Holomorphic-topological twist of 3d $\mathcal{N}=2$ theories & (Boundary) Vertex Algebras

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Mainly based on [Costello, Dimofte, Gaiotto]
(also 1908.05791 (3d $\mathcal{N}=2$ 2-bracket), 2112.01559, 1512.00821 (Poisson algebras), 1804.06460) (Gaiotto et al. top. twist)

Plan
(i) Introduction & Motivation
(ii) Twisted 3d $\mathcal{N}=2$ theories & Vertex Algebras
(iii) Bulk-to-boundary map (+ Line ops perspective)
(iv) Example

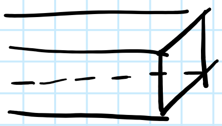
(i) In this talk we want to discuss the algebraic structure arising in the holomorphic-topological sector of 3d $\mathcal{N}=2$ theories.
Indeed, susy theories can be twisted by choosing a nihilpotent supercharge and restricting the local operators to be valued in the cohomology thereof.
This generates a closed subsector of the theory that is often easier to analyse and that presents still a rich and interesting structure. In our case the algebraic structure that arises is the one of a (commutative) Poisson VOA.
On top of being interesting already on its own, there is a further motivation. Indeed, local operators are not the mere content of a physical theories.
There is a plethora of extended operators that couple to the bulk theory producing interesting new structures (defect OPE's ...). Inserting those operators modify the bulk topology and enrich the algebraic structure of the theory.

If the extend operator is cod 1 ($d=2$), then it usually acts as a domain wall at an interface between different theories with given ∂ conditions. This is equivalent to consider the 3d theory on a manifold w/ boundary. Fixed a set of boundary condition for the fields of the theory we want to describe how the structure of the bulk is related to the one living on the ∂. We will see that, in favourable cases, we can fully reconstruct the bulk VOA having the mere knowledge of the ∂ one.

(ii) Let's now describe precisely the (partial) hol-top twist.

Consider a 3dimensional theory defined on a manifold $\mathcal{M}$ that we take, for now, to be such that it admits a <u>transverse holomorphic foliation</u>, i.e. we can think that locally $\mathcal{M}$ looks like $\underset{z,\bar{z}}{\mathbb{C}} \times \underset{t}{\mathbb{R}}$.

When we put boundary conditions this will get modified to $\mathbb{C} \times \mathbb{R}_{\geq 0}$



Given that we are just interested in the algebra of local ops, the global geometry of $\mathcal{M}$ does not play a role. Hence, we can safely work in $\mathcal{M} \equiv \mathbb{C} \times \mathbb{R}$ or $\mathbb{C} \times \mathbb{R}_{\geq}$ flat space.

[This story can be repeated for any $\mathbb{C} \times \mathbb{R}^d$ where we take all $\mathbb{R}^d$ to be topological]

$\boxed{\text{Recall}}$ 3d $\mathcal{N}=2$ SUSY in flat <u>Euclidean</u> space (4 supercharges)

$$\{Q_+, \bar{Q}_+\} = -2i\partial_{\bar{z}} \; ; \qquad \{Q_-, \bar{Q}_-\} = 2i\partial_z$$

$$\{Q_+, \bar{Q}_-\} = \{Q_-, \bar{Q}_+\} = i\partial_t \; ; \; \{Q_\pm, Q_\pm\} = \{\bar{Q}_\pm, \bar{Q}_\pm\} = 0$$

To perform a twist of the theory, we start by identifying a nilpotent supercharge of which we can consider the cohomology. In lieu of the algebra above, we find that that nilpotent supercharges are all of the form:

$$Q^{(1)} = a Q_+ + b Q_- \quad \text{or} \quad c \bar{Q}_- + d \bar{Q}_+ = Q^{(2)}$$

Hence, the nilpotency variety is a cone over $\mathbb{CP}^1 \sqcup \mathbb{CP}^1$ as the overall scaling is not important.

The cohomology of $Q^{(1)} = a Q_+ + b Q_-$ makes the theory topological along the line identified by $a/b \in \mathbb{CP}^1$ and anti-holomorphic in the transverse plane thereof as the derivatives are $Q^{(1)}$ exact.

