbounded control operators. It is known that input-to-state stability is equivalent to admissibility (together with exponential stability). We will show that integral input-to-state stability in fact implies *zero-class admissibility* [JPP09], [XLY08], which is slightly stronger than admissibility.

In this paper we study systems $\Sigma(A,B)$ of the form

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = x_0, \quad t \geq 0, \quad (1)$$

where A generates a C_0-semigroup on a Hilbert space X and B is a linear, unbounded operator defined on the input space U. This class of systems covers in particular linear partial differential equations with boundary control. Furthermore, will restrict our study to input-to-state stability and integral input-to-state stability with respect to L^∞. We remark that the corresponding questions for L^p, $p \in [1,\infty)$, are less interesting as the notions coincide then, see e.g. [JNPS16].

By the relation to admissibility, input-to-state stability follows from integral input-to-state stability. We prove that integral input-to-state stability moreover implies zero-class admissibility, Proposition 2.9.

We then consider parabolic diagonal systems, that is, we assume that A possesses a Riesz basis of eigenvectors with eigenvalues lying in a sector in the open left half-plane and that the input space U is one-dimensional. Our main result states that, for such systems, integral input-to-state stability is equivalent to input-to-state stability and equivalent to the fact that B is a linear bounded operator from U to the extrapolation space X_{-1}, see Theorem 3.1.
Finally, we illustrate the obtained results by an example of a heat equation with boundary control.

II. Definitions

We study systems $\Sigma(A,B)$ of the form in (1) where B is a linear and bounded operator from a Hilbert space U to the extrapolation space X_{-1}. Note that B is possibly unbounded from U to X. Here X_{-1} is the completion of X with respect to the norm $\|x\|_{X_{-1}} = \|(\beta - A)^{-1}x\|_X$ for some β in the resolvent set $\rho(A)$ of A. The semigroup $(T(t))_{t\geq 0}$ extends uniquely to a C_0-semigroup $(T_{-1}(t))_{t\geq 0}$ on X_{-1} whose generator A_{-1} is an extension of A, see e.g. [EN00]. Thus we may consider Equation (1) on the Hilbert space X_{-1}. For $u \in L^1_{loc}(0,\infty;U)$ the mild solution of (1) is given by the variation of parameters formula

$$x(t) = T(t)x_0 + \int_0^t T_{-1}(t-s)Bu(s)ds, \qquad t \geq 0. \quad (2)$$

The notion of *admissibility* of the system $\Sigma(A,B)$ guarantees that the state $x(t)$ lies in X.

[1]B. Jacob and R. Nabiullin are with Functional Analysis group, School of Mathematics and Natural Sciences, University of Wuppertal, 42119 Wuppertal, Germany {jacob,nabiullin}@math.uni-wuppertal.de,

[2]Jonathan Partington is with School of Mathematics, University of Leeds, Leeds LS2 9JT, Yorkshire, United Kingdom j.r.partington@leeds.ac.uk

[3]F. Schwenninger is with Department of Mathematics, University of Hamburg, Bundesstraße 55, D-20146 Hamburg, Germany, felix.schwenninger@uni-hamburg.de

Definition 2.1: System $\Sigma(A,B)$ is called *admissible* if

$$\forall t>0, u\in L^\infty(0,t;U): \quad \int_0^t T_{-1}(s)Bu(s)\;ds \in X. \quad (3)$$

It follows that if $\Sigma(A,B)$ is admissible, then all mild solutions (2) are in X and by the Closed Graph Theorem there exists a constant $c(t)$ (take the infimum over all possible constants) such that

$$\left\|\int_0^t T_{-1}(s)Bu(s)\;ds\right\| \le c(t)\|u\|_{L^\infty(0,t;U)}.$$

If (3) holds for $t=\infty$, then $\Sigma(A,B)$ is called *infinite-time admissible*.

