Universität Hamburg
Fachbereich Mathematik
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Fourier Analysis - Exercise sheet 1

In the following let $L^p(\Omega, \mathcal{A}, \mu)$ always refer to the usual L^p spaces (of equivalence classes) of *complex-valued* μ-measurable functions with integrable p-th power of the modulus ((and the usual modification for $p = \infty$), where $(\Omega, \mathcal{A}, \mu)$ is a measure space. With the L^p-norm, these spaces are Banach spaces. Here, we will mainly be focused on the following choices; the torus $\Omega = \mathbb{T} = [0, 2\pi)$ — *which we identify with* $\mathbb{R}/2\pi\mathbb{Z}$ [1] — with the Lebesgue σ-algebra $\mathcal{B}$ and Lebesgue measure λ, $L^p(\mathbb{T}) = L^p(\mathbb{T}, \mathcal{B}, \lambda)$. Similarly, $C(\mathbb{T})$ refers to the 2π-periodic, complex-valued continuous functions on $\mathbb{R}$. Furthermore, we consider $\Omega = \mathbb{Z}$ and its discrete σ-algebra, with the counting measure $|\cdot|$ leading to $\ell^p(\mathbb{Z}) = L^p(\mathbb{Z}, |\cdot|)$.

In Lecture 1 we have encountered the operator

$$T : L^1(\mathbb{T}) \to \ell^\infty(\mathbb{Z}), f \mapsto \hat{f}$$

which maps integrable functions to their Fourier coefficients $\hat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} f(t) \mathrm{e}^{-int}\, dt$. The following exercises continue our (first) investigations on properties of the relation between a function and its Fourier series.

Ex 1:
Show that the statement

If sequence (f_n) converges to f in $L^p(\mathbb{T})$, it follows that $f_n \to f$ pointwise a.e.

is wrong for $p \in [1, \infty)$ and prove a suitable adaption of this statement which is true (hint: subsequence). What can we conclude for Fourier series of functions in $L^2(\mathbb{T})$ regarding pointwise convergence?

Ex 2:
Give an alternative proof of Theorem 1.6 (2), that is, the injectivity of the mapping $T : L^1(\mathbb{T}) \to \ell^\infty(\mathbb{Z}), f \mapsto \hat{f}$. Do this along the following steps

(1) Given that $\hat{f} = 0$ show that $\int_0^{2\pi} f(t) g(t)\, dt = 0$ for all $g \in L^\infty(\mathbb{T})$.
(2) Use fundamental result(s) from functional analysis to conclude that $f = 0$.

From the injectivity of T conclude that the following statement holds.

$\hat{f} \in \ell^1(\mathbb{Z})$ implies that the Fourier series of f,

$$S(f) = \sum_{n=-\infty}^{\infty} \hat{f}(n) \mathrm{e}^{int},$$

converges to $f(t)$ for almost every $t \in \mathbb{T}$. Hence, f is equal to a continuous function a.e..

Ex 3:
Consider the operators

$$\tilde{S}_N : \ell^\infty(\mathbb{Z}) \to L^1(\mathbb{T}), (a_n)_{n \in \mathbb{Z}} \mapsto (t \mapsto \sum_{n=-N}^{N} a_n \mathrm{e}^{int})$$

with $N \in \mathbb{N}$ and show that they are bounded, but not uniformly bounded, i.e. $\sup_{N \in \mathbb{N}} \|T_N\| = \infty$. Conclude that the operator $T : L^1(\mathbb{T}) \to \ell^\infty(\mathbb{Z}), f \mapsto \hat{f}$ is not surjective.

Ex 4:
Compute the (complex) Fourier series of the following functions $f : \mathbb{T} \to \mathbb{C}$

(a) $f(x) = |x|$

[1] That particularly means that we regard functions on $\mathbb{T}$ has 2π-periodic and defined on the whole of $\mathbb{R}$!

(b) $f = \chi_{[0,\pi]} - \chi_{[\pi,2\pi]}$, where χ_E denotes the indicator function on the set $E \subset \mathbb{R}$

(c) $f(x) = \cos^2(x) - \sin^2(x)$

Ex 5:
For $s \in \mathbb{T}$ let τ_s denote the shift operator

$$\tau_s : X \to X, f \mapsto f(\cdot + s)$$

which we consider for the following choices of spaces X:

$$X = L^p(\mathbb{T}) \text{ and } p \in [1, \infty], \quad C(\mathbb{T}).$$

(a) Convince yourself that τ_s is a bounded linear operator from X to X. Moreover, show that τ_s an isometric isomorphism (of Banach spaces).

(b) Let $p \in [1, \infty)$ and let X be either $L^p(\mathbb{T})$ or $C(\mathbb{T})$. Show that for any fixed $f \in X$, $s \mapsto \tau_s f$ is continuous as mapping from $\mathbb{T}$ to X (*hint: first consider* $X = C(\mathbb{T})$ *and reduce the other case to that one*).

(c) Show that the continuity in (b) is not *uniform* in f. In other words, show that $s \mapsto \tau_s$ is not continuous as operator from $\mathbb{T}$ to $\mathcal{B}(X)$[2]

(d) Show that the assertion in (b) is not true for $X = L^\infty(\mathbb{T})$.

(e) Show that $(\widehat{\tau_s f})(n) = \mathrm{e}^{ins} \hat{f}(n)$ and $(\widehat{\mathrm{e}^{i(\cdot)k} f}) = (\tau_k \hat{f})$ for $s \in \mathbb{T}$ and $k \in \mathbb{Z}$.

[2]here $\mathcal{B}(X)$ denotes the bounded linear operators from X to X.