FUSION 2-CATEGORIES

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Notes accompanying a talk given on Sections 2.1.1 and 2.1.2 of Christopher L. Douglas and David J. Reutter's "Fusion 2-categories and a state-sum invariant for 4-manifolds" [DR18] in Ingo Runkel and Christoph Schweigert's "Research Seminar Algebra and Mathematical Physics" at Hamburg University on Tuesday, January 24, 2023. None of this is my own work.

Objective of the talk. Defining fusion 2-categories and illustrating their graphical calculus.

The definition of fusion 2-categories

We now add a "monoidal structure" to the 2-categories studied so far. To that end, a short aside about "higher monoidality".

Remark. (a) *"Higher monoidality":* (for all but finitely many n yet undefined notion that) a *k-tuply monoidal* n-dimensional category is one equipped with k additional binary operations, interchanging via specified n-morphisms. Examples:
  - 0-tuply monoidal 0-dimensional category: set
  - 1-tuply monoidal 0-dimensional category: monoid
  - 2-tuply monoidal 0-dimensional category: commutative monoid
  - 0-tuply monoidal 1-dimensional category: category
  - 1-tuply monoidal 1-dimensional category: monoidal category ...

(b) *"Delooping hypothesis":* (consistency condition for potential definitions of higher categories, demanding an) $(n+k)$-dimensional equivalence between k-tuply monoidal n-dimensional categories and pointed $(k-1)$-connected $(n+k)$-dimensional categories (i.e., with singled out 0-morphisms and such that for any $j < k$ any two parallel j-morphisms are equivalent).

Whereas we could afford to be fairly cavalier about distinguishing between weak and strict 2-categories, from now on we need to be more careful.

Remark. (a) *Fusion 2-categories* are to be Vect-enriched monoidal bicategories with certain special properties (not additional structure).

(b) *Monoidal bicategories* are equivalent to pointed 0-connected tricategories (see [Sch09, Section 2.1] for the significance of being pointed).

(c) *Tricategories*, fully weak 3-dimensional categories, were defined in [GPS95, Definition 2.2].

(d) Optimal strictification result: Any tricategory is triequivalent to a Gray category ("strict (op-)cubical" tricategory) [GPS95, Theorem 8.1].
(e) *Gray categories* are categories enriched in 2-categories – but with respect to Gray monoidal structure (strong version of [Gra76]), as opposed to the Cartesian monoidal structure (which would yield 3-*categories* instead).

Gray categories are almost 3-categories. They are, and in a unique way, if their interchange 3-morphism is an identity.

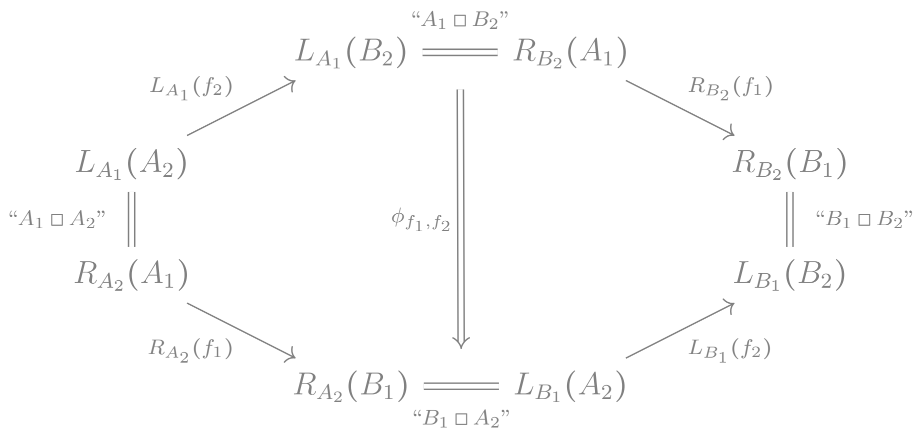
Definition. [DR18, Definition 2.1.1] A *Gray monoid* is any quintuple $(\mathcal{C}, I, L, R, \phi)$ such that
(i) $\mathcal{C}$ is a (strict) 2-category, the *underlying* 2-*category*,
(ii) I is a 0-morphism of $\mathcal{C}$, the *monoidal unit*,
(iii) L and R are families of (strict) 2-endofunctors of $\mathcal{C}$, each indexed by the 0-morphisms of $\mathcal{C}$, with