If the semigroup $(T(t))_{t\ge 0}$ is exponentially stable, that is, there exist constants $M,\omega>0$ such that

$$\|T(t)\|\le Me^{-\omega t}, \quad t\ge 0, \quad (4)$$

then $c=\sup_{t\ge 0} c(t)<\infty$, and it is easy to see that infinite-time admissibility is equivalent to admissibility. Beside classic admissibility, we are also interested in the following refinement.

Definition 2.2: We call the system $\Sigma(A,B)$ *zero-class admissible* if the system is admissible and $\lim_{t\to 0} c(t)=0$.

Remark 2.3: If $\Sigma(A,B)$ is zero-class admissible, then for every $x_0\in X$ and every $u\in L^\infty(0,\infty;U)$ the mild solution of (1), given by (2), satisfies $x\in C([0,\infty);X)$. This is proved similarly to Proposition 2.3 in [Wei89], see [JNPS16].

We will need the following well-known function classes from Lyapunov theory.

$$\begin{aligned}
\mathcal{K} &= \{\mu:\mathbb{R}_0^+\to\mathbb{R}_0^+ \mid \mu(0)=0, \mu \text{ continuous},\\
&\quad \text{strictly increasing}\},\\
\mathcal{K}_\infty &= \{\theta\in\mathcal{K} \mid \lim_{x\to\infty}\theta(x)=\infty\},\\
\mathcal{L} &= \{\gamma:\mathbb{R}_0^+\to\mathbb{R}_0^+ \mid \gamma \text{ continuous},\\
&\quad \text{strictly decreasing}, \lim_{t\to\infty}\gamma(t)=0\},\\
\mathcal{KL} &= \{\beta:(\mathbb{R}_0^+)^2\to\mathbb{R}_0^+ \mid \beta(\cdot,t)\in\mathcal{K}\ \forall t \wedge \beta(s,\cdot)\in\mathcal{L}\ \forall s\}.
\end{aligned}$$

Definition 2.4:

1) A system $\Sigma(A,B)$ is called *input-to-state stable* if the mild solution $x(t)$ lies in X for every $t\ge 0$ and there exist functions $\beta\in\mathcal{KL}$ and $\mu\in\mathcal{K}_\infty$ such that

$$\|x(t)\|\le\beta(\|x_0\|,t)+\mu(\|u\|_\infty) \quad (5)$$

for every $t\ge 0$, $x_0\in X$ and $u\in L^\infty(0,\infty;U)$.

2) A system $\Sigma(A,B)$ is called *integral input-to-state stable* if the mild solution $x(t)$ lies in X for every $t\ge 0$ and there exist functions $\beta\in\mathcal{KL}$, $\theta\in\mathcal{K}_\infty$ and $\mu\in\mathcal{K}$ such that

$$\|x(t)\|\le\beta(\|x_0\|,t)+\theta\left(\int_0^t\mu(\|u(s)\|)ds\right) \quad (6)$$

for every $t\ge 0$, $x_0\in X$ and $u\in L^\infty(0,\infty;U)$.

It follows immediately that A generates an exponentially stable C_0-semigroup if the system $\Sigma(A,B)$ is (integral) input-to-state stable. The following results are easily seen from the definition of admissibility and input-to-state stability. Proofs and related results can be found in [MW15, Thm. 4, Thm. 6 and Prop. 7] and [JNPS16].

Proposition 2.5: Suppose B is a bounded operator from U to X and A generates an exponentially stable C_0-semigroup. Then the system $\Sigma(A,B)$ is input-to-state stable, integral input-to-state stable, infinite-time admissible and zero-class admissible.

Remark 2.6: Let $\Sigma(A,B)$ as in Proposition 2.5. Then the system $\Sigma(A,B)$ is input-to-state stable with the following choices for the functions β and μ

$$\beta(s,t):=Me^{-\omega t}s \quad\text{and}\quad \mu(s):=\frac{M}{\omega}\|B\|s,$$

and integral input-to-state stable with

$$\beta(s,t):=Me^{-\omega t}s, \qquad \mu(s):=s, \quad\text{and}\quad \theta(s):=sM\|B\|.$$

Here the constants M and ω are given by (4).

For unbounded B, we still have the following result.

