Representation categories of vertex algebras

ZMP Seminar

Vertex operator algebras and topological field theories from twisted QFTs in 3d and 4d

Overview

1. Basic definitions
2. The big picture of vertex algebra representation categories
3. How to find representations

Definition: Vertex algebra (VA)

Data:

- vector space V
- vacuum vector $\Omega \in V$
- translation operator $T : V \to V$
- field map $Y : V \otimes V \to V((z))$

Axioms:

- vacuum axiom:
  $Y(\Omega, z) = \mathrm{id}_V$ and
  $Y(A, z)\Omega = A + zV[[z]], \ \forall A \in V$
- translation axiom:
  $T\Omega = 0$ and
  $[T, Y(A, z)] = \partial_z Y(A, z)$
- locality: $\forall A, B \in V$, $\exists n \in \mathbb{N}$
  $(z - w)^n [Y(A, z), Y(B, w)] = 0$

Consequence/Proposition

For all $A, B, C \in V$

$$Y(A, z)Y(B, w)C \in V((z))((w))$$

$$Y(B, w)Y(A, z)C \in V((w))((z))$$

$$Y(Y(A, z - w)B, w)C \in V((w))((z - w))$$

Expansion of same element in $V[[z, w]][z^{-1}, w^{-1}, (z - w)^{-1}]$.

The previous slide shows vertex algebras are close to associative commutative unital $\mathbb{C}$-algebras with a derivation. We also see that the fields $Y(A,z)$ are essentially an action of V on itself.

Definition: Vertex algebra module

Let (V,Ω,T,Y) be a vertex algebra. A V-module is a pair (M,Y_M): M a vector space and $Y_M : V\otimes M \to M((z))$ such that

- $Y_M(\Omega,z) = \mathrm{id}_M$
- $Y_M(TA,z) = \partial_z Y_M(A,z)$
- For all $A,B\in V$ and $C\in M$ the expansions
  $Y_M(A,z)Y_M(B,w)C \in M((z))((w))$
  $Y_M(B,w)Y_M(A,z)C \in M((w))((z))$
  $Y_M(Y(A,z-w)B,w)C \in M((w))((z-w))$
  can be identified in $M[[z,w]][z^{-1},w^{-1},(z-w)^{-1}]$.

Many additional assumtions can be added to the above definition. E.g. bounded conformal weights, finite weight spaces, semi simplicity, etc.

Let (V, Ω, T, Y) be a vertex algebra and $(M_1, Y_{m_1}), (M_2, Y_{M_2}), \dots$ be V-modules. In order use V for conformal field theory, one needs to consider chiral correlation functions:

$$\langle Y(A,z)\phi(m_1,x_1)\phi(m_2,x_2)\cdots\rangle$$

$$\langle \phi(Y(A,z-x_1)m_1,x_1)\phi(m_2,x_2)\cdots\rangle$$

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These are essentially V-multilinear maps.

Definition: Intertwining operator

Let (V, Ω, T, Y) be a vertex algebra and $(M_1, Y_{m_1}), (M_2, Y_{M_2}), (M_3, Y_{M_3})$ be V-modules. An intertwining operator of type $\binom{M_3}{M_1,\ M_2}$ is a map $\mathcal{Y} : M_1 \otimes M_2 \to M_3 x$ such that for all $m_i \in M_i$

- $\mathcal{Y}(m_1, z)m_2$ truncates below.
- $\mathcal{Y}(Tm_1, z) = \partial_z \mathcal{Y}(m_1, z)$.
- The expansions $Y_{M_3}(A, z)\mathcal{Y}(m_1, x)m_2 \sim \mathcal{Y}(Y_{M_1}(A, z-x)m_1, x)m_2 \sim \mathcal{Y}(m_1, x)Y_{M_2}(A, z)m_2$ can be identified (via VA version of Jacobi identity).

Observations:

- The field map Y is an intertwining operator of type $\binom{V}{V,\ V}$.
- The action Y_M is an intertwining operator of type $\binom{M}{V,\ M}$.
- Intertwining operators are V-bilinear maps. All intertwining operators of a given type form a vector space. The field map Y and the action Y_M have a distinguished normalisation due to $Y_*(\omega, z) = \mathrm{id}_*$.

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Tensor products pull multilinear algebra back to linear algebra!

