Hopf algebras, quantum groups and topological field theory

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Contents

Literature:

Some of the Literature I used to prepare the course:

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The current version of these notes can be found under
`http://www.math.uni-hamburg.de/home/schweigert/ws12/hskript.pdf`
as a pdf file.

Please send comments and corrections to `christoph.schweigert@uni-hamburg.de`!

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1 Introduction

1.1 Braided vector spaces

Let us study the following ad hoc problem:

Definition 1.1.1
Let $\mathbb{K}$ be a field. A braided vector space *is a $\mathbb{K}$-vector space V, together with an invertible $\mathbb{K}$-linear map*

$$c : V \otimes V \to V \otimes V$$

which obeys the equation

$$(c \otimes \mathrm{id}_V) \circ (\mathrm{id}_V \otimes c) \circ (c \otimes \mathrm{id}_V) = (\mathrm{id}_V \otimes c) \circ (c \otimes \mathrm{id}_V) \circ (\mathrm{id}_V \otimes c)$$

in $\mathrm{End}(V \otimes V \otimes V)$.

Remark 1.1.2.
Let $(v_i)_{i\in I}$ be a $\mathbb{K}$-basis of V. This allows us to describe $c \in \mathrm{End}(V \otimes V)$ by a family $(c_{ij}^{kl})_{i,j,k,l\in I}$ of scalars:

$$c(v_i \otimes v_j) = \sum_{k,l} c_{ij}^{kl} v_k \otimes v_l \, .$$

If c is invertible, then c describes a braided vector space, if and only if the following equation holds:

$$\sum_{p,q,y} c_{ij}^{pq} c_{qk}^{yn} c_{py}^{lm} = \sum_{y,q,r} c_{jk}^{qr} c_{iq}^{ly} c_{yr}^{mn} \quad \text{for all} \quad l, m, n, i, j, k \in I \, .$$

This is a complicated set of non-linear equations, called the Yang-Baxter equation. In this lecture, we will see how to find solutions to this equation (and why this is an interesting problem at all).

Examples 1.1.3.

(i) For any $\mathbb{K}$-vector space V denote by

$$\begin{aligned} \tau_{V,V} : V \otimes V &\to V \otimes V \\ v_1 \otimes v_2 &\mapsto v_2 \otimes v_1 \end{aligned}$$

the map that switches the two copies of V. The pair (V, τ) is a braided vector space, since the following relation holds in the symmetric group S_3 for transpositions:

$$\tau_{12}\tau_{23}\tau_{12} = \tau_{23}\tau_{12}\tau_{23} \, .$$

(ii) Let V be finite-dimensional with ordered basis $(e_1, \dots, e_n)$. We choose some $q \in \mathbb{K}^\times$ and define $c \in \mathrm{End}(V \otimes V)$, by

$$c(e_i \otimes e_j) = \begin{cases} q\, e_i \otimes e_i & \text{if } \; i = j \\ e_j \otimes e_i & \text{if } \; i < j \\ e_j \otimes e_i + (q - q^{-1}) e_i \otimes e_j & \text{if } \; i > j \, . \end{cases}$$

For $n = \dim_{\mathbb{K}} V = 2$, the vector space $V \otimes V$ has the basis $(e_1 \otimes e_1, e_2 \otimes e_2, e_1 \otimes e_2, e_2 \otimes e_1)$ which leads to the following matrix representation for c:

$$\begin{pmatrix} q & 0 & 0 & 0 \\ 0 & q & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & q - q^{-1} \end{pmatrix} .$$

The reader should check by direct calculation that the pair (V, c) is a braided vector space. Moreover, we have

$$(c - q\,\mathrm{id}_{V\otimes V})(c + q^{-1}\mathrm{id}_{V\otimes V}) = 0\,.$$

For $q = 1$, one recovers example (i). For this reason, example (ii) is called a one-parameter deformation of example (i).

1.2 Braid groups

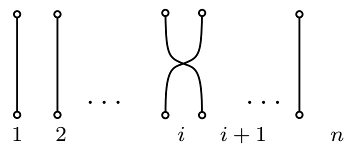
Definition 1.2.1
Fix an integer $n \geq 3$. The braid group B_n on n strands is the group with $n-1$ generators $\sigma_1 \dots \sigma_{n-1}$ and relations

$$\begin{array}{rcll} \sigma_i\sigma_j &=& \sigma_j\sigma_i & \text{for} \quad |i-j| > 1. \\ \sigma_i\sigma_{i+1}\sigma_i &=& \sigma_{i+1}\sigma_i\sigma_{i+1} & \text{for} \quad 1 \leq i \leq n-2 \end{array}$$

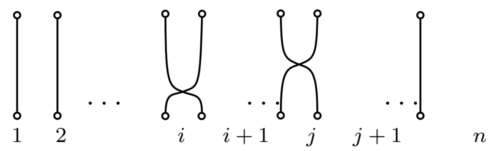
We define for $n = 2$ the braid group B_2 as the free group with one generator and we let $B_0 = B_1 = \{1\}$ be the trivial group.

Remarks 1.2.2.
(i) The following pictures explain the name braid group:

$\sigma_i =$

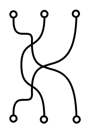


$\sigma_j\sigma_i =$



$= \sigma_i\sigma_j$

$\sigma_1\sigma_2\sigma_1 =$



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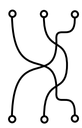


$= \sigma_2\sigma_1\sigma_2$

(ii) There is a canonical surjection from the braid group to the symmetric group:

$$\begin{array}{rcl} \pi : B_n & \to & S_n \\ \sigma_i & \mapsto & \tau_{i,i+1}\,. \end{array}$$

There is an important difference between the symmetric group S_n and the braid group B_n: in the symmetric group S_n the relation $\tau^2_{i,i+1} = \mathrm{id}$ holds. In contrast to the symmetric group, the braid group is an infinite group without any non-trivial torsion elements, i.e. without elements of finite order.