Proposition 2.7: Suppose A generates an exponentially stable C_0-semigroup. Then the following statements are equivalent.

1) System $\Sigma(A,B)$ is input-to-state stable,
2) System $\Sigma(A,B)$ is infinite-time admissible,
3) System $\Sigma(A,B)$ is admissible.

Remark 2.8: If one of the equivalent conditions of Proposition 2.7 hold, then the system $\Sigma(A,B)$ is input-to-state stable with the following choices for the functions β and μ

$$\beta(s,t):=Me^{-\omega t}s \quad\text{and}\quad \mu(s):=cs,$$

where M and ω are given by (4) and $c=\sup_{t\ge 0}c(t)$.

Proposition 2.9: If the system $\Sigma(A,B)$ is integral input-to-state stable, then $\Sigma(A,B)$ is zero-class admissible.

Proof: There exist $\theta\in\mathcal{K}_\infty$ and $\mu\in\mathcal{K}$ such that

$$\frac{1}{\|u\|_\infty}\left\|\int_0^t T_{-1}(s)Bu(s)\,ds\right\| \le \theta\left[\int_0^t \mu\left[\frac{\|u(s)\|_U}{\|u\|_\infty}\right]ds\right] \quad (7)$$

for all $t>0$, $u\in L^\infty(0,t;U)$, $u\neq 0$. Since the function μ is monotonically increasing and $\|u(s)\|_U\le\|u\|_\infty$ a.e., the right-hand side of (7) is bounded above by $\theta(t\mu(1))$ which converges to zero as $t\searrow 0$. ■

By the above results, it is clear that integral input-to-state stability implies input-to-state stability.

The relations of the different stability notions discussed above are illustrated in the diagram depicted in Figure 1.

III. Diagonal Systems

In this section we assume that $U=\mathbb{C}$ and that the operator A possesses a Riesz basis of eigenvectors $(e_n)_{n\in\mathbb{N}}$ with eigenvalues $(\lambda_n)_{n\in\mathbb{N}}$ lying in a sector in the open left half-plane $\mathbb{C}_-$. More precisely, let $(e_n)_{n\in\mathbb{N}}$ be a Riesz basis of X, that is, a basis such that, for some constants $c_1,c_2>0$ we have

$$c_1\sum_k|a_k|^2\le\left\|\sum_k a_ke_k\right\|^2\le c_2\sum_k|a_k|^2$$

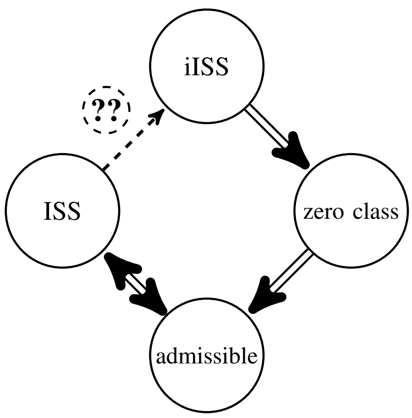

Fig. 1. Relation between the different stability notions for a system $\Sigma(A, B)$ (where we assume that the semigroup is exponentially stable). ISS refers to input-to-state stability, iISS to integral input-to-state stability, "zero-class" to zero-class admissibility and "admissible" to admissibility.

for all sequences (a_k) in ℓ^2. Thus without loss of generality we can assume that $X = \ell^2$ and that $(e_n)_{n\in\mathbb{N}}$ is the canonical basis of ℓ^2. We further assume that the sequence $(\lambda_n)_{n\in\mathbb{N}}$ lies in $\mathbb{C}$ with $\sup_n \operatorname{Re}(\lambda_n) < 0$ and that there exists a constant $k > 0$ such that $|\operatorname{Im}\lambda_n| \le k|\operatorname{Re}\lambda_n|$, $n \in \mathbb{N}$. Then the linear operator $A : D(A) \subset \ell^2 \to \ell^2$ is given by