Definition: Fusion product aka vertex algebra tensor product

Let (V, Ω, T, Y) be a vertex algebra and $(M_1, Y_{m_1}), (M_2, Y_{M_2})$ be V-modules. A fusion product is a triple $(M_1 \boxtimes M_2, Y_{M_1 \boxtimes M_2}, \mathcal{Y}_{M_1,M_2})$, where $(M_1 \boxtimes M_2, Y_{M_1 \boxtimes M_2})$ is a V-module and $\mathcal{Y}_{M_1,M_2}$ is an intertwining operator of type $\binom{M_1 \boxtimes M_2}{M_1,\ M_2}$ such that the following universal property holds: For every V-module (X, Y_X) and intertwining operator $\mathcal{Y}_X$ of type $\binom{X}{M_1,M_2}$

$$\begin{tikzcd} M_1 \otimes M_2 \arrow[r, "\mathcal{Y}_{M_1,M_2}"] \arrow[dr, "\mathcal{Y}_X"'] & M_1 \boxtimes M_2\{z\} \arrow[d, dashed, "\exists! f"] \\ & X\{z\} \end{tikzcd}$$

In contrast to linear algebra (or ring theory) constructing $M_1 \boxtimes M_2$ and decomposing into a direct sum of indecomposable modules is *extremely* hard.

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Example: Heisenberg vertex algebra. Let F_μ, be the Fock space of weight $\mu \in \mathbb{C}$. Then $\dim \binom{F_\rho}{F_\mu,\ F_\nu} = \delta_{\rho,\mu+\nu}$ for all $\rho, \mu, \nu \in \mathbb{C}$.
$\binom{F_{\mu+\nu}}{F_\mu,\ F_\nu}$ is spanned by

$$\mathcal{Y}_{F_\mu,F_\nu}(p|\mu\rangle, z)q|\nu\rangle = z^{\mu\nu} S_\mu \prod_{m\geq 1} \exp\left(\mu \frac{a_{-m}}{m} z^m\right) Y_{F_\nu}(p|0\rangle, z) \cdot \prod_{m\geq 1} \exp\left(-\mu \frac{a_m}{m} z^{-m}\right) q|\nu\rangle,$$

where S_μ is the shift operator.

Well chosen categories of modules are tensor categories with respect to $\boxtimes$ with the following structures.

- For module homomorphimsm $f : X \to Z$, $g : U \to W$, $f \boxtimes g$ is uniquely characterised by $(f \boxtimes g)\mathcal{Y}_{X\boxtimes U} = \mathcal{Y}_{Z\boxtimes W} \circ (f \otimes g)$
- V is the tensor identity and the unit isomorphisms are uniquely characterised by $\ell_M\left(\mathcal{Y}_{V,M}(a,z)m\right) = Y_M(a,z)m$ and $r_M\left(\mathcal{Y}_{M,V}(m,z)a\right) = e^{zT}Y_M(a,-z)m$.
- associativity isomorphisms (hardest part!) $A_{M_1,M_2,M_3}\left(\mathcal{Y}_{M_1,M_2\boxtimes M_3}(m_1,x_1)\mathcal{Y}_{M_2,M_3}(m_2,x_2)m_3\right) = \mathcal{Y}_{M_1\boxtimes M_2,M_3}(\mathcal{Y}_{M_1,M_2}(m_1,x_1)m_2,x_2)m_3$
All analytic details hidden.
- Braiding isomorphisms uniquely characterised by $c_{M_1,M_2}\left(\mathcal{Y}_{M_1,M_2}(m_1,x_1)m_2\right) = e^{zT}\mathcal{Y}_{M_2,M_1}(m_2,e^{i\pi}z)m_1$

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If the vertex algebra V is conformal (a vertex operator algebra) and the modules are chosen to be compatible with this conformal structure, then there is also a twist $\theta_M = e^{2\pi i L_0}|_M$, which satisfies the balancing equation
$\theta_{M_1 \boxtimes M_2} = c_{M_1,M_2} \circ c_{M_2,M_1} \circ (\theta_{M_1} \boxtimes \theta_{M_2})$

Tensor categories of vertex operator algebra modules depend only very weakly on the conformal structure. Only the twist and the dual (not discussed) depend on the conformal structure.

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Theorem [Huang '04]: The Verlinde Conjecture

Let (V, Ω, ω, Y) be a vertex operator algebra and $\mathrm{Adm}\, V$ be the category of admissible V-modules. If

1. $\dim V_0 = 1, \dim V_{-n} = 0, \dim V_n < \infty,\ n \in \mathbb{N}$,
2. V is simple as a module over itself,
3. $V \cong V'$, self-dual,
4. $\dim V/c_2(V) < \infty$,
5. $\mathrm{Adm}(V)$ is semisimple,

then $\mathrm{Adm}\, V$ is a modular tensor category. Further the action of the modular group on the category (which determines Verlinde's formula) is equal (after a renormalisation) to the action of the modular group on module characters.