Let (V, c) be a braided vector space. For $1 \leq i \leq n-1$, define an automorphism of $V^{\otimes n}$ by

$$c_i := \begin{cases} c \otimes \mathrm{id}_{V^{\otimes(n-2)}} & \text{for} \quad i = 1 \\ \mathrm{id}_{V^{\otimes(i-1)}} \otimes c \otimes \mathrm{id}_{V^{\otimes(n-i-1)}} & \text{for} \quad 1 < i < n-1 \\ \mathrm{id}_{V^{\otimes(n-2)}} \otimes c & \text{for} \quad i = n-1\,. \end{cases}$$

We deduce from the axioms of a braided vector space that this defines a linear representation of the braid group B_n on the vector space $V^{\otimes n}$:

Proposition 1.2.3.
Let (V, c) with $c \in$ Aut $(V \otimes V)$ be a braided vector space. We have then for any $n > 0$ a unique homomorphism of groups

$$\begin{array}{rcl} \rho_n^c : B_n & \to & \text{Aut } (V^{\otimes n}) \\ \sigma_i & \mapsto & c_i \quad \text{for} \quad i = 1, 2, \dots n-1 \, . \end{array}$$

Proof.
The relation $c_i c_j = c_j c_i$ for $|i - j| \geq 2$ holds, since the linear maps c_i and c_j act on different copies of the tensor product. The relation $c_i c_{i+1} c_i = c_{i+1} c_i c_{i+1}$ is part of the axioms of a braided vector space in definition 1.1.1. □

Let us explain why the braid group is interesting: consider the subset $Y_n \subset \mathbb{C}^n = \mathbb{C} \times \cdots \times \mathbb{C}$ consisting of all n-tuples $(z_1, \dots, z_n) \in \mathbb{C}^n$ of pairwise distinct points, i.e. such that

$$z_i \neq z_j \quad \text{for} \quad i \neq j \, .$$

The symmetric group S_n acts on Y_n by permutation of entries. The orbit space $X_n = Y_n / S_n$ is called the configuration space of n different points in the complex plane $\mathbb{C}$. Fix the point $p = (1, 2, \dots n) \in Y_n$ and the quotient topology on X_n.

Theorem 1.2.4 (Artin)**.**
The fundamental group $\pi_1(X_n, p)$ of the configuration space X_n is isomorphic to the braid group B_n.

Proof.
We only give a group homomorphism

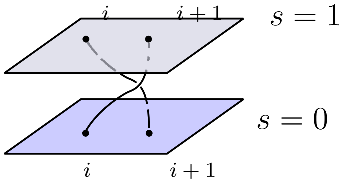
$$B_n \to \pi_1(X_n, p) \; .$$

We assign to the element $\sigma_i \in B_n$ the continuous path in the configuration space X_n described by the map

$$f = (f_1, \dots, f_n) : [0, 1] \to \mathbb{C}^n$$

given by

$$\begin{array}{rcl} f_j(s) & = & j \qquad \text{for} \qquad j \neq i \text{ and } j \neq i+1 \\ f_i(s) & = & \frac{1}{2}(2i + 1 - e^{\mathrm{i}\pi s}) \\ f_{i+1}(s) & = & \frac{1}{2}(2i + 1 + e^{\mathrm{i}\pi s}) \end{array}$$



Since we identified points, this describes a closed path in the configuration space X_n. Denote the class of f in the fundamental group $\pi_1(X_n, p)$ by $\hat{\sigma}_i$. One verifies that the classes $\hat{\sigma}_i$ obey the relations of the braid group. Hence there is a unique homomorphism

$$B_n \to \pi_1(X, p) \, .$$

We omit in these lectures the proof that the homomorphism is even an isomorphism.

□

In physics, the braid group appears in the description of (quasi-)particles in low-dimensional quantum field theories. In this case, more general statistics than Bose or Fermi statistics is possible.

For the sake of completeness, we finally present

Definition 1.2.5

(i) A braid with n strands is a continuous embedding of n closed intervals into $\mathbb{C} \times [0,1]$ whose image L_f has the following properties:

  (i) The boundary of L_f is the set $\{1, 2, \dots n\} \times \{0, 1\}$

  (ii) For any $s \in [0,1]$, the intersection $L_f \cap (\mathbb{C} \times \{s\})$ contains precisely n different points.

(ii) Braids can be concatenated.

(iii) There is an equivalence relation on the set of braids, called isotopy such that the set of equivalence classes with a composition derived from the concatenation of braids is isomorphic to the braid group.

One of our goals is to present a general mathematical framework in which representations of the braid group can be produced. This framework will incidentally allow to describe a variety of physical phenomena:

- Universality classes of low-dimensional gapped systems.
- Candidates for implementations of quantum computing.
- Quantum groups also describe symmetries in a variety of integrable systems, including in particular sectors of Yang-Mills theories.

It also produces representation theoretic structures that arise in many fields of mathematics, ranging from algebraic topology to number theory. In particular, it is clear that when one closes a braid, one obtains a knot, hence there is a relation to knot theory.

2 Hopf algebras and their representation categories

2.1 Algebras and modules

Definition 2.1.1

1. *Let $\mathbb{K}$ be a field. A unital $\mathbb{K}$-algebra is a pair (A, μ) consisting of a $\mathbb{K}$-vector space A and a $\mathbb{K}$-linear map*

$$\mu : A \otimes A \to A$$

such that there is a $\mathbb{K}$-linear map

$$\eta : \mathbb{K} \to A \ ,$$

called the unit, such that

(a) $\mu \circ (\mu \otimes \mathrm{id}_A) = \mu \circ (\mathrm{id}_A \otimes \mu)$ *(associativity)*
(b) $\mu \circ (\eta \otimes \mathrm{id}_A) = \mu \circ (\mathrm{id}_A \otimes \eta) = \mathrm{id}_A$ *(unitality)*

In the first identity, the identification $(A \otimes A) \otimes A \cong A \otimes (A \otimes A)$ of tensor products of vector spaces is tacitly understood. Similarly, in the second equation, we identify the tensor products $\mathbb{K} \otimes A \cong A \cong A \otimes \mathbb{K}$. We also write $a \cdot b := \mu(a, b)$.