$$Ae_n = \lambda_n e_n, \qquad n \in \mathbb{N},$$

and $D(A) = \{(x_n) \in \ell^2 \mid \sum |x_n\lambda_n|^2 < \infty\}$. A generates an analytic, exponentially stable C_0-semigroup $(T(t))_{t\ge0}$ on ℓ^2, which is given by $T(t)e_n = e^{t\lambda_n}e_n$. The extrapolation space $(\ell^2)_{-1}$ is given by

$$(\ell^2)_{-1} = \left\{ x = (x_n)_{n\in\mathbb{N}} \mid \sum_n \frac{|x_n|^2}{|\lambda_n|^2} < \infty \right\},$$
$$\|x\|_{X_{-1}} = \|A^{-1}x\|_{\ell^2}.$$

Thus any linear bounded operator B from $\mathbb{C}$ to $(\ell^2)_{-1}$ can be identified with a sequence $(b_n)_{n\in\mathbb{N}}$ in $\mathbb{C}$ satisfying

$$\sum_{n\in\mathbb{N}} \frac{|b_n|^2}{|\lambda_n|^2} < \infty.$$

Thanks to the sectoriality condition for $(\lambda_n)_{n\in\mathbb{N}}$ this is equivalent to

$$\sum_{n\in\mathbb{N}} \frac{|b_n|^2}{|\operatorname{Re}\lambda_n|^2} < \infty.$$

The following result shows that, under these assumptions, the system $\Sigma(A, B)$ is integral input-to-state stable. Thus for this class of systems all stability notions introduced in the previous section are equivalent to $B \in (\ell^2)_{-1}$, that is, to $\sum_n \frac{|b_n|^2}{|\lambda_n|^2} < \infty$.

Theorem 3.1: Let $U = \mathbb{C}$, and assume that the operator A possesses a Riesz basis of X consisting of eigenvectors $(e_n)_{n\in\mathbb{N}}$ with eigenvalues $(\lambda_n)_{n\in\mathbb{N}}$ lying in a sector in the open left half-plane $\mathbb{C}_-$ and $B \in \mathcal{L}(U, X_{-1})$. Then the system $\Sigma(A, B)$ is integral input-to-state stable, and hence also input-to-state stable and zero-class admissible.

Lemma 3.2: Let $\Sigma(A, B)$ be as in Theorem 3.1. Then there exists $M > 0$ and $\mu \in \mathcal{K}_\infty$ such that

$$\left\| \int_0^t T_{-1}(s)Bu(s)\, ds \right\|^2 \le M + \int_0^t \mu(|u(s)|)\, ds, \qquad (8)$$

for all $t > 0$ and all $u \in L^1(0, t)$ with $\int_0^t \mu(|u(s)|)\, ds < \infty$.

Proof: We may assume that $X = \ell^2$ with the canonical basis $(e_n)_{n\in\mathbb{N}}$. Let $f\colon (0,\infty) \to [0,\infty)$ be defined by

$$f(s) = \sum_{n\in\mathbb{N}} \frac{|b_n|^2}{|\operatorname{Re}\lambda_n|} e^{\operatorname{Re}\lambda_n s}.$$

Then it is easy to see that f is smooth, strictly decreasing, belongs to $L^1(0,\infty)$, and satisfies $\lim_{s\searrow0} f(s) = \infty$ and $\lim_{s\to\infty} f(s) = 0$.

We remark that boundedness of $(\operatorname{Re}\lambda_n)_{n\in\mathbb{N}}$ implies boundedness of $(\lambda_n)_{n\in\mathbb{N}}$. Thus if the sequence $(\operatorname{Re}\lambda_n)_{n\in\mathbb{N}}$ is bounded or $b_n = 0$ for all but finitely many $n \in \mathbb{N}$, then B is a bounded operator from $\mathbb{C}$ to ℓ^2 and therefore $\Sigma(A, B)$ is integral input-to-state stable by Proposition 2.5. Moreover, the series defining the function f is absolutely convergent and

$$\frac{|b_n|^2}{|\operatorname{Re}\lambda_n|} e^{\operatorname{Re}\lambda_n s} + \frac{|b_m|^2}{|\operatorname{Re}\lambda_m|} e^{\operatorname{Re}\lambda_m s} = \frac{|b_n|^2 + |b_m|^2}{|\operatorname{Re}\lambda_n|} e^{\operatorname{Re}\lambda_n s}$$

