Quantum groups

+ non ssi 3d TFT

Cob w/ constant background field

modular functor?

3d + 4d

twisted SUSY gauge theory

on 3 mfd w/ THF

$\mathbb{C}_{z,\bar{z}} \times \mathbb{R}_t$

Vertex algebras

representation categories

conformal blocks

non-ssi

- Two remarks
- Two groups of questions

Remark 1: Transverse holomorphic foliations

Question: On what 3d mfds can we consider the twisted SYM?

A 3-manifold admits a THF if we can choose local coordinates (t, z) where $t \in \mathbb{R}$ and $z \in \mathbb{C}$. Moreover, as we go from patch to patch the transformations

are of the type:

$$t'(t, z, \bar{z}), \quad z'(z), \quad \bar{z}'(\bar{z}). \tag{2.1}$$

Compact 3-manifolds with THF have been classified [9, 10, 11] and admit a finite number of deformation parameters each (see [6], section 5, for a more details). They are all either Seifert manifolds, i.e. circle fibrations over a Riemann surface, or T^2 fibrations over S^1. In this sense 3-manifolds with THF are analogous to Riemann surfaces, whose space of complex structures is finite dimensional ($3g-3$ complex dimensional for a Riemann surface of genus g).

(Aganagic, Costello, Mc Namara, Vafa 1706.09977)

Application :

$q \in \mathbb{C}^\times$ $\quad$ $S^2 \times_q S^1 = S^2 \times \mathbb{R} \big/ (z,n) \sim (zq, n+1)$

The operators of the bulk chiral algebra $\mathcal{V}$ are precisely those counted by the supersymmetric index, or $S^2 \times_q S^1$ partition function, of 3d $\mathcal{N} = 2$ theories [38–40]. Explicitly, the graded character of $\mathcal{V}$ should coincide with the index:

$$\chi[\mathcal{V}] := \mathrm{Tr}_{\mathcal{V}} e^{i\pi R} q^J = I(q), = Z(S^2 \times_q S^1) \tag{1.2}$$

where J, R measure the spin and R-charge in $\mathcal{V}$. In this sense, $\mathcal{V}$ categorifies the index.

(Costello, Dimofte, Gaiotto 2005.00083)

Question : 3-mflds in TFT vs. THF ?

Question : How does constant background field in TFT appear

(BLLRR)

of the $(z, \bar{z})$ plane so that it continues to represent a $\mathbb{Q}_i$-cohomology class. Within the plane, though, we can accomplish this task using the $\mathbb{Q}_i$-exact, twisted $\widehat{\mathfrak{sl}(2)}$ of the previous subsection. In particular, because the twisted anti-holomorphic translation generator $\widehat{L}_{-1}$ is a $\mathbb{Q}_i$ anti-commutator and the holomorphic translation generator L_{-1} is $\mathbb{Q}_i$-closed, we can define the *twisted-translated* operators

$$\mathcal{O}(z, \bar{z}) = e^{zL_{-1} + \bar{z}\widehat{L}_{-1}}\, \mathcal{O}(0)\, e^{-zL_{-1} - \bar{z}\widehat{L}_{-1}}\ , \tag{2.27}$$

where $\mathcal{O}(0)$ is a Schur operator. One way of thinking about this prescription for the translation of local operators is as the consequence of introducing a constant, complex background gauge field for the $\mathfrak{sl}(2)_R$ symmetry that is proportional to the $\mathfrak{sl}(2)$ raising operator. By construction, the

Remark 2 : Descendants and the (shifted) Poisson structure

The 3d $\mathcal{N}=2$ SUSY algebra in flat space $\mathbb{C}_{z,\bar{z}} \times \mathbb{R}_t$ has four odd generators $Q_\pm$, $\overline{Q}_\pm$ satisfying $\{Q_\alpha, \overline{Q}_\beta\} = i\sigma^\mu_{\alpha\beta}\partial_\mu$, or in components

$$\{Q_+, \overline{Q}_+\} = -2i\partial_{\bar{z}}\,, \qquad \{Q_-, \overline{Q}_-\} = 2i\partial_z\,,$$
$$\{Q_+, \overline{Q}_-\} = \{Q_-, \overline{Q}_+\} = i\partial_t\,. \tag{2.1}$$