2. *A morphism of algebras $(A, \mu, \eta) \to (A', \mu', \eta')$ is a $\mathbb{K}$-linear map*

$$\varphi : A \to A' ,$$

such that

$$\varphi \circ \mu = \mu' \circ (\varphi \otimes \varphi) \quad \text{and} \quad \varphi \circ \eta = \eta' .$$

3. *Consider again the flip map*

$$\begin{array}{rcl} \tau_{A,A} : A \otimes A & \to & A \otimes A \\ u \otimes v & \mapsto & v \otimes u \end{array}$$

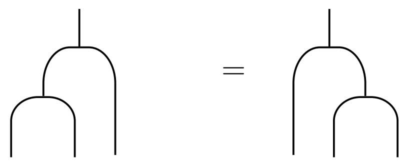
The opposite algebra A^{opp} is the triple $(A, \mu^{\mathrm{opp}} = \mu \circ \tau_{A,A}, \eta)$. Thus $a \cdot_{\mathrm{opp}} b = b \cdot a$.

4. *An algebra is called commutative, if $\mu^{\mathrm{opp}} = \mu$ holds, i.e. if $a \cdot b = b \cdot a$ for all $a, b \in A$.*

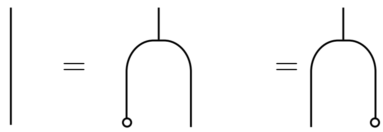
Examples 2.1.2.

1. The unit η is unique, if it exists.
2. The ground field $\mathbb{K}$ itself is a $\mathbb{K}$-algebra.
3. For any $\mathbb{K}$-vector space M, the vector space $\mathrm{End}_{\mathbb{K}}(M)$ of $\mathbb{K}$-linear endomorphisms of M is a $\mathbb{K}$-algebra. The product is composition of linear maps.
4. Let $\mathbb{K}$ be a field and G a group. Denote by $\mathbb{K}[G]$ the vector space freely generated by G. It has a basis labelled by elements of G which we denote by a slight abuse of notation by $(g)_{g \in G}$. The multiplication on basis elements $g \cdot h = gh$ is inherited from the multiplication of G. It is thus associative, and the neutral element $e \in G$ provides a unit.

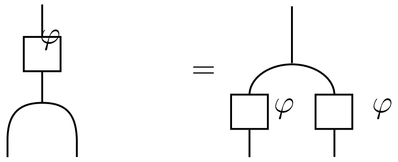
We introduce a graphical calculus in which associativity reads

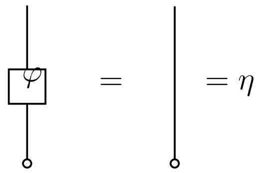


Our convention is to read such a diagram from below to above. Lines here represent the algebra A, trivalent vertices with two ingoing and one outgoing line the multiplication morphism μ. The juxtaposition of lines represents the tensor product. We have identified again the tensor products $(A \otimes A) \otimes A \cong A \otimes (A \otimes A)$.

Similarly, we represent unitality by 

where we identified again the tensor products $\mathbb{K} \otimes A \cong A \cong A \otimes \mathbb{K}$. Invisible lines denote the ground field $\mathbb{K}$. Note that we have required that the unit element $1_A := \eta(1_\mathbb{K}) \in A$ is both a left and a right unit element. If it exists, such an element is unique.

A morphism φ of unital algebras obeys 

and 

Alternatively, we can characterize associativity by the following commutative diagram

$$\begin{tikzcd} A \otimes A \otimes A \arrow[r, "\mu\otimes \mathrm{id}"] \arrow[d, "\mathrm{id}\otimes\mu"'] & A \otimes A \arrow[d, "\mu"] \\ A \otimes A \arrow[r, "\mu"'] & A \end{tikzcd}$$

while unitality reads

$$\begin{tikzcd} \mathbb{K} \otimes A \arrow[r, "\eta\otimes \mathrm{id}"] \arrow[d] & A \otimes A \arrow[d, "\mu"] & A \otimes \mathbb{K} \arrow[l, "\mathrm{id}\otimes\eta"'] \arrow[d] \\ A \arrow[r, equal] & A \arrow[r, equal] & A \end{tikzcd}$$

Examples 2.1.3.

1. We give another important example of a $\mathbb{K}$-algebra: let V be a $\mathbb{K}$-vector space. The tensor algebra over V is the associative unital $\mathbb{K}$-algebra

$$T(V) = \bigoplus_{r \geq 0} V^{\otimes r}.$$

with the tensor product as multiplication:

$$(v_1 \otimes v_2 \otimes \cdots \otimes v_r) \cdot (w_1 \otimes \cdots \otimes w_t) := v_1 \otimes \cdots \otimes v_r \otimes w_1 \otimes \cdots \otimes w_t \,.$$

The tensor algebra is a $\mathbb{Z}_+$-graded algebra: with the homogeneous component $T^{(r)} := V^{\otimes r}$ we have

$$T^{(r)} \cdot T^{(s)} \subset T^{(r+s)} \,.$$

The tensor algebra is infinite-dimensional, even if V is finite-dimensional. In this case, obviously

$$\dim T^{(r)} = \dim V^{\otimes r} = (\dim V)^r \ .$$

On the homogenous subspace $V^{\otimes r}$, it carries an action of the symmetric group S_r.

2. Denote by $I_+(V)$ the two-sided ideal of $T(V)$ that is generated by all elements of the form $x \otimes y - y \otimes x$ with $x, y \in V$. The quotient

$$S(V) := T(V)/I_+(V)$$

with its natural algebra structure is called the symmetric algebra over V. The symmetric algebra is a $\mathbb{Z}_+$-graded algebra, as well. It is infinite-dimensional, even if V is finite-dimensional.