if $\operatorname{Re}\lambda_n = \operatorname{Re}\lambda_m$. Thus without loss of generality we may assume that $\operatorname{Re}\lambda_n < \operatorname{Re}\lambda_m$ for $m < n$, $b_n \ne 0$ for $n \in \mathbb{N}$ and B is unbounded. By Remark 178 in [Kno28] there is a strictly increasing unbounded sequence $(h_n)_{n\in\mathbb{N}}$ of positive numbers such that the series

$$\sum_{n\in\mathbb{N}} \frac{h_n|b_n|^2}{|\operatorname{Re}\lambda_n|^2}$$

converges. We define the smooth, strictly decreasing function $g\colon (0,\infty) \to [0,\infty)$ by

$$g(s) = \sum_{n\in\mathbb{N}} \frac{h_n|b_n|^2}{|\operatorname{Re}\lambda_n|} e^{\operatorname{Re}\lambda_n s},$$

for $s > 0$. Clearly, $g \in L^1(0,\infty)$. The function $\eta\colon [0,\infty) \to (0,\infty)$, $\eta(s) = g'(s)/f'(s)$, is strictly decreasing, see [JNPS16]. In particular the following limit exists

$$a := \lim_{s\to\infty} \frac{g'(s)}{f'(s)} \ge 0.$$

We define the smooth function $\Phi\colon [0,\infty) \to [0,\infty)$ by $\Phi(0) = 0$ and $\Phi(f(s)) = g(s) - af(s)$. Φ is a Young function, that is, $\Phi'(0) = 0$, $\lim_{s\to\infty} \Phi'(s) = \infty$ and Φ is strictly increasing and strictly convex, see [JNPS16].

Define $\Phi^*\colon [0,\infty) \to [0,\infty)$ by

$$\Phi^*(s) = \int_0^s (\Phi')^{-1}(t)\, dt$$

and $\mu\colon [0,\infty) \to [0,\infty)$ by $\mu(s) := \Phi^*(s^2)$. The function Φ^* is continuous, strictly increasing and unbounded. Thus $\mu \in \mathcal{K}_\infty$.

Let $u \in L^1(0,t)$ such that $\int_0^t \mu(|u(s)|)\, ds < \infty$. We have that

$$\left\|\int_0^t T_{-1}(s)Bu(s)\, ds\right\|^2 = \sum_{n\in\mathbb{N}} |b_n|^2 \left|\int_0^t e^{\lambda_n s} u(s)\, ds\right|^2$$
$$\le \sum_{n\in\mathbb{N}} |b_n|^2 \left(\int_0^t e^{\operatorname{Re}\lambda_n s}|u(s)|\, ds\right)^2$$
$$= \sum_{n\in\mathbb{N}} \frac{|b_n|^2}{(\operatorname{Re}\lambda_n)^2}\left(\int_0^t |\operatorname{Re}\lambda_n| e^{\operatorname{Re}\lambda_n s}|u(s)|\, ds\right)^2$$
$$\le \sum_{n\in\mathbb{N}} \frac{|b_n|^2}{(\operatorname{Re}\lambda_n)^2}\left(\int_0^t |\operatorname{Re}\lambda_n| e^{\operatorname{Re}\lambda_n s}|u(s)|^2\, ds\right)$$
$$\left(\int_0^t |\operatorname{Re}\lambda_n| e^{\operatorname{Re}\lambda_n s}\, ds\right)$$
$$\le \sum_{n\in\mathbb{N}} \frac{|b_n|^2}{(\operatorname{Re}\lambda_n)^2}\left(\int_0^t |\operatorname{Re}\lambda_n| e^{\operatorname{Re}\lambda_n s}|u(s)|^2\, ds\right)$$
$$= \int_0^t \sum_{n\in\mathbb{N}} \frac{|b_n|^2}{|\operatorname{Re}\lambda_n|} e^{\operatorname{Re}\lambda_n s}|u(s)|^2\, ds$$
$$= \int_0^t f(s)|u(s)|^2\, ds,$$