Analogously the $Q^{(2)}$ cohomology is topological along a direction $c/d \in \mathbb{CP}^1$ and holomorphic in the transverse plane. Since $SO(3)$ rotations rotates each copy of $\mathbb{CP}^1$, we can always fix the splitting $M \simeq \mathbb{C} \times \mathbb{R}$ s.t. the cohomology is hol/anti-hol on $\mathbb{C}$ and top along t, so that, upon further restricting to the holomorphic case, we are left just with either $\bar{Q}_+$ or $\bar{Q}_-$. Choosing either of the two is equivalent up to a "PT" discrete symmetry acting as the antipodal map on $\mathbb{CP}^1$.

Hence, we can arbitrarily take the nilpotent charge $Q := \bar{Q}_+$ as any other choice is simply related to this by space-time + discrete isometries.

The Q-cohomology of local operators $\mathcal{V} = H^\bullet(\mathrm{Ops}, Q)$ has the structure of a Chiral algebra.

Indeed, in the cohomology of Q both $\partial_{\bar{z}}$ and ∂_t are exact, and therefore any correlation function of local operators is topological in t and independent of $\bar{z}$, i.e. it has just a non-trivial holomorphic dependence on the coordinate z. This is the origin of the name (partial) Hol-top twist.

$\langle O_1(z_1,\bar z_1,t_1)\cdots O_n(z_n,\bar z_n,t_n)\rangle = \langle O_1(z_1)\cdots O_n(z_n)\rangle$

i.e. $\partial_t \langle O_1(p_1)\cdots O_n(p_n)\rangle = 0 = \partial_{\bar z}\langle O_1(p_1)\cdots O_n(p_n)\rangle$

$p_i\in\mathbb{C}\times\mathbb{R}$

The bulk algebra is naturally graded by $\mathbb{Z}_R\times\mathbb{Z}_J$

where $J = J_0 - \frac{R}{2}$ is a twisted spin defined s.t.

J_0 spin for $Spin(2)_E \subset Spin(3)_E$

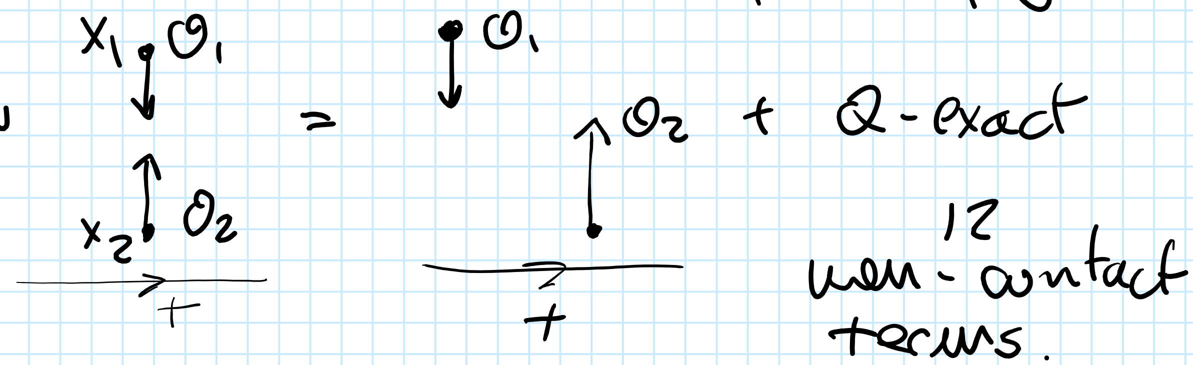
	Q	∂_z	
$U(1)_R$	$+1$	0	cohomological grading
$U(1)_J$	0	1	twisted spin

The VA $\mathcal{V}$ is an ABELIAN VA, i.e. all the OPEs are non-singular ($\mathcal{Z}(\mathcal{V})=\mathcal{V}$)