$$A_1 \,\square\, A_2 \coloneqq L_{A_1}(A_2) = R_{A_2}(A_1)$$

for any 0-morphisms A_1 and A_2, the *left and right monoidal products*,
(iv) ϕ is a family of invertible 2-cells of $\mathcal{C}$, indexed by pairs of 1-morphisms of $\mathcal{C}$, such that for any 1-morphisms f_1 and f_2, if $f_i\colon A_i \to B_i$ for each $i \in \{1, 2\}$, then

$$\phi_{f_1,f_2}\colon R_{B_2}(f_1) \circ L_{A_1}(f_2) \Rightarrow L_{B_1}(f_2) \circ R_{A_2}(f_1),$$

the *(monoidal product) interchange*,



and such that
(a) $L_I = R_I$ and both are the identity 2-functor on $\mathcal{C}$,
(b) for any 0-morphisms A_1, A_2 and A_3, as compositions of 2-functors,

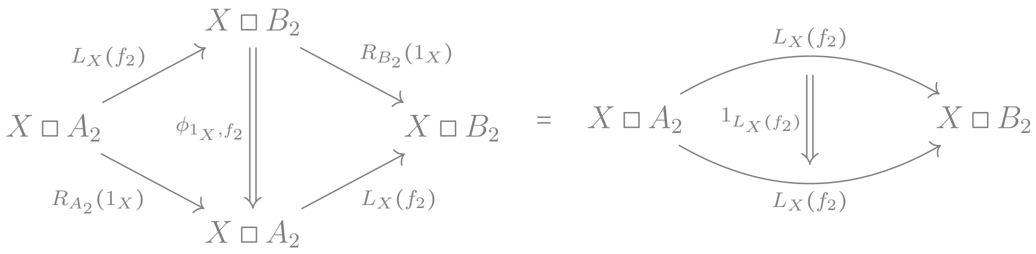
$$L_{A_1} L_{A_2} = L_{L_{A_1}(A_2)} \quad \text{and} \quad R_{R_{A_2}(A_1)} = R_{A_2} R_{A_1}$$

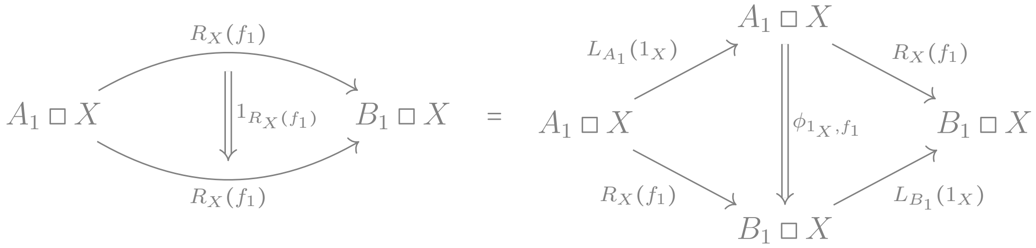
and also

$$L_{A_1} R_{A_3} = R_{A_3} L_{A_1},$$

(c) for any 1-morphisms f_1 and f_2 and any 0-morphism X,

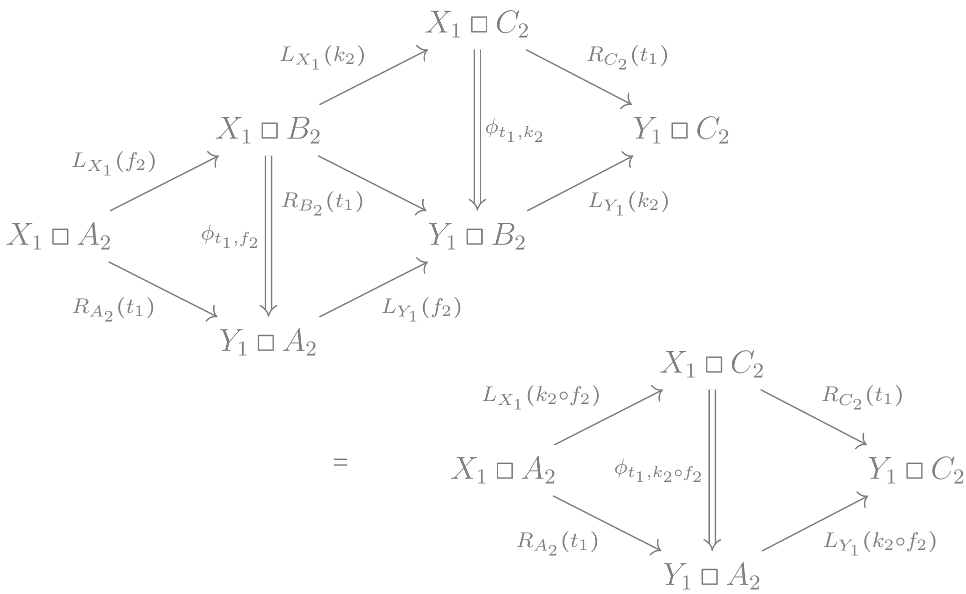
$$\phi_{1_X,f_2} = 1_{L_X(f_2)} \quad \text{and} \quad 1_{R_X(f_1)} = \phi_{f_1,1_X},$$