where we have used Cauchy-Schwarz with respect to the measure given by $|\operatorname{Re}\lambda_n| e^{\operatorname{Re}\lambda_n s}\, ds$. By Young's inequality (see e.g. [Ada75, Page 264]), we can further conclude that

$$\left\|\int_0^t T_{-1}(s)Bu(s)\, ds\right\|^2 \le \int_0^t f(s)|u(s)|^2\, ds$$
$$\le \int_0^t \left(\int_0^{f(s)} \Phi'(r)\, dr + \int_0^{|u(s)|^2} (\Phi')^{-1}(r)\, dr\right) ds$$
$$= \int_0^t \Phi(f(s))\, ds + \int_0^t \mu(|u(s)|)\, ds.$$

This shows (8) with $M := \|g - af\|_{L^1(0,\infty)}$. ■

Remark 3.3: Lemma 3.2 shows that $\Sigma(A,B)$ is *uniformly bounded energy bounded state (UBEBS)*, a weakened form of integral input-to-state stability introduced in [ASW99].

Proof-sketch of Theorem 3.1: By Lemma 3.2, the following choice for $\theta\colon [0,\infty) \to [0,\infty)$ seems to be a suitable candidate to show (6).

$$\theta(\alpha)^2 = \sup\left\{\left\|\int_0^t T_{-1}(s)Bu(s)\, ds\right\|^2 : u \in L^\infty(0,\infty),\ t \ge 0, \int_0^t \mu(|u(s)|)\, ds \le \alpha\right\}.$$

In fact, $\theta(\alpha) < \infty$ for all $\alpha \ge 0$ and θ is non-decreasing. It is easy to see that there exists a continuous, strictly increasing function $\tilde\theta$ such that $\theta \le \tilde\theta$ pointwise. Then the definition of θ yields that

$$\left\|\int_0^t T_{-1}(s)Bu(s)\, ds\right\| \le \tilde\theta\left(\int_0^t \mu(|u(s)|)\, ds\right)$$

for all $t \ge 0$, $u \in L^\infty(0,\infty)$. To conclude that $\Sigma(A,B)$ is integral input-to-state stable, we need that $\lim_{t\searrow 0}\tilde\theta(t) = 0$.

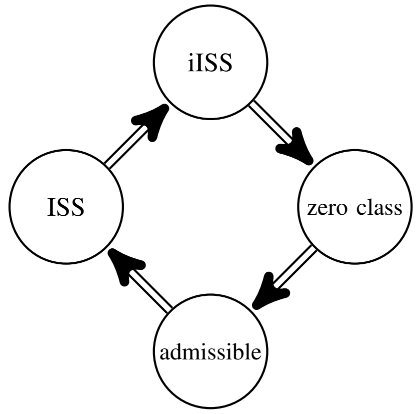

Fig. 2. Relations between the different stability notions for parabolic diagonal system (assuming that the semigroup is exponentially stable).

As it is not clear whether $\lim_{t\searrow 0}\theta(t) = 0$, the choice of μ and θ has to be revisited. In fact, the issue can be resolved by a slight adaption in the choice of μ and incorporating the theory of Orlicz spaces. We refer to [JNPS16] for details. ■

The relations of the different stability notions for parabolic diagonal systems are summarized in the diagram shown in Figure 2.

