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Fourier Analysis – Exercise sheet 4
(to be discussed on May 28)

Note that we may also discuss Ex. 3.3 from sheet 3.

Ex 4.1: *(Local Behavior of Fourier series)*
The goal is to show that if two functions $f, g \in L^1(\mathbb{T})$ coincide in an open interval of $\mathbb{T}$, then the Fourier series either both converge pointwise on this interval to the same limit or both diverge. In order to conclude this show the following for functions $f \in L^1(\mathbb{T})$

(a) If $\int_{-\varepsilon}^{\varepsilon} \left| \frac{f(s)}{s} \right| ds < \infty$ for some $\varepsilon > 0$ then $\lim_{n\to\infty}(D_n * f)(0) \to 0$.
Hint: You may want to use the representation of D_n in terms of sines and use elementary facts on trigonometric functions.

(b) If $\int_{-\varepsilon}^{\varepsilon} \left| \frac{f(t+s)-f(t)}{s} \right| ds < \infty$ for some $\varepsilon > 0$ then $\lim_{n\to\infty}(D_n * f)(t) \to f(t)$.

(c) Conclude the above statement.

(d) Refine the statement in the following way: On any compact subinterval of the considered interval the pointwise convergence of the Fourier series of $f - g$ to 0 is uniform.
Hint: this requires a "uniform version" of the Riemann–Lebesgue lemma in the sense that the Fourier coefficients of a compact set of $L^1(\mathbb{T})$-functions tend to 0 (at ∞) uniformly).

Ex. 4.2: Show that for $X = C^k(\mathbb{T})$, $k \in \mathbb{N}$, the partial sums of the Fourier series $S(f)$ do not converge to f in X for general $f \in X$.

Ex 4.3: *(The Poisson kernel)*

(a) Show that

$$P_r(t) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} r^{|k|} e^{ikt}$$

defines an approximate identity $(P_r)_{r\in(0,1)} \subset L^1(\mathbb{T})$ indexed by $(0,1)$[1]. Here you may first show that

$$P_r(t) = \frac{1}{2\pi} \frac{1 - r^2}{1 - 2r\cos(t) + r^2}, \qquad \forall t \in \mathbb{T}, r \in (0,1).$$

(b) Conclude that for any homogeneous Banach space X we have that

$$P_r * f \to f \text{ in } X$$

as $r \to 1$ for any $f \in X$ (this is the version of the main result for approximate identities $(k_n)_{n\in\mathbb{N}}$).

(c) Show that $(P_r * f)(t) = \sum_{k=-\infty}^{\infty} r^{|k|} \hat{f}(k) e^{ikt}$.

(d) Show the following properties (analogous to the Fejér kernel) for all $r \in (0,1), t \in \mathbb{T}$.

$$P_r(t) \geq 0, \qquad P_r(t) = P_r(-t), \qquad \lim_{n\to\infty} \sup_{s\in[\delta,2\pi-\delta]} |P_r(s)| = 0$$

Furthermore show that $t \mapsto P_r(t)$ is decreasing for $t \in (0, \pi)$.

[1] this means that in the definition of an approximate identity we replace the sequence $(k_n)_n$ by the function $k : (0,1) \to L^1(\mathbb{T}), r \mapsto k_r$ where the conditions "$\forall n$" get replaced by $\forall r \in (0,1)$ and $\lim_{n\to\infty}$ by $\lim_{r\to 1}$.

(e) Show that if $f \in L^1(\mathbb{T})$ and $t_0 \in \mathbb{T}$ such that $L = \lim_{s\to 0} f(t_0+s) + f(t_0-s)$ exists, it follows that
$$\lim_{r\to 1}(P_r * f)(t_0) = \frac{L}{2}.$$
Hint: Start in a similarly as in the proof for the pointwise convergence of the Fejér means, Section 4), but then exploit the properties from Ex. 4.3(d)

<u>Ex 4.4:</u> Let $(P_r)_{r\in(0,1)}$ be the Poisson kernel from Ex. 4.3. We want to show that for $f \in L^1(\mathbb{T})$
$$\lim_{r\to 1^-} \sum_{k=-\infty}^{\infty} r^{|k|}\hat{f}(k)\mathrm{e}^{ikt} = \lim_{r\to 1^-}(P_r * f)(t) = f(t) \quad \text{for a.e. } t \in \mathbb{T}.$$
For that consider the steps:

(a) Show that if $f \in L^1(\mathbb{T})$, $t_0 \in \mathbb{T}$ and $L_{t_0} \in \mathbb{C}$ are such that
$$(*) \qquad \lim_{h\to 0^+} \frac{1}{h}\int_0^h \frac{1}{2}(f(t_0+s)+f(t_0-s)) - L_{t_0}\, ds = 0,$$
then $(P_r * f)(t_0) \to L_{t_0}$ as $r \to 1$. Note that in the corresponding result on the pointwise convergence of the Fejér means $F_n * f$ from the lecture, Section 4, we have assumed a slightly stronger assumption than $(*)$.
(Hint: This follows the same lines as the proof for the Fejér kernel — however, fill the "gap" we have left in the lecture and observe why we can use this weaker condition here..)

(b) Show that for any $f \in L^1(\mathbb{T})$ it holds that $(*)$ is satisfied with $L_{t_0} = f(t_0)$ for almost every t_0.

<u>Ex 4.5:</u> *Fourier coefficients of bounded linear functionals.*
For a homogeneous Banach space X consider the dual space X' of bounded linear functionals $\mu : X \to \mathbb{C}$. Let us assume that $\mathrm{e}^{in\cdot} \in X$ for all $n \in \mathbb{Z}$. For $\mu \in X'$ define
$$\hat{\mu}(n) = \frac{1}{2\pi}\overline{\langle \mathrm{e}^{in\cdot}, \mu\rangle_{X',X}} := \frac{1}{2\pi}\overline{\mu(\mathrm{e}^{in\cdot})} \qquad n \in \mathbb{Z}.$$

(a) Show that $|\hat{\mu}(n)| \le \frac{1}{2\pi}\|\mu\|_{X'}\|\mathrm{e}^{in\cdot}\|_B$ for all $n \in \mathbb{Z}$ and $\mu \in X'$.

(b) Show that this definition is consistent with our definition of Fourier coefficients in the case of $X = L^p(\mathbb{T}, p \in [1,\infty)$ (in which case X' is isomorphic to $L^q(\mathbb{T})$ with the Hölder conjugate q).

(c) Show that the following holds for $f \in X$ and $\mu \in X'$. The limit
$$\lim_{N\to\infty} \frac{1}{2\pi}\sum_{k=-N}^{N}(1-\frac{|k|}{N+1})\hat{f}(k)\hat{\mu}(k)$$
exists and equals $\langle f, \mu\rangle_{X,X'}$.

(d) Show that if $\hat{\mu}(n) = 0$ for all $n \in \mathbb{Z}$ then $\mu = 0 \in X'$.