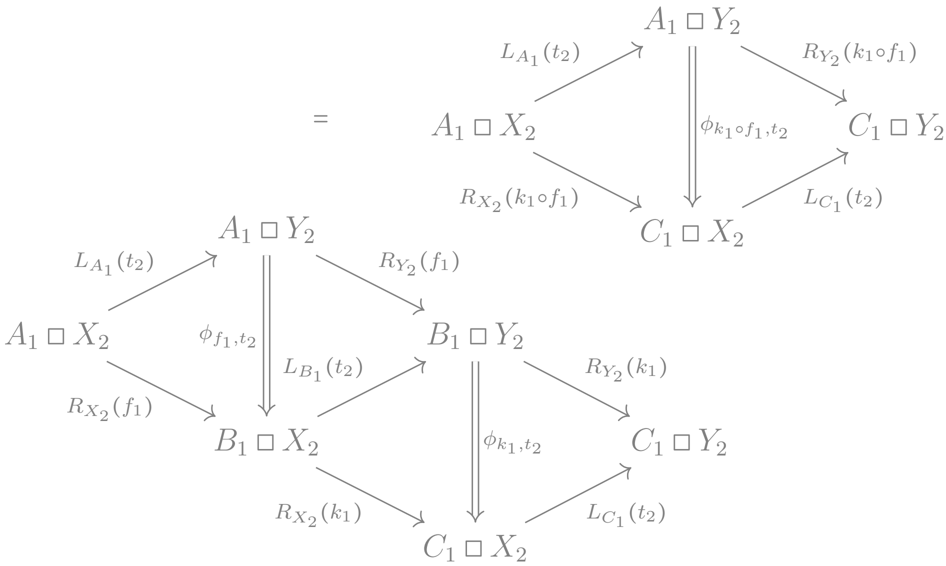
(d) given any 0-morphisms A_i, B_i, C_i, X_i and Y_i and any 1-morphisms $f_i\colon A_i \to B_i$ and $k_i\colon B_i \to C_i$ as well as $t_i\colon X_i \to Y_i$ for each $i \in \{1,2\}$,

$$\phi_{t_1,k_2\circ f_2} = (1_{L_{Y_1}(k_2)} \circ \phi_{t_1,f_2}) \cdot (\phi_{t_1,k_2} \circ 1_{L_{X_1}(f_2)})$$