We are interested in the cohomology of the supercharge $\boxed{Q := \overline{Q}_+}$. Since the derivatives $\partial_{\bar{z}}$

Suppose that (local) operator $\mathcal{O}$ in the bulk is Q-closed :

$$\{Q, \mathcal{O}\} = 0$$

$$\mathcal{O}^{(1)} := \frac{i}{2} Q_+ \mathcal{O}\, d\bar{z} - i Q_- \mathcal{O}\, dt$$

1-form, not holomorphic

$$Q\big(\underbrace{dz \wedge \mathcal{O}^{(1)}}\big) = d\big(dz \wedge \mathcal{O}\big)$$

2-form, not Q-closed (→ full complex)

$$Q\,(\underbrace{dz \wedge \theta^{(1)}}_{\text{2-form}}) = d\,(dz \wedge \theta)$$

Lemma

Γ 2-cycle in 3-mfd $\qquad \mathcal{O}_\Gamma := \int_\Gamma dz \wedge \theta^{(1)}$

1. $\mathcal{O}_\Gamma$ is Q-closed, $Q\,\mathcal{O}_\Gamma = 0$

2. $\mathcal{O}_\Gamma$ depends only on homology class of Γ

3. $\mathcal{O}_\Gamma$ depends only on Q-cohomology class of θ.

Poisson bracket

Now, given two Q-closed local operators $\mathcal{O}_1, \mathcal{O}_2$, their bracket at $\lambda = 0$ is defined by integrating $dz \wedge \mathcal{O}_1^{(1)}$ around a small sphere S^2 surrounding the insertion point of $\mathcal{O}_2$. Explicitly, if we insert $\mathcal{O}_2$ at $(w, \bar{w}, s)$,

$$\{\!\{\mathcal{O}_1, \mathcal{O}_2\}\!\}(w, \bar{w}, s) := \oint_{S^2} dz \wedge \mathcal{O}_1^{(1)}(z, \bar{z}, t)\, \mathcal{O}_2(w, \bar{w}, s)\,. \tag{2.12}$$

Note that, since the S^2 can be made arbitrarily small, the LHS is again a Q-closed local operator at the *same* point as $\mathcal{O}_2$. The generalization to arbitrary λ is obtained by replacing $dz \to e^{\lambda z} dz$. This should be thought of as a generating function for an infinite collection of brackets $\{\!\{\mathcal{O}_1, \mathcal{O}_2\}\!\}^{(n)}$ associated to one-forms $z^n dz$. Various algebraic properties of the

(CDG)

$$\{\!\{\mathcal{O}_1, \mathcal{O}_2\}\!\}_\lambda(w, \bar{w}, s) := \oint_{S^2} e^{\lambda z}\, dz \wedge \mathcal{O}_1^{(1)}(z, \bar{z}, t)\, \mathcal{O}_2(w, \bar{w}, s)\,.$$

Stress energy tensor

$$T_{\mu\bar{z}} = T_{\bar{z}\mu} = \tfrac{i}{2} Q(G_{+\mu})\,, \qquad T_{\mu t} = T_{t\mu} = -iQ(G_{-\mu})\,. \tag{2.13}$$

Therefore, given any Q-closed local operator $\mathcal{O}$, its holomorphic derivative is obtained as

$$\partial_w \mathcal{O}(w) = \oint_{S^2} *(T_{z\mu} dx^\mu)\mathcal{O}(w) = -i \oint_{S^2} T_{zz}(z) dz \wedge dt\, \mathcal{O}(w) + (Q\text{-exact})\,, \tag{2.14}$$

T is not a Q-closed local operator, but a descendant of a supercharge

$$G := -\frac{i}{2}\, \bar{G}_{-z}$$

$$dz \wedge G^{(1)} = dz \wedge \left(\tfrac{1}{4} Q_+(\overline{G}_{-z}) d\bar{z} - \tfrac{1}{2} Q_-(\overline{G}_{-z}) dt\right) = -iT_{zz} dz \wedge dt + (Q\text{-exact})\,. \tag{2.15}$$

Therefore, G belongs to the chiral algebra $\mathcal{V}$; and for any other $\mathcal{O} \in \mathcal{V}$ we have

$$\partial_z \mathcal{O} = \{\!\{G, \mathcal{O}\}\!\}\,. \tag{2.16}$$

There is more than a vertex algebra

$$\left[\int_{\mathbb{R}} \mathcal{O}_1^{(1)}(z, \bar{z}, t)\right] \mathcal{O}_2(w, \bar{w}, s) \sim \sum_n \frac{\{\!\{\mathcal{O}_1, \mathcal{O}_2\}\!\}^{(n)}(w, \bar{w}, s)}{(z-w)^{n+1}} + \cdots \tag{2.17}$$

Questions for the physicists

1. Questions on the description of the (twisted) gauge theories

Question : Is a Lagrangian known that resembles Chern-Simons ?