$\mathcal{Z}(\mathcal{V}) = \{O\in\mathcal{V}$ s.t. $O\,O^*$ has non-singular OPE $\forall O^*\in\mathcal{V}\}$

Indeed, any singular term in an OPE is generated by CONTACT TERMS. But since ∂_t is topological (euclidean)

contact term ~



there are no contact terms up to Q-exact terms. $\left[\lim_{(z,\bar z,t)\to(z',\bar z',t')} f(z,\bar z,t,z',\bar z',t') = \lim_{(z\to z')} f(z,z')\Big|_{t\neq t'} = \text{non sing.}\right.$

Then we can deduce a very important result. Namely $T_{zz}\notin\mathcal{V}$. To see that, recall that in any 3d $\mathcal{N}=2$ the S.E. tensor and the supercurrents $G_{\pm\mu}, \bar G_{\pm\mu}$ sit in the same multiplet with $T_{\mu\nu}$ being the h.w. rep. [cfr. Bertolini SISSA 4.6.2].

then, it follows that some of the components of $T_{\mu\nu}$ are Q-exact. [$\mathcal{N}=1$ $\{\bar{Q}_{\dot\alpha}, S_{\alpha\nu}\} \sim 2\sigma^{\mu}_{\alpha\dot\alpha} T_{\mu\nu}$]

Explicitly $T_{\mu\bar{z}} = \frac{i}{2} Q(G_{+\mu})$, $T_{\mu t} = -i\, Q(G_{-\mu})$.

this can also be understood by noting that $T_{\mu\bar{z}}$ and $T_{\mu t}$ generates $\partial_{\bar{z}}$ ∂_t that are Q-exact.

Rather ∂_z is not Q-exact, its action on any local op. $\mathcal{O}(z)$ can be expressed as: [$[Q, \mathcal{O}] = \exp(i \oint_{S^{d-1}} * j^{(1)})$]

$$\partial_w \mathcal{O}(w) = \oint_{S^2} *(T_{z\mu}\, dx^\mu)\mathcal{O}(w) = -i \oint_{S^2} T_{zz}\, dz \wedge dt\, \mathcal{O}(w) + Q\text{-exact}$$

where S^2 is a sphere centered in w.

But this implies that $\partial_w \mathcal{O}$ must appear in the singular part of the OPE between T_{zz} and $\mathcal{O}$ (Cauchy thm)

But since $\mathcal{O} \in \mathcal{V}$ that is abelian, $T_{zz} \notin \mathcal{V}$ (disappointing)

And therefore there seems to be no way to generate z-translation through an operator in $\mathcal{V}$.

Yet, the structure of $\mathcal{V}$ turns out to be richer compared to a standard VA: $\mathcal{V}$ is a POISSON (abelian) VA.

The poisson structure provides (as we will see in a moment) us with a "secondary" energy tensor $\in \mathcal{V}$ that indeed generates z-translations within $\mathcal{V}$.

Let us explain this in some details.

In addition to the "primary" product of local ops (pointwise operation) $\mathcal{O}'(x)\mathcal{O}(y)$, we can construct more (non-pointwise) operations. Those are called secondary products and involve integrals over sphere of suitably defined differential forms constructed out of local ops. these higher structure exists in any dimension (not only in 3d), here we illustrate the one that is relevant for our discussion: the λ-bracket.

[This construction always word with $\mathbb{R}^d \times \mathbb{C}$ w/ all $\mathbb{R}^d$ being topological.]

the λ-bracket is constructed in the following way: given two local ops $\mathcal{O}_1(z_1,\bar{z}_1,t_1)$, $\mathcal{O}_2(z_2,\bar{z}_2,t_2)$ their bracket at $\mathcal{O}_2$ is defined as:
$\mathcal{O}_1(z_1) = \mathcal{O}_1(z_1,\bar{z}_1,t_1)$, $\mathcal{O}_2(z_2) = \mathcal{O}_2(z_2,\bar{z}_2,t_2)$

$$\{\{\mathcal{O}_1 \; {}_\lambda \; \mathcal{O}_2\}\}(z_2) = \oint_{S^2_{(z_2,\bar{z}_2,t)}} e^{\lambda(z_1-z_2)}\, dz \wedge \mathcal{O}^{(1)}(z_1)\; \mathcal{O}_2(z_2).$$