IV. An Example

Let us consider the following boundary control system given by the one-dimensional heat equation on the spatial domain $[0,1]$ with Neumann boundary control at the point 1,

$$\frac{\partial}{\partial t}x(\xi,t) = \frac{\partial^2}{\partial \xi^2}x(\xi,t), \quad \xi \in (0,1), t > 0,$$
$$\frac{\partial}{\partial \xi}x(0,t) = 0, \quad \frac{\partial}{\partial \xi}x(1,t) = u(t), \quad t > 0,$$
$$x(\xi,0) = x_0(\xi),$$

see e.g., [JPP14, Example 3.6]. It can be shown that this system can be written in the form $\Sigma(A,B)$ in (1). Here $X = L^2(0,1)$ and

$$Af = \frac{\partial^2}{\partial \xi^2}f, \quad f \in D(A),$$
$$D(A) = \left\{f \in L^2(0,1) : f, \frac{\partial}{\partial \xi}f \text{ are absolutely continuous}, \frac{\partial^2}{\partial \xi^2}f \in L^2(0,1), \frac{\partial}{\partial \xi}f(0) = \frac{\partial}{\partial \xi}f(1) = 0\right\}.$$

Moreover, with $\lambda_n = -\pi^2 n^2$,

$$Ae_n = \lambda_n e_n, \quad n \in \mathbb{N},$$

where the functions $e_0 = 1$ and $e_n = \sqrt{2}\cos(n\pi\cdot)$, $n \ge 1$, form an orthonormal basis of X. With respect to this basis, the operator $B = b$ can be identified with $(b_n)_{n\in\mathbb{N}}$ for $b_n = 1$, $n \in \mathbb{N}$. Therefore,

$$\sum_{n\in\mathbb{N}} \frac{|b_n|^2}{|\lambda_n|^2} < \infty,$$

which shows that $b \in X_{-1}$. By Theorem 3.1, we conclude that the system is integral input-to-state stable.

ERRATA: Note that the above A does not generate an exponentially stable semigroup. Thus $\Sigma(A, B)$ can not be integral input-to-state stable. However, the example can be repaired by considering

$$\frac{\partial}{\partial t} x(\xi, t) = \frac{\partial^2}{\partial \xi^2} x(\xi, t) - x(\xi, t), \quad \xi \in (0, 1), t > 0,$$
$$\frac{\partial}{\partial \xi} x(0, t) = 0, \quad \frac{\partial}{\partial \xi} x(1, t) = u(t), \quad t > 0,$$
$$x(\xi, 0) = x_0(\xi),$$

instead, which results in $A = \frac{\partial^2}{\partial \xi^2} - \pi^2 I$. Thus, we have the same eigenvalues λ_n as before, for $n > 1$. In fact, A generates an analytic, exponentially stable semigroup and, by the same reasoning as above, $\Sigma(A, B)$ is integral input-to-state stable.

A choice of functions β, μ, θ satisfying (6) is given by

$$\beta(s, t) := e^{-\pi^2 t} s, \quad \mu(s) := s^p, \quad \text{and} \quad \theta(s) := c \cdot s^{\frac{1}{p}},$$

for $p \geq \frac{4}{3}$ and some constant $c = c(p) > 0$. This follows from the fact that $\Sigma(A, B)$ is even L^p-admissible for $p \geq \frac{4}{3}$, see [JPP14, Example 3.6]. However, we remark that there exists examples of parabolic diagonal systems satisfying the assumptions of Theorem 3.1, but such that they are not L^p-admissible for any $p < \infty$.

V. Conclusions and Outlook

In this paper we have studied the relation between input-to-state stability and integral input-to-state stability for linear infinite-dimensional systems with an unbounded control operator and inputs in L^∞. We have shown that for parabolic diagonal systems and scalar input, both notions coincide and are equivalent to admissibility.

Among possible directions for future research are the investigation of the non-analytic case and the relation of zero-class admissibility, input-to-state stability and admissibility with respect to Orlicz spaces. Some of these topics are addressed in the [JNPS16].

VI. Acknowledgment

The authors would like to thank Andrii Mironchenko for pointing out a gap in a previous version of the proof of Theorem 3.1.

References

[Ada75] R.A. Adams. *Sobolev spaces*. Academic Press, New York-London, 1975. Pure and Applied Mathematics, Vol. 65.

[ASW99] D. Angeli, E.D. Sontag, and Y. Wang. Further equivalences and semiglobal versions of integral input to state stability. 1999.

[DM13a] S. Dashkovskiy and A. Mironchenko. Input-to-state stability of infinite-dimensional control systems. *Mathematics of Control, Signals, and Systems*, 25(1):1–35, August 2013.