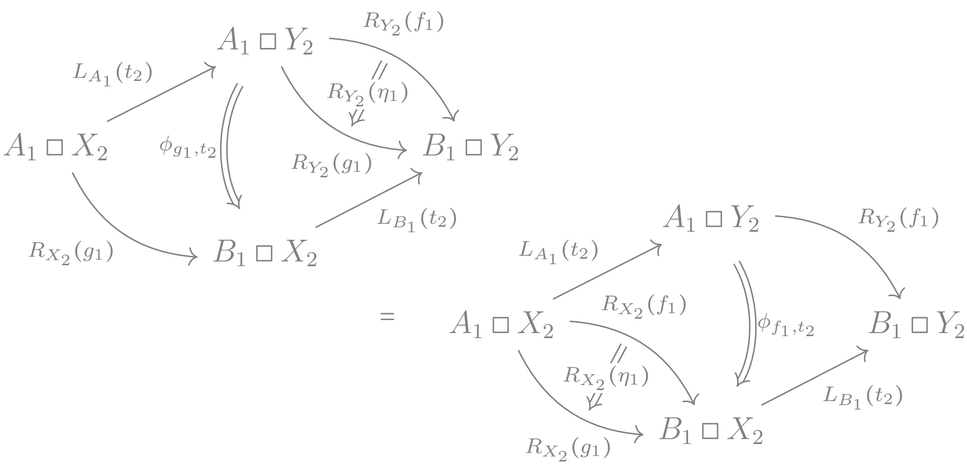
and

$$\phi_{k_1\circ f_1,t_2} = (\phi_{k_1,t_2} \circ 1_{R_{X_2}(f_1)}) \cdot (1_{R_{Y_2}(k_1)} \circ \phi_{f_1,t_2}),$$



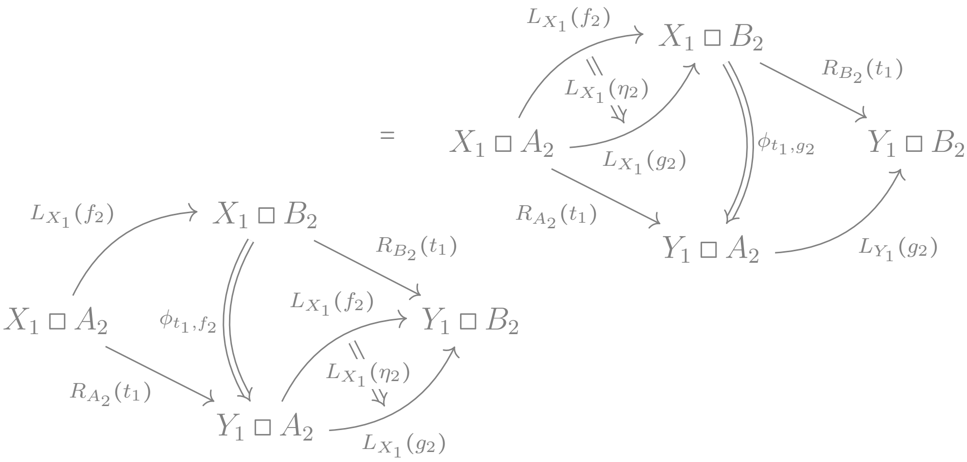
(e) given any 0-morphisms A_i, B_i, X_i and Y_i, any 1-morphisms $f_i\colon A_i \to B_i$ and $g_i\colon A_i \to B_i$ as well as $t_i\colon X_i \to Y_i$ and any 2-morphisms $\eta_i\colon f_i \Rightarrow g_i$ for each $i \in \{1,2\}$,

$$\phi_{g_1,t_2} \cdot (R_{Y_2}(\eta_1) \circ 1_{L_{A_1}(t_2)}) = (1_{L_{B_1}(t_2)} \circ R_{X_2}(\eta_1)) \cdot \phi_{f_1,t_2}$$