3. Similarly, denote by $I_-(V)$ the two-sided ideal of $T(V)$ that is generated by all elements of the form $x \otimes y + y \otimes x$ with $x, y \in V$. The quotient

$$\Lambda(V) := T(V)/I_-(V)$$

with its natural algebra structure is called the alternating algebra or exterior algebra over V. The alternating algebra is a $\mathbb{Z}_+$-graded algebra, as well. If V is finite-dimensional, $n := \dim V$, it is finite-dimensional. The dimension of the homogeneous component is

$$\dim \Lambda^r(V) = \begin{pmatrix} n \\ r \end{pmatrix}$$

A central notion for this lecture is the one of a module:

Definition 2.1.4
Let A be a $\mathbb{K}$ algebra. A left A-module is a pair (M, ρ), consisting of a $\mathbb{K}$-vector space M and a map of $\mathbb{K}$-algebras

$$\rho : A \to \mathrm{End}_{\mathbb{K}}(M) \ .$$

Remark 2.1.5.

1. We also write

$$a.m := \rho(a)\, m \quad \text{for all} \quad a \in A \ \text{ and } \ m \in M$$

and thus obtain a $\mathbb{K}$-linear map which by abuse of notation we also denote by ρ:

$$\begin{array}{rcl} \rho : A \otimes M & \to & M \\ a \otimes m & \mapsto & a.m \end{array}$$

such that for all $a, b \in A$ and $m, n \in M$ and $\lambda, \mu \in \mathbb{K}$ the following identities hold:

$$\begin{array}{rcl} a.(\lambda m + \mu n) & = & \lambda(a.m) + \mu(a.n) \\ (\lambda a + \mu b).m & = & \lambda(a.m) + \mu(b.m) \\ (a \cdot b).m & = & a.(b.m) \\ 1.m & = & m \end{array}$$

(The first two lines just express that ρ is $\mathbb{K}$-bilinear.) For the properties of this map, one can again use a graphical representation and write down the two commuting diagrams:

$$\begin{tikzcd}
A \otimes A \otimes M \arrow[r, "\mu\otimes \mathrm{id}_M"] \arrow[d, "\mathrm{id}\otimes\rho"'] & A \otimes M \arrow[d, "\rho"'] \\
A \otimes M \arrow[r, "\rho"'] & A
\end{tikzcd}$$

while unitality reads

$$\begin{tikzcd}
\mathbb{K} \otimes M \arrow[r, "\eta\otimes \mathrm{id}_M"] \arrow[d] & A \otimes M \arrow[d, "\rho"] & M \otimes \mathbb{K} \arrow[l, "\mathrm{id}_M\otimes\eta"'] \arrow[d] \\
M \arrow[r, equal] & M \arrow[r, equal] & M
\end{tikzcd}$$

2. A right A-module is a left A^{opp}-module (M, ρ) with $\rho : A^{\mathrm{opp}} \to \mathrm{End}(M)$. We write $m.a := \rho(a)m$ and find the relations:

$$\begin{array}{rcl}
(\lambda m + \mu n).a &=& \lambda(m.a) + \mu(n.a) \\
m.(\lambda a + \mu b) &=& \lambda(m.a + \mu(m.b) \\
m.(a \cdot b) &=& (m.a).b \\
m.1 &=& m
\end{array}$$

for all $a, b \in A$ and $\lambda, \mu \in K$ and $m, n \in M$. This explains the word "right module". This also becomes evident in the graphical notation.

3. To give a module
$$\rho : \mathbb{K}[G] \to \mathrm{End}(M)$$
over a group algebra $\mathbb{K}[G]$, it is sufficient to specify the algebra morphism ρ on the basis $(g)_{g\in G}$ of $\mathbb{K}[G]$. This amounts to giving a group homomorphism into the invertible $\mathbb{K}$-linear endomorphisms:
$$\rho_G : G \to \mathrm{GL}(M) := \{\varphi \in \ \mathrm{End}_{\mathbb{K}}(M), \varphi \text{ invertible } \}\,.$$
The pair (M, ρ_G) is called a representation of the group G.

Remarks 2.1.6.

1. Any $\mathbb{K}$-vector space M carries a representation of its automorphism group $\mathrm{GL}(V)$ by $\rho = \mathrm{id}_{\mathrm{GL}(V)}$. This representation is called the defining representation of $\mathrm{GL}(V)$.

2. Any vector space M becomes a representation of any group G by the trivial operation $\rho(g) = \mathrm{id}_M$ for all $g \in G$.

3. A representation (M, ρ) of the free abelian group $\mathbb{Z}$ amounts to an automorphism $A \in \mathrm{GL}(M)$, namely $A = \rho(1)$. Then $\rho(n) = A^n$.

4. A representation of the cyclic group $\mathbb{Z}/2\mathbb{Z}$ on a $\mathbb{K}$-vector space V amounts to an automorphism $A : V \to V$ such that $A^2 = \mathrm{id}_V$.

   If $\mathrm{char}\mathbb{K} \neq 2$, V is the direct sum of eigenspaces of A to the eigenvalues ± 1,
$$V = V^+ \oplus V^-\,,$$

since any vector $v \in V$ can be decomposed as

$$v = \frac{1}{2}(v + Av) + \frac{1}{2}(v - Av) \ .$$

Since

$$A\frac{1}{2}(v \pm Av) = \frac{1}{2}(Av \pm A^2 v) = \pm\frac{1}{2}(v \pm Av) \quad ,$$

these are eigenvectors of A to the eigenvalues ± 1. This decomposition can be seen to be unique.