[DM13b] S. Dashkovskiy and A. Mironchenko. Input-to-State Stability of Nonlinear Impulsive Systems. *SIAM Journal on Control and Optimization*, 51(3):1962–1987, May 2013.

[EN00] K.-J. Engel and R. Nagel. *One-parameter semigroups for linear evolution equations*, volume 194 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2000.

[JLR08] B. Jayawardhana, H. Logemann, and E.P. Ryan. Infinite-dimensional feedback systems: the circle criterion and input-to-state stability. *Commun. Inf. Syst.*, 8(4):413–414, 2008.

[JNPS16] B. Jacob, R. Nabiullin, J.R. Partington, and F. Schwenninger. Infinite-dimensional input-to-state stability and Orlicz spaces. *preprint*, 2016.

[JPP09] B. Jacob, J.R. Partington, and S. Pott. Zero-class admissibility of observation operators. *Systems Control Lett.*, 58(6):406–412, 2009.

[JPP14] B. Jacob, J.R. Partington, and S. Pott. Applications of Laplace-Carleson embeddings to admissibility and controllability. *SIAM J. Control Optim.*, 52(2):1299–1313, 2014.

[KK16] I. Karafyllis and M. Krstic. ISS in different norms for 1-D parabolic PDEs with boundary disturbances. Preprint available at arXiv: 1605.01364, 2016.

[Kno28] K. Knopp. Theory and application of infinite series. London: Blackie & Sons. XII, 571 pp., 1928.

[Log13] H. Logemann. Stabilization of well-posed infinite-dimensional systems by dynamic sampled-data feedback. *SIAM J. Control Optim.*, 51(2):1203–1231, 2013.

[MI14] A. Mironchenko and H. Ito. Integral input-to-state stability of bilinear infinite-dimensional systems. In *Proc. of the 53th IEEE Conference on Decision and Control*, pages 3155–3160, 2014.

[MI15] A. Mironchenko and H. Ito. Construction of Lyapunov Functions for Interconnected Parabolic Systems: An iISS Approach. *SIAM Journal on Control and Optimization*, 53(6):3364–3382, 2015.

[Mir16] A. Mironchenko. Local input-to-state stability: Characterizations and counterexamples. *Systems & Control Letters*, 87:23–28, 2016.

[MW15] A. Mironchenko and F. Wirth. A note on input-to-state stability of linear and bilinear infinite-dimensional systems. In *Proc. of the 54th IEEE Conference on Decision and Control*, pages 495–500, 2015.

[MW16] A. Mironchenko and F. Wirth. Restatements of input-to-state stability in infinite dimensions: what goes wrong. In *Proc. of 22th International Symposium on Mathematical Theory of Systems and Networks (MTNS 2016)*, 2016.

[Sal84] D. Salamon. *Control and Observation of Neutral Systems*, volume 91 of *Research Notes in Math.* Pitman, Boston, London, 1984.

[Son89] E.D. Sontag. Smooth stabilization implies coprime factorization. *IEEE Trans. Automat. Control*, 34(4):435–443, 1989.

[Son98] E.D. Sontag. Comments on integral variants of ISS. *Systems Control Lett.*, 34(1-2):93–100, 1998.

[Son08] E.D. Sontag. Input to state stability: basic concepts and results. In *Nonlinear and optimal control theory*, volume 1932 of *Lecture Notes in Math.*, pages 163–220. Springer Berlin, 2008.

[TW09] M. Tucsnak and G. Weiss. *Observation and control for operator semigroups*. Birkhäuser Advanced Texts: Basler Lehrbücher. Birkhäuser Verlag, Basel, 2009.

[Wei89] G. Weiss. Admissibility of unbounded control operators. *SIAM J. Control Optim.*, 27(3):527–545, 1989.

[XLY08] G.Q. Xu, C. Liu, and S.P. Yung. Necessary conditions for the exact observability of systems on Hilbert spaces. *Systems Control Lett.*, 57(3):222–227, 2008.