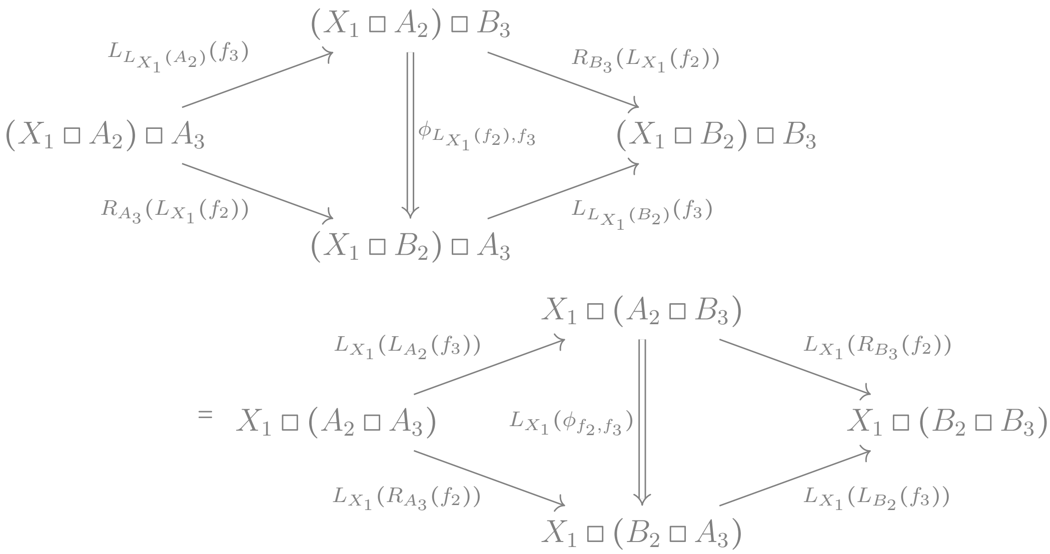
and

$$\phi_{t_1,g_2} \cdot (1_{R_{B_2}(t_1)} \circ L_{X_1}(\eta_2)) = (L_{Y_1}(\eta_2) \circ 1_{R_{A_2}(t_1)}) \cdot \phi_{t_1,f_2},$$



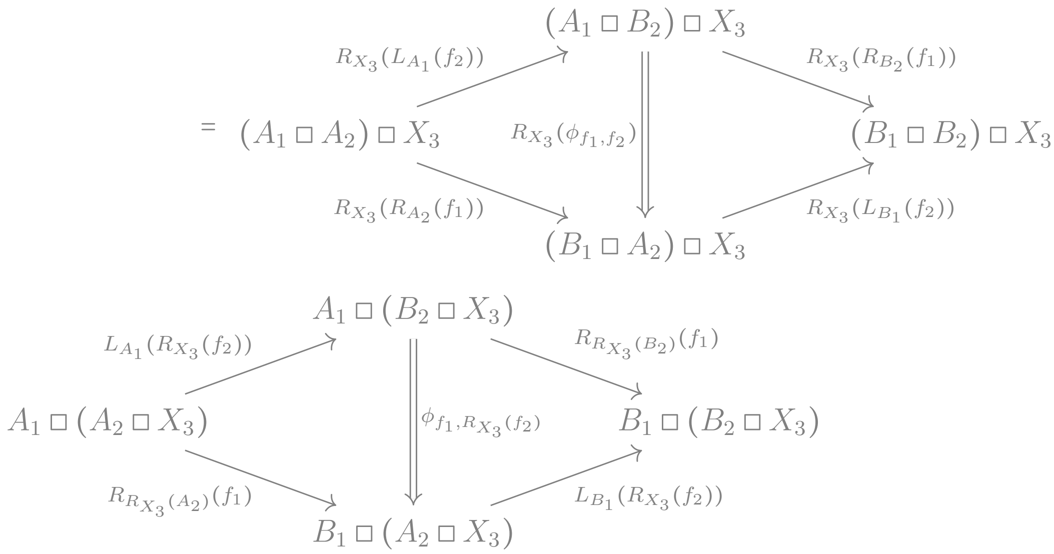
(f) given any 0-morphism X_i and any 1-morphism f_i for any $i \in \{1,2,3\}$,

$$\phi_{L_{X_1}(f_2),f_3} = L_{X_1}(\phi_{f_2,f_3})$$



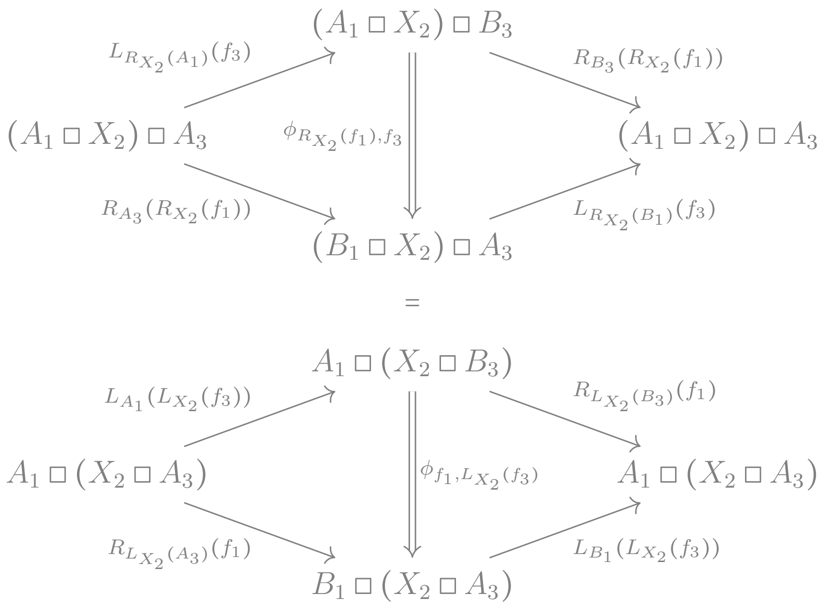
and

$$R_{X_3}(\phi_{f_1,f_2}) = \phi_{f_1,R_{X_3}(f_2)}$$



and also

$$\phi_{R_{X_2}(f_1),f_3} = \phi_{f_1,L_{X_2}(f_3)}.$$



It is not surprising that tricategories cannot be strictified to 3-categories, given the next example. After all, there are even symmetric monoidal categories which are not equivalent to one whose symmetry is an identity.