Summary of what we've discussed so far

- Vertex algebras are almost commutative unital algebras with derivations.
- The conformal vector is a choice/structure: there can be 0, 1 or many.
- Vertex algebras admit modules. "Good choices" of module categories admit a tensor (aka fusion) product.
- With the exception of associators, the tensor structure morphisms follow from natural constructions and are easy to obtain.

Practical matters

- Nothing presented so far helps with actually finding modules.
- For commutative algebras, the regular module can be used to present any finitely generated module.
- This fails for vertex (operator) algebras. The vertex algebra as a module over itself is often assumed/required to be simple. In general almost all modules cannot be presented from sums of the vertex algebra.
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Definition/Proposition: Zhu's associative algebra, [Zhu '96]

Let (V, Ω, ω, Y) be a vertex operator algebra and consider the two binary operations

$$a \circ b = \operatorname{Res} Y(a,z)b\frac{(1+z)^{h_a}}{z^2}, \quad a * b = \operatorname{Res} Y(a,z)b\frac{(1+z)^{h_a}}{z}.$$

Let $O(V) = \operatorname{span}\{a \circ b | \forall a, b \in V\}$. Then the following hold.

1. $A(V) = V/O(V)$ is a unital associative algebra under the binary operation $*$.
2. The class of the vacuum vector $[\Omega] = \Omega + O(V)$ is the identity element.
3. The class of the Virasoro vector $[\omega] = \omega + O(V)$ lies in the centre.
4. Let M be a V-module with ground state space $\overline{M}$. On $\overline{M}$ $(a * b)_0 = a_0 b_0$ for all $a, b \in V$.
5. If $\overline{M}$ is a (left) module over $A(V)$ then it can be induced to a V-module with $\overline{M}$ as the space of ground states.

Virasoro algebra example:

- For the universal virasoro vertex operator algebra (V_c, Ω, ω, Y), of central charge $c \in \mathbb{C}$ we have $\mathbb{C}[X] \cong A(V_c)$, where the isormophism is given by $X \mapsto [\omega]$.
- At minimal model central charges $c_{p,q} = 1 - 6\frac{(p-q)^2}{pq}$, $p, q \geq 2$, $\gcd(p,q) = 1$, there is a singular vector $\chi \in V_{c_{p,q}}$ at degree $(p-1)(q-1)$. Under the above isomorphism $[\chi] \in A(V_{c_{p,q}})$ corresponds to some $f(X)$ and $A(V_{c_{p,q}}/\langle\chi\rangle) \cong C[X]/\langle f(X)\rangle$.
- For the Yang-Lee minimal model $p = 2, q = 5$
  $\chi = (L_{-2}^2 - \frac{3}{5}L_{-4})\Omega$
  $Y(\chi; z) = : T(z)^2 : - \frac{3}{10}\partial^2 T(z)$

Affine example:

- Let $\mathfrak{g}$ be a complex finite dimensional simple Lie algebra and let $V_k(\mathfrak{g})$ be the universal affine vertex operator algebra (conformal vector given by the Sugawara construction). Then $U(\mathfrak{g}) \cong A(V_k(\mathfrak{g}))$, where isomorphism is given by $x \mapsto [x_{-1}\Omega]$.
- If $k \in \mathbb{Z}_{\geq 0}$, $\left(e^{\theta}_{-1}\right)^{k+1}\Omega$ is singular and generates the maximal ideal. $A(V_k(\mathfrak{g})/\langle\left(e^{\theta}_{-1}\right)^{k+1}\Omega\rangle) \cong U(\mathfrak{g})/\langle\left(e^{\theta}\right)^{k+1}\rangle$ is finite dimensional and semi simple.
- $\mathfrak{sl}_2$ at $k = -\frac{4}{3}$. The singular vector $\chi = \left(h_{-3} + 3e_{-2}f_{-1} - 3e_{-1}f_{-2} + \frac{9}{2}h_{-1}e_{-1}f_{-1} + \frac{9}{8}h^3_{-1} - \frac{9}{4}h_{-2}h_{-1}\right)\Omega$ generates the unique non-trivial ideal.

Literature

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