$\mathcal{O}^{(1)}(z_1)$ is the 1-form descendant built out of the (0-form) local op $\mathcal{O}(z_1)$ acting with the differential

$$\mathcal{O}^{(1)}(x) = Q^1\,\mathcal{O}(x) \;;\; Q^1 = \tfrac{i}{2}Q_+ \, d\bar{z} - i\, Q_-\, dt.$$

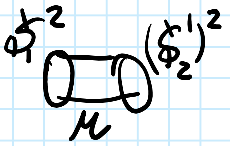
constructed s.t. $\{Q, Q^1\} = d^1 = \partial_{\bar{z}}\, d\bar{z} + \partial_t\, dt$ (exterior der)

Then $Q\,(dz \wedge \mathcal{O}^{(1)}(x)) = Q(dz \wedge Q^1\mathcal{O}(x)) = d(dz \wedge \mathcal{O}(x))$

where d is the total exterior der $d = \partial_z dz + \partial_{\bar{z}} d\bar{z} + \partial_t dt$

then the integral in $\{\{\;{}_\lambda\;\}\}$ depends only topologically on the choice of sphere as

$$\oint_{(S_1)^2_z} (\cdots) - \oint_{(S_2)^2_z} (\cdots) = \oint_{M^3} d(\cdots) = Q\int_{M^3} (\cdots) \equiv 0$$



where M^3 is s.t. $\partial M^3 = S^2 \cup (S^1)^2$.

So that, indeed, the λ-bracket does not depend on the choice of S^2.

The λ-bracket structure is not inherited from the cohomology one and is:

(i) Sesquilinear $\{\{\partial_z a \;{}_\lambda\; b\}\} = -\lambda\,\{\{a \;{}_\lambda\; b\}\}$;
$\{\{a \;{}_\lambda\; \partial_z b\}\} = (\lambda + \partial_z)\,\{\{a \;{}_\lambda\; b\}\}$

(ii) Skew-symmetric $\{\{a \;{}_\lambda\; b\}\} = -\{\{b \;{}_{-\lambda-\partial}\; a\}\}$

(iii) Satisfies the Jacobi identity
$$\{\{a \;{}_\lambda\; \{\{b \;{}_\mu\; c\}\}\}\} - \{\{b \;{}_\mu\; \{\{a \;{}_\lambda\; c\}\}\}\} = \{\{\{\{a \;{}_\lambda\; b\}\}_{\lambda+\mu}\; c\}\}$$

Furthermore, it is straightforward to show that the λ-bracket satisfies a Leibniz rule w.r.t. the pointwise multiplication $\{\{a \;{}_\lambda\; bc\}\} = \{\{a \;{}_\lambda\; b\}\}c + b\,\{\{a \;{}_\lambda\; c\}\}$

This makes $(\mathcal{V}, \{\{\;{}_\lambda\;\}\})$ a Poisson (abelian) VOA.

As promised this structure plays a crucial role. Indeed, if we take $G = -\frac{i}{2}\bar{G}_{-z}$, $QG = 0$ and $dz\wedge G^{(1)} = dz\wedge Q'G = -i\,T_{zz}\,dz\wedge dt + Q$-exact. Then if we set $\lambda \equiv 0$ (and $\{\{\cdot\,{}_0\,\cdot\}\} := \{\cdot\,,\cdot\}$)

$$\partial_w \mathcal{O}(w) = -i\oint_{S^2} T_{zz}\,dz\wedge dt\,\mathcal{O}(w) + Q\text{-exact}$$

$$= \oint_{S^2} dz\wedge G^{(1)}\,\mathcal{O}(w) = \{\{G, \mathcal{O}\}\}(w)$$

So that there exist an operator $G\in\mathcal{V}$ generating ∂_z translations through the λ-bracket. (So G is a secondary stress-energy tensor) You can think of $dz\wedge G^{(1)}$ as a stress-energy tensor in the chiral algebra.

(iii) Having described the algebraic structure of the Bulk algebra, we want to explore the effect of considering M^3 to be a manifold w/ a (single) boundary, i.e. $M \simeq \mathbb{C}\times\mathbb{R}_{\geq 0}$ locally.

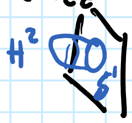
We want to consider susy boundary conditions that preserve Q and the $U(1)_R$ symmetry. This fixes uniquely the boundary theory to be a 2d $\mathcal{N}=(0,2)$ theory generated by $\{Q, Q_+\} = -2i\partial_{\bar z}$. ($\frac{1}{2}$-BPS) (same chirality "+").

The algebra of the boundary local ops is

$\mathcal{V}_\partial = H^\bullet(\mathrm{Ops}_\partial, Q)$ and is again a chiral algebra.

This time, since we do no longer have a topological direction, the OPE might be singular and $\mathcal{V}_\partial$ is, in general, not abelian. In general, there is no boundary stress-energy tensor $\in\mathcal{V}$ as in the presence of a boundary the $\partial_w\mathcal{O}(w)$ ($\mathcal{O}^\partial$ boundary local op.) is given by:

$$\partial_w\mathcal{O}(w) = \underbrace{\int_{H^2} Q'G\,\mathcal{O}(w)}_{\text{non-trivial}} + \oint_{S^1} *_\partial\left(T^\partial_{zz}\,dz + T^\partial_{z\bar z}\,d\bar z\right)\cdot\mathcal{O}(w)$$



So, even in absence of a boundary S.E tensor, we can still generate derivatives on the boundary.

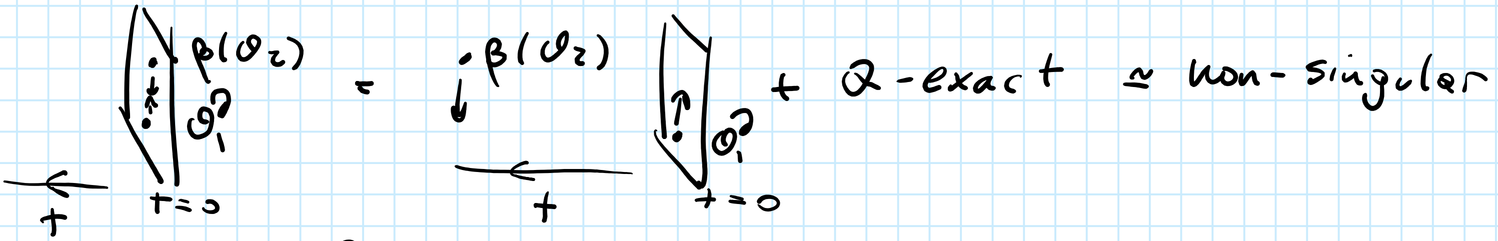
[Indeed, as we will see shortly, the presence of non-trivial ops in the bulk obstruct the existence of a S.E. tensor in the boundary.]

We want now to understand the relation between $\mathcal{V}$ and $\mathcal{V}_\partial$. Since the t direction is topological, we can construct a bulk-to-boundary map $\beta : \mathcal{V} \to \mathcal{V}_\partial$ that factors

$$\begin{tikzcd} \mathcal{V} \arrow[r] \arrow[d] & \mathcal{V}_\partial \\ \mathcal{Z}(\mathcal{V}_\partial) \arrow[ur] & \end{tikzcd}$$

Since $\mathcal{V}$ is commutative, β factors through the center:

$\mathcal{Z}(\mathcal{V}_\partial) = \{ \mathcal{O} \in \mathcal{V}_\partial$ s.t. the OPE $\mathcal{O}\mathcal{O}^*$ is not-singular $\forall \mathcal{O}^* \in \mathcal{V}_\partial \}$



$\mathrm{Im}\,\beta \subseteq \mathcal{Z}(\mathcal{V}_\partial) \hookrightarrow \mathcal{V}_\partial$. Furthermore, this map can be always be made surjective onto $\mathcal{Z}(\mathcal{V}_\partial)$. Indeed, if there are non trivial ops in $\mathcal{Z}(\mathcal{V}_\partial)$ that do not come from bulk ops, we can enhance the bulk algebra to couple it to a bigger bulk theory consistently. For instance we could tensor $\mathcal{V}_\partial$ with any 2d TQFT, and this would enlarge the center $\mathcal{Z}(\mathcal{V}_\partial) \to \mathcal{Z}(\mathcal{V}_\partial) \otimes$ 2d TQFT, but we could do the same on the bulk algebra.

Then, we might hope that $\beta : \mathcal{V} \twoheadrightarrow \mathcal{Z}(\mathcal{V}_\partial)$ is actually an isomorphism. Yet this is delicate. Indeed, if we take $\mathcal{O}_1, \mathcal{O}_2 \in \mathcal{V}$ with $\mathcal{O}_1 \neq \mathcal{O}_2$, it might happen that, once dragged to the boundary, $\mathcal{O}_1|_\partial = \mathcal{O}_2|_\partial + Q$-exact as boundary ops. So, in general, we expect $\ker\beta \neq 0$. [Note $\ker\beta \subsetneq \mathcal{V}_\partial$ as $\beta(\mathbb{1}) := \mathbb{1}_\partial \neq 0$]

This motivates us to try to use the higher λ-bracket structure to resolve $\ker\beta$.

Indeed, if two operators are different ops in the cohomology, then $\oint_{H^2} e^{\lambda x} dz \wedge \mathcal{O}_1^{(1)}(x)\, \mathcal{O}^\partial(0)$ and $\oint_{H^2} e^{\lambda x} dz \wedge \mathcal{O}_2^{(1)}\, \mathcal{O}^\partial(0)$

$$\shortparallel \qquad\qquad\qquad \shortparallel$$

same, i.e. $\{\{\mathcal{O}_1 \,{}_\lambda \mathcal{O}^\partial\}\}(0) \quad \neq \quad \{\{\mathcal{O}_2 \,{}_\lambda \mathcal{O}^\partial\}\}(0)$.

are not expected to define the same operator once dragged to the boundary, that we define as following $\tilde{\beta}(\mathcal{O}_1) = \mathcal{O}_1^\partial$ s.t. $\{\{\mathcal{O}_1^\partial \,{}_\lambda \mathcal{O}^\partial\}\}(z) = \{\{\mathcal{O}_1 \,{}_\lambda \mathcal{O}^\partial\}\}(z)$ $\forall \mathcal{O}^\partial \in \mathcal{V}^\partial$ $\forall z \in \mathbb{C}$

the idea behind the statement $\ker \tilde{\beta} = 0$, is that, given that the dependence on S^2 is topological, if two operators give rise to the same λ-bracket for any choice of $\mathcal{O}^\partial \in \mathcal{V}^\partial$ and any point z, then, they are the same, identical op$\mathcal{V}$ (e.g. this is similar to say that if $\int_a^b f = \int_a^b g \quad \forall\, a,b \Rightarrow f = g$.

The map as defined is a derived map (in the sense of homological algebra) $\beta_{der}: \mathcal{V} \longrightarrow \mathcal{Z}_{der}(\mathcal{V}_\partial)$.

The conjecture, motivated by 3d CS & consideration above, is that β_{der} indeed furnishes an isomorphism

$$\boxed{\mathcal{V} \cong \mathcal{Z}_{der}(\mathcal{V}_\partial)} \quad \text{for "sufficiently rich b.c."}$$

This is a remarkable result as it implies that in favorable cases one is able to fully reconstruct the bulk algebra with the mere knowledge of the 2d boundary theory. Cfr. 3.1 of Costello - Gaiotto for thorough explanation.

Another perspective to the same quest is thinking about line ops. Indeed, in 3d $\mathcal{N}=2$ theories $\frac{1}{2}$-BPS line & vortex operators supported on $\mathbb{C} \ni \{0\} \times \mathbb{R}_+$ are compatible w/ the Hol-Top twist.

Line ops form a category $\mathcal{C}$ and can terminate on module of the boundary algebras [e.g. fundamental Wilson line terminates on fundamental quarks etc.]. Hence, we can construct a functor
$\rho : \mathcal{C} \to \mathcal{V}_B$ that associate to a line its "endpoint" local op.
Note that the $\mathcal{V}_\partial$-module associated w/ the trivial line is nothing but the vacuum module, i.e. the whole $\mathcal{V}_\partial$. This sheds a new light on the bijection above. Indeed, bulk local operators are self-interfaces of the identity line $\underset{\mathbb{1}}{-}\!\!-\!\!\bullet\!\!-\!\!\underset{\mathbb{1}}{-}$ that are then computed as $\mathrm{Ext}^\bullet_{\mathcal{C}}(\mathbb{1}, \mathbb{1})$
(Equivalence classes of extensions)
Such that we can rephrase the conjecture above by saying that ρ furnishes an isomorphism

$$\mathcal{V} = \mathrm{Ext}^\bullet_{\mathcal{C}}(\mathbb{1}, \mathbb{1}) \underset{\rho}{\cong} \mathrm{Ext}^\bullet(\underbrace{\mathcal{V}_\partial, \mathcal{V}_\partial}_{\text{vacuum } \mathcal{V}_\partial\text{-module}})$$

that serves now as a concrete model for $\mathcal{Z}_{\mathrm{der}}(\mathcal{V}_\partial)$ [For sufficiently rich b.c.]

(iv) Example Free chiral multiplet.

Schematically a chiral multiplet Φ and $\bar{\Phi}$ it's opposite R charge conjugate. $\Phi \supset \phi$, $\psi_\pm$ and auxiliary F.

scalar F. Among those, $Q\phi = Q\psi = 0$ $\psi := \bar{\psi}_-$

So that, $\mathcal{V} = \langle\langle \phi, \psi \rangle\rangle = \mathbb{C}[\phi, \partial_z\phi, \partial_z^2\phi, \ldots, \psi, \partial_z\psi, \ldots]$

Given that the theory is trivial the OPEs

$\phi(x) \times \phi(y) \sim \frac{1}{x-y} + \text{reg}$ and $\psi(x) \times \psi(y) \sim \frac{1}{x-y} + \text{reg.}$

Then, the λ-brackets $\{\{\phi\,{}_\lambda\,\phi\}\}$ and $\{\{\psi\,{}_\lambda\,\psi\}\}$ are both 0 as ϕ is regular inside the OPE appearing in $\oint_\phi z\, dz \wedge Q'\phi(x)\phi(y)$. ($\psi$ is analogous)

the λ-bracket between $\{\{\psi\,{}_\lambda\,\phi\}\}$ can be computed explicitly: $Q'\psi = (\partial_z dt - d\bar{z}\,\partial_t)\phi$ (from susy variations)

from which it follows that $\{\{\psi\,{}_\lambda\,\phi\}\} = 1$.

The relevant component of the supercurrent is $G = \psi\partial\phi - \phi\partial\psi$ from which we compute explicitly that $\{G, \phi\} = \partial_z\phi$ and $\{G, \psi\} = \partial_z\psi$ as expected since G is the secondary S-E tensor.

If we put the theory on a manifold w/ boundary we can either impose:

Neumann b.c. $\psi|_\partial = 0$ or Dirichlet b.c. $\phi|_\partial = 0$

$\mathcal{V}_N = \langle \phi \rangle = \mathcal{Z}(\mathcal{V}_N)$ $\mathcal{V}_D = \langle \psi \rangle = \mathcal{Z}(\mathcal{V}_D)$

both algebras are entirely central (there is no SE tensor) (trivial OPE). Furthermore, in the reference is proven that $\mathrm{Ext}^\bullet_{\mathcal{V}_{N/D}}(\mathbb{1}, \mathbb{1}) = \mathcal{V}$, in accordance w/ the conjecture.