Examples. (a) In accordance with the delooping hypothesis, a *braided strict monoidal category* is evidently the same thing as a Gray monoid with a single 0-morphism. (In fact, the tricategory of pointed tricategories with (up to isomorphism) a single 0-morphism and a single 1-morphism and pointed tri-homomorphisms etc. is triequivalent to the the category of braided monoidal categories and braided monoidal functors etc. [GPS95, Propositions 8.6, 8.7].)

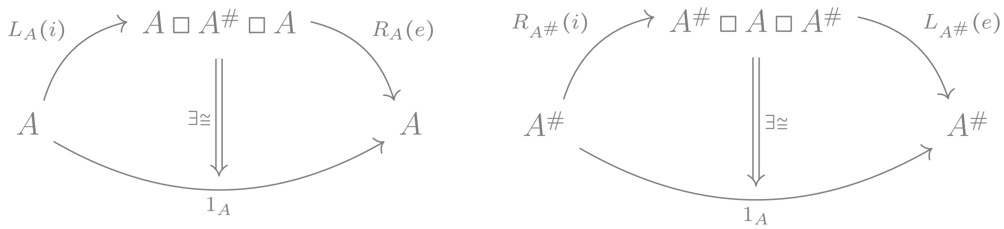
(b) On any (strict) 2-category the strict 2-endfunctors, pseudonatural transformations and modifications can be assembled into a Gray monoid, where the monoidal product comes from the composition of 2-functors.

The delooping hypothesis also motivates a definition of dual 0-morphisms in Gray monoids.

Remark. (a) *"Higher dual":* (for all but finitely many n undefined notion that) any 0-morphism $A^{\#}$ in a 1-tuply monoidal n-dimensional category is a right *n-dimensional dual* of any given 0-morphism A if $A^{\#}$ is a right $(n+1)$-dimensional adjoint to A in the delooping.

(b) *"Higher adjoints":* (for all but finitely many m undefined notion that) any 1-morphism g in any m-dimensional category is a right *m-dimensional adjoint* to any 1-morphism f if there are 2-morphisms $\varepsilon\colon f \circ g \Rightarrow 1$ and $\eta\colon 1 \Rightarrow g \circ f$ which satisfy the unit-co-unit equations up to 3-morphisms which are $(m-2)$-dimensional equivalences.

Definition. In any Gray monoid $(\mathcal{C}, I, L, R, \phi)$, any 0-morphism $A^{\#}$ is called a *right dual* of any 0-morphism A (or, equivalently, A a *left dual* of $A^{\#}$) if there exist 1-morphisms $e\colon L_A(A^{\#}) \to I$ and $i\colon I \to R_A(A^{\#})$ such that $R_A(e) \circ L_A(i)$ is 2-isomorphic to 1_A and $L_{A^{\#}}(e) \circ R_{A^{\#}}(i)$ to $1_{A^{\#}}$.



Remark. If the underlying 2-category of a Gray monoid admits left adjoints and right adjoints for any 1-morphisms, then also the evaluation and co-evaluation 1-morphisms e and i have "duals".

Graphical calculus of fusion 2-categories

Versions of a graphical calculus based on stratified 3-dimensional manifolds for Gray categories were developed independently in [Hum12] ("surface diagrams") and [BMS12]. A similar approach applicable to Gray monoids was pursued in [Bar14] ("wire diagrams").

Next time ...

Vect-enriched Gray monoids, possibly with duals, can be defined in the usual way.

Definition. A *fusion 2-category* is any Vect-enriched Gray monoid with duals and the property that the underlying Vect-enriched 2-category is finite semisimple and that there the monoidal unit is simple.

A range of examples will be presented by David Jaklitsch on January 25, 2023.

References

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