If $\mathrm{char}\mathbb{K} = 2$, the only possible eigenvalue is $+1$. Because of $A^2 = \mathrm{id}_V$, the minimal polynomial of A divides $X^2 - 1 = (X-1)^2$. Hence the Jordan blocks of the automorphism A have size 1 or 2. Indeed, we find for a Jordan block of size 2:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} .$$

Definition 2.1.7

Let A be a $\mathbb{K}$-algebra and M, N be A-modules. A $\mathbb{K}$-linear map $\varphi : M \to N$ is called a morphism of A-modules or, equivalently, an A-linear map, if

$$\varphi(a.m) = a.\varphi(m) \quad \textit{for all} \quad m \in M, a \in A \ .$$

As a diagram, this reads

$$\begin{array}{ccc} A \otimes M & \xrightarrow{\mathrm{id}_A \otimes \varphi} & A \otimes N \\ {\scriptstyle \rho_M}\downarrow & & \downarrow{\scriptstyle \rho_N} \\ M & \xrightarrow[\varphi]{} & N \ . \end{array}$$

If A is a group algebra, A-linear maps are also called intertwiners of G-representations.

One goal of this lecture is to obtain insights on representations of groups and to generalize them to a class of algebraic structures much larger than groups. To this end, it is convenient to have more terminology available to talk about all modules over a given algebra A at once: they form a category.

Definition 2.1.8

1. *A category $\mathcal{C}$ consists*

   (a) *of a class of objects $\mathrm{Obj}(\mathcal{C})$, whose entries are called the objects of the category.*

   (b) *a class $\mathrm{Hom}(\mathcal{C})$, whose entries are called morphisms of the category*

   (c) *Maps*

$$\begin{aligned} \mathrm{id} &: \mathrm{Obj}(\mathcal{C}) \to \mathrm{Hom}\,(\mathcal{C}) \\ s, t &: \mathrm{Hom}(\mathcal{C}) \to \mathrm{Obj}(\mathcal{C}) \\ o &: \mathrm{Hom}(\mathcal{C}) \times_{\mathrm{Obj}\,(\mathcal{C})} \mathrm{Hom}(\mathcal{C}) \to \mathrm{Hom}(\mathcal{C}) \end{aligned}$$

   such that

   (a) $s(\mathrm{id}_V) = t(\mathrm{id}_V) = V$ *for all* $V \in \mathrm{Obj}(\mathcal{C})$

(b) $\mathrm{id}_{t(f)} \circ f = f \circ \mathrm{id}_{s(f)} = f$ *for all* $f \in \mathrm{Hom}(\mathcal{C})$

(c) *for all* $f, g, h \in \mathrm{Hom}(\mathcal{C})$ *with* $t(f) = s(g)$ *and* $t(g) = s(h)$ *the associativity identity* $(h \circ g) \circ f = h \circ (g \circ f)$ *holds.*

2. *We write for* $V, W \in \mathrm{Obj}(\mathcal{C})$

$$\mathrm{Hom}_{\mathcal{C}}(V, W) = \{f \in \mathrm{Hom}(\mathcal{C}) \mid s(f) = V, t(f) = W\}$$

and $\mathrm{End}_{\mathcal{C}}(V)$ *for* $\mathrm{Hom}_{\mathcal{C}}(V, V)$*. For any pair* V, W*, we require* $\mathrm{Hom}_{\mathcal{C}}(V, W)$ *to be a set. Elements of* $\mathrm{End}_{\mathcal{C}}(V)$ *are called* <u>*endomorphisms*</u> *of* V*.*

3. *A morphism* $f \in \mathrm{Hom}(V, W)$ *which we also write* $V \xrightarrow{f} W$ *or in the form* $f : V \to W$ *is called an* <u>*isomorphism*</u>*, if there exists a morphism* $g : W \to V$*, such that*

$$g \circ f = \mathrm{id}_V \quad \text{and } f \circ g = \mathrm{id}_W \ .$$

Two objects V, W *of a category are called* <u>*isomorphic*</u>*, if there is an isomorphism* $V \to W$*. Being isomorphic is an equivalence relation; the equivalence classes of the category* $\mathcal{C}$ *are denoted by* $\pi_0(\mathcal{C})$*.*

Remark 2.1.9.
There is an important rule: never require two objects of a category to be *equal* - rather require them to be *isomorphic*. For example, any finite-dimensional vector space is isomorphic to its dual vector space, but there is no distinguished such isomorphism (for example, one has to chose a basis to exhibit such an isomorphism).

As a more subtle example, consider finite-dimensional representations of the compact Lie group $\mathrm{SU}(n)$. We should not ask whether the defining n-dimensional complex representation equals it dual, but rather whether it is isomorphic to its dual; then we can ask refined questions about the isomorphism, leading e.g. to the distinction of real and pseudoreal representations.

Examples 2.1.10.

1. Any set X can be endowed with a trivial structure of a category $\underline{X}$ in which the only morphisms are the identity morphisms.

2. The category $\mathrm{Cob}_{1,0}$ has as objects sets of finitely many oriented points and as morphisms arrows (or, rather, oriented one-dimensional manifolds up to diffeomorphism). The set of isomorphism classes of objects of $\mathrm{Cob}_{1,0}$ is $\pi_0(\mathrm{Cob}_{1,0}) =: \Omega_0^{SO} \cong \mathbb{Z}$. This category (or rather its higher-dimensional analogues) is central for topological field theory. They contain much information on the collection of all manifolds.

3. Vector spaces over a field $\mathbb{K}$, together with linear maps, form a category $\mathrm{vect}(\mathbb{K})$. It is a particular feature of this category that its Hom-sets are not only sets, but $\mathbb{K}$-vector spaces, and that composition is $\mathbb{K}$-bilinear. We say that the category $\mathrm{vect}(\mathbb{K})$ is <u>enriched</u> over the category $\mathrm{vect}(\mathbb{K})$.

4. More generally, left modules over a ring R form a category R-mod. Complex representations of a given group G, together with intertwiners, form a category that is enriched over the field of complex numbers.

5. Consider a category with a single object $*$; this category is completely described by the set $\mathrm{End}(*)$ which has the structure of an (associative, unital) monoid.

6. A category in which all morphisms are isomorphisms is a called a groupoid. A groupoid with single object $*$ is completely described by the monoid $G := \mathrm{End}(*)$ which is a group. We write $*//G$ for this groupoid.

   More generally, we can consider for any associative unital $\mathbb{K}$-algebra A the category $*//A$ with a single object and morphisms given by A. Its Hom-spaces are *enriched* over the category of $\mathbb{K}$-vector spaces.

7. An important example of a groupoid is the fundamental groupoid $\Pi_1(M)$ of a topological space M: its objects are the points of the space M, a morphism from $p \in M$ to $q \in M$ is a homotopy class of paths from p to q. For this groupoid $\mathrm{End}(x) =: \pi_1(X, x)$ is the fundamental group for the base point $x \in X$. The isomorphism classes of $\Pi_1(M)$ are the path-connected components of M.

8. Let G be a group and X a set, together with an action

$$\begin{array}{rrcl} \rho: & G \times X & \to & X \\ & (g, x) & \mapsto & g.x \end{array}$$

   of G on X, i.e. $(gh).x = g.(h.x)$. Define a category, the action groupoid, $X//G$ whose objects are elements $x \in X$ and which has a morphism $x \to g.x$ for every pair $(g, x) \in G \times X$. (We use the somewhat counterintuitive notation $X//G$ for a *left* action.) The equivalence classes are the orbits, thus $\pi_0(X//G) = X/G$ with X/G the orbit set, the set-theoretic quotient.

9. The category Man has as objects smooth finite-dimensional manifolds and as morphisms smooth maps of manifolds. All manifolds in this lecture will be smooth manifolds.

For the next observation, we need the following notion:

Proposition 2.1.11.
Let (A, μ_A, η_A) and (B, μ_B, η_B) be unital associative $\mathbb{K}$-algebras. Then the tensor product $A \otimes B$ has a natural structure of an associative unital algebra determined by

$$(a \otimes b) \cdot (a' \otimes b') := aa' \otimes b \cdot b' \quad \text{for all} \quad a, a' \in A, b, b' \in B$$

and $\eta_{A \otimes B} := \eta_A \otimes \eta_B$.

Put differently, the multiplication $\mu_{A \otimes B}$ is the map

$$A \otimes B \otimes A \otimes B \xrightarrow{\mathrm{id}_A \otimes \tau \otimes \mathrm{id}_B} A \otimes A \otimes B \otimes B \xrightarrow{\mu_A \otimes \mu_B} A \otimes B \ ,$$

with τ the twist map $\tau : a \otimes b \mapsto b \otimes a$ from Example 1.1.3 (i).

Observation 2.1.12.
Modules over a group algebra have more structure.

- Let V, W be $\mathbb{K}[G]$-modules. Then the ground field $\mathbb{K}$, the tensor product $V \otimes_{\mathbb{K}} W$ and the dual vector space $V^* := \mathrm{Hom}_{\mathbb{K}}(V, \mathbb{K})$ can be turned into $\mathbb{K}[G]$-modules as well by

$$\begin{array}{rcl} g.1 & := & 1 \quad \text{for all } g \in G \\ g.(v \otimes w) & := & g.v \otimes g.w \quad \text{for all } g \in G, v \in V \text{ and } w \in W \\ (g.\phi)(v) & := & \phi(g^{-1}.v) \quad \text{for all } g \in G, v \in V \text{ and } \phi \in V^* \ . \end{array}$$

  (In physics, the representation on the ground field $\mathbb{K}$ is used to describe invariant states, and the representation on $V \otimes W$ corresponds to "coupling systems" for symmetries leading to multiplicative quantum numbers.)

- We want to encode this information in additional algebraic structure on the group algebra $\mathbb{K}[G]$. To this end, we note the following simple fact:

  Let $\varphi : A \to A'$ be a morphism of $\mathbb{K}$-algebras and M an A'-module $\rho' : A' \to \mathrm{End}(M)$. Then
  $$A \xrightarrow{\varphi} A' \xrightarrow{\rho'} \mathrm{End}(M)$$
  is an A-module, denoted by $\varphi^* M$. The action is
  $$a.m := \varphi(a).m \quad \text{for all} \quad a \in A, m \in M \ .$$
  The operation is called restriction of scalars, even if A is not a subalgebra of A'. One also calls the A-module $\varphi^* M$ the pullback of M along the algebra morphism φ.

- Now suppose that (M, ρ) and (M', ρ') are two A'-modules and $M \xrightarrow{f} M'$ is a morphism of A'-modules. Then the linear map f is also a morphism $\varphi^* f : \varphi^* M \to \varphi^* M'$ of A-modules.

- In the case of the tensor product $V \otimes W$, consider the morphism of algebras
  $$\begin{array}{rcl} \Delta : \mathbb{K}[G] & \to & \mathbb{K}[G] \otimes \mathbb{K}[G] \\ g & \mapsto & g \otimes g \qquad \text{for all} \ \ g \in G \ . \end{array}$$
  The $\mathbb{K}[G]$-module structure on $V \otimes W$ is then obtained from
  $$\begin{array}{ccccccc} \mathbb{K}[G] & \xrightarrow{\Delta} & \mathbb{K}[G] \otimes \mathbb{K}[G] & \xrightarrow{\rho_V \otimes \rho_W} & \mathrm{End}(V) \otimes \mathrm{End}(W) \to \mathrm{End}(V \otimes W) \\ g & \mapsto & g \otimes g & \mapsto & \rho_V(g) \otimes \rho_W(g) \end{array} .$$
  We thus get the $\mathbb{K}[G]$-module structure on $V \otimes W$ as the pullback along Δ of the natural $\mathbb{K}[G] \otimes \mathbb{K}[G]$-module structure on $V \otimes W$.

  For the case of the ground field, consider the algebra morphism
  $$\begin{array}{rcl} \epsilon : \mathbb{K}[G] & \to & \mathbb{K} \\ g & \mapsto & 1 \quad \text{for all} \ \ g \in G \end{array}$$
  The $\mathbb{K}[G]$-module structure on $\mathbb{K}$ is then obtained from
  $$\mathbb{K}[G] \xrightarrow{\epsilon} \mathbb{K} \cong \mathrm{End}_{\mathbb{K}}(\mathbb{K}) \ .$$
  Finally, for the dual vector space, consider the algebra morphism
  $$\begin{array}{rcl} S : \mathbb{K}[G] & \to & \mathbb{K}[G]^{\mathrm{opp}} \\ g & \mapsto & g^{-1} \quad \text{for all} \ \ g \in G \end{array}$$
  The $\mathbb{K}[G]$-module structure on V^* is then obtained via the transpose from
  $$\mathbb{K}[G] \xrightarrow{S} \mathbb{K}[G] \xrightarrow{\rho^t} \mathrm{End}(V^*) \ .$$

The same type of algebraic structure is present on another class of associative algebras. To this end, we first introduce Lie algebras:

Definition 2.1.13

1. *A Lie algebra over a field $\mathbb{K}$ is a $\mathbb{K}$-vector space, $\mathfrak{g}$ together with a bilinear map, called the Lie bracket,*
$$\begin{array}{rcl} [-,-]:\mathfrak{g}\otimes\mathfrak{g} & \to & \mathfrak{g} \\ x\otimes y & \mapsto & [x,y] \end{array}$$
which is antisymmetric, i.e. $[x,x]=0$ for all $x\in\mathfrak{g}$, and for which the Jacobi identity
$$[x,[y,z]]+[y,[z,x]]+[z,[x,y]]=0$$
holds for all $x,y,z\in\mathfrak{g}$.

2. *A morphism of Lie algebras $\varphi:\mathfrak{g}\to\mathfrak{g}'$ is a $\mathbb{K}$-linear map which preserves the Lie bracket,*
$$\varphi([x,y])=[\varphi(x),\varphi(y)] \quad \text{for all} \quad x,y\in\mathfrak{g}\ .$$

3. *Given a Lie algebra $\mathfrak{g}$, we define the opposed Lie algebra $\mathfrak{g}^{\mathrm{opp}}$ as the Lie algebra with the same underlying vector space and Lie bracket*
$$[x,y]_{\mathrm{opp}}:=-[x,y]=[y,x] \quad \text{for all} \quad x,y\in\mathfrak{g}\ .$$

Examples 2.1.14.

1. For any $\mathbb{K}$-vector space V, the vector space $\mathrm{End}_{\mathbb{K}}(V)$ is endowed with the structure of a Lie algebra by the commutator
$$[f,g]:=f\circ g-g\circ f\ .$$
We denote this Lie algebra by $\mathrm{gl}(V)$.

2. More generally, any associative $\mathbb{K}$-algebra A inherits a structure of a Lie algebra by using the commutator:
$$[a,b]:=a\cdot b-b\cdot a \quad \text{for all } a,b\in A\ .$$

3. Let $\mathbb{K}$ be finite-dimensional. The subspace $\mathrm{sl}(V)$ of endomorphisms with vanishing trace is a Lie subalgebra of $\mathrm{gl}(V)$.

4. Consider the algebra $\mathrm{End}_{\mathbb{K}}(A)$ of $\mathbb{K}$-linear endomorphisms of a $\mathbb{K}$-algebra. An endomorphism $\varphi:A\to A$ is called a derivation, if it obeys the Leibniz rule:
$$\varphi(a\cdot b)=\varphi(a)\cdot b+a\cdot\varphi(b) \quad \text{for all } a,b\in A\ .$$
Denote by $\mathrm{Der}(A)\subset\mathrm{End}_{\mathbb{K}}(A)$ the subspace of derivations. It is a Lie subalgebra of $\mathrm{End}_{\mathbb{K}}(A)$:
$$\begin{array}{rcl} [\varphi,\psi](a\cdot b) & = & \varphi(a\psi(b)+\psi(a)b)-\psi(\varphi(a)b+a\varphi(b)) \\ & = & \varphi\psi(a)b+a\varphi\psi(b)-\psi\varphi(a)b-a\psi\varphi(b) \\ & = & [\varphi,\psi](a)\cdot b+a\cdot[\varphi,\psi](b) \end{array}$$

5. Examples of Lie algebras are abundant. In particular, the smooth vector fields on a smooth manifold form a Lie algebra.

Remarks 2.1.15.

- To any Lie algebra $\mathfrak{g}$, one can associate a unital associative algebra, the universal enveloping algebra. It is constructed as a quotient of the tensor algebra

$$T(\mathfrak{g}) := \bigoplus_{n=0}^{\infty} \mathfrak{g}^{\otimes n}$$

by the two-sided ideal $I(\mathfrak{g})$ that is generated by all elements of the form

$$x \otimes y - y \otimes x - [x, y] \quad \text{with } x, y \in \mathfrak{g}$$

i.e.

$$\mathrm{U}(\mathfrak{g}) = T(\mathfrak{g})/I(\mathfrak{g})\,.$$

Consider the map

$$\iota_{\mathfrak{g}} : \mathfrak{g} \to T(\mathfrak{g}) \stackrel{\pi}{\twoheadrightarrow} T(\mathfrak{g})/I(\mathfrak{g}) = \mathrm{U}(\mathfrak{g})$$

which is a morphism of Lie algebras. Since the ideal $I(\mathfrak{g})$ is not homogeneous, we only have a filtration: define $\mathrm{U}^r(\mathfrak{g})$ as the image of

$$\mathrm{U}^r(\mathfrak{g}) := \pi(\oplus_{i=0}^{r} T(\mathfrak{g})) \subset \mathrm{U}(\mathfrak{g})\ .$$

Then we have an increasing series of subspaces

$$\mathbb{K} \subset \mathrm{U}^1(\mathfrak{g}) \subset \mathrm{U}^2(\mathfrak{g}) \subset \ldots \subset \mathrm{U}^r(\mathfrak{g}) \subset \mathrm{U}^{r+1}(\mathfrak{g}) \subset \ldots$$

with $\cup_{i=1}^{\infty} \mathrm{U}^i(\mathfrak{g}) = \mathrm{U}(\mathfrak{g})$ which is is compatible with the multiplication:

$$\mathrm{U}^r(\mathfrak{g}) \cdot \mathrm{U}^s(\mathfrak{g}) \subset \mathrm{U}^{r+s}(\mathfrak{g})\ .$$

- As an example, take V to be any vector space. It is turned into a Lie algebra by $[v, w] = 0$ for all $v, w \in V$. Such a Lie algebra is called abelian. In this case, the universal enveloping algebra is just the symmetric algebra $S(V)$. In this case, the algebra is not only filtered, but even graded.

- If the Lie algebra $\mathfrak{g}$ has a totally ordered basis (x_i), the Poincaré-Birkhoff-Witt theorem gives a $\mathbb{K}$-basis of $\mathrm{U}(\mathfrak{g})$. It consists of the elements $\iota(x_{i_1}) \cdot \iota(x_{i_2}) \ldots \iota(x_{i_k})$ with $k = 0, 1, \ldots$ and $i_1 \leq i_2 \leq \ldots$.

  In particular, the elements $(\iota(x_i))$ generate $\mathrm{U}(\mathfrak{g})$ as an associative algebra. As a consequence of the Poincaré-Birkhoff-Witt theorem, the map $\iota : \mathfrak{g} \to \mathrm{U}(\mathfrak{g})$ is an *injective* map of Lie algebras.

- The following universal property holds: for any associative $\mathbb{K}$-algebra A and any $\mathbb{K}$-linear map

$$\varphi : \mathfrak{g} \to A\,,$$

that is a morphism of Lie algebras,

$$\varphi([x, y]) = [\varphi(x), \varphi(y)]$$

with the Lie algebra structure from example 2.1.14.2, there is a *unique* morphism of associative algebras $\widetilde{\varphi} : \mathrm{U}(\mathfrak{g}) \to A$ such that the diagram

$$\begin{tikzcd}
\mathfrak{g} \arrow[rr, "\iota_{\mathfrak{g}}"] \arrow[dr, "\varphi"'] & & \mathrm{U}(\mathfrak{g}) \arrow[dl, "\exists!\tilde{\varphi}"] \\
& A &
\end{tikzcd}$$

of morphisms of Lie algebras commutes. This means that any morphism $\varphi : \mathfrak{g} \to A$ of Lie algebras can be uniquely extended to a morphism $\tilde{\varphi} : \mathrm{U}(\mathfrak{g}) \to A$ of associative algebras. As a consequence, it is possible to construct algebra morphisms out of the universal enveloping $\mathrm{U}(\mathfrak{g})$ algebra into an associative algebra by giving a morphism $\mathfrak{g} \to A$ of Lie algebras.

For example, the linear map underlying the morphism $\iota_{\mathfrak{g}} : \mathfrak{g} \to \mathrm{U}(\mathfrak{g})$ of Lie algebras can also be seen as a morphism of Lie algebra $\mathfrak{g}^{\mathrm{opp}} \to \mathrm{U}(\mathfrak{g})^{\mathrm{opp}}$, where on the codomain we take the opposed algebra structure. It extends to a map of algebras $\mathrm{U}(\mathfrak{g}^{\mathrm{opp}}) \to \mathrm{U}(\mathfrak{g})^{\mathrm{opp}}$ which can be shown to be an isomorphism.

Lie algebras have representations as well:

Definition 2.1.16
Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{K}$. A representation of $\mathfrak{g}$ is a pair (M, ρ), consisting of a $\mathbb{K}$-vector space M and morphism of Lie algebras

$$\rho : \mathfrak{g} \to \mathrm{gl}(M) \ .$$

Remark 2.1.17.
We also write

$$x.m := \rho(x)\, m \quad \text{for all} \quad x \in \mathfrak{g} \ \text{and} \ m \in M$$

and thus obtain a $\mathbb{K}$-linear map

$$\begin{array}{ccc} \mathfrak{g} \otimes M & \to & M \\ x \otimes m & \mapsto & x.m \end{array}$$

such that for all $x, y \in \mathfrak{g}$ and $m, n \in M$ the following identities hold:

$$\begin{array}{rcl} x.(\lambda m + \mu n) & = & \lambda(x.m) + \mu(x.n) \\ (\lambda x + \mu y).m & = & (\lambda x.m) + (\mu x.m) \\ ([x, y]).m & = & x.(y.m) - y.(x.m) \ . \end{array}$$

Again, the first two lines express that we have a $\mathbb{K}$-bilinear map.

Definition 2.1.18
Let $\mathfrak{g}$ be a $\mathbb{K}$-Lie algebra and let M, N be representations of $\mathfrak{g}$. A $\mathbb{K}$-linear map $\varphi : M \to N$ is called a morphism of representations of $\mathfrak{g}$, if

$$\varphi(x.m) = x.\varphi(m) \quad \textit{for all} \quad m \in M \ \textit{and} \ x \in \mathfrak{g} \ .$$

This defines the category $\mathfrak{g}{-}\mathrm{rep}$ of representations of $\mathfrak{g}$.

Using the universal property of the universal enveloping algebra, every representation $\rho : \mathfrak{g} \to \mathrm{End}_{\mathbb{K}}(M)$ of a Lie algebra $\mathfrak{g}$ extends uniquely to a representation $\tilde{\rho} : \mathrm{U}(\mathfrak{g}) \to \mathrm{End}_{\mathbb{K}}(M)$ of the universal enveloping algebra:

$$\begin{tikzcd} \mathfrak{g} \arrow[rr, "\iota_{\mathfrak{g}}"] \arrow[dr, "\rho"'] & & \mathrm{U}(\mathfrak{g}) \arrow[dl, dashed, "\exists! \tilde{\rho}"] \\ & \mathrm{End}_{\mathbb{K}}(M) & \end{tikzcd}$$

We have thus proven:

Proposition 2.1.19.
There is a canonical bijection between representations of the Lie algebra $\mathfrak{g}$ and modules over its universal enveloping algebra $\mathrm{U}(\mathfrak{g})$. One can show that morphisms of representations of