$q = \exp(i\pi/k)$. The theories $\mathcal{T}^A_{n,k}$ are defined as topological twists of certain 3d $\mathcal{N} = 4$ Chern-Simons-matter theories, which also admit string/M-theory realizations. They may be thought of as $SU(n)_{k-n}$ Chern-Simons theories, coupled to a twisted $\mathcal{N} = 4$ matter sector (the source of non-semisimplicity). We show that $\mathcal{T}^A_{n,k}$ admits holomorphic boundary condi-

!!!

(Creutzig, Dimofte, Garner, Geer)

Question : How do 2-groups appear? Which 2-groups ?

The theory $\mathcal{T}_{G,k}$ also gains a discrete one-form "center symmetry" $Z(G)$ [133]. Indeed, a more refined analysis following [134–136] (closely related to examples in [137–139]) shows that the full global symmetry of $\mathcal{T}_{G,k}$ is a 2-group, with one-form part $Z(G)$, zero-form part $\widetilde{G}^\vee$, and a nontrivial 2-group structure such that only $Z(G)$ and $\widetilde{G}^\vee/Z(\widetilde{G}^\vee) = G^\vee$ act as independent 1-form and 0-form symmetries.

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2. Questions on the categories of line operators

Question :

A feature of the 3d QFT's $\mathcal{T}^A_{G,k}$, common to most theories defined via topological twists, is that its category of line operators is intrinsically a dg (differential graded) category (*cf.* [37–40]). Only the dg category makes sense physically, and behaves well under dualities, such as 3d mirror symmetry. This is why the equivalence of categories we are proposing involves line operators in $\mathcal{T}^A_{G,k}$ and *derived* categories of $U_q(\mathfrak{g})$ modules and VOA modules. This

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3.1. **Definition of an A_∞-algebra.** Let k be a field. An A_∞*-algebra over* k (also called a 'strongly homotopy associative algebra' or an 'sha algebra') is a $\mathbf{Z}$-graded vector space

$$A = \bigoplus_{p\in\mathbf{Z}} A^p$$

endowed with graded maps (=homogeneous k-linear maps)

$$m_n : A^{\otimes n} \to A\,,\ n \geq 1\,,$$

of degree $2-n$ satisfying the following relations

- We have $m_1\, m_1 = 0$, i.e. (A, m_1) is a differential complex.

- We have
$$m_1\, m_2 = m_2\,(m_1 \otimes \mathbf{1} + \mathbf{1} \otimes m_1)$$
as maps $A^{\otimes 2} \to A$. Here $\mathbf{1}$ denotes the identity map of the space A. So m_1 is a (graded) derivation with respect to the multiplication m_2.
- We have
$$\begin{aligned} &m_2(\mathbf{1} \otimes m_2 - m_2 \otimes \mathbf{1}) \\ &\qquad = m_1 m_3 + m_3\,(m_1 \otimes \mathbf{1} \otimes \mathbf{1} + \mathbf{1} \otimes m_1 \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{1} \otimes m_1) \end{aligned}$$
as maps $A^{\otimes 3} \to A$. Note that the left hand side is the associator for m_2 and that the right hand side may be viewed as the boundary of m_3 in the morphism complex $\mathsf{Hom}^{\bullet}{}_k\,(A^{\otimes 3}, A)$ (the definition of the morphism complex is recalled below). This implies that m_2 is associative up to homotopy.
- More generally, for $n \geq 1$, we have
$$\sum (-1)^{r+st}\, m_u\,(\mathbf{1}^{\otimes r} \otimes m_s \otimes \mathbf{1}^{\otimes t}) = 0$$
where the sum runs over all decompositions $n = r + s + t$ and we put $u = r + 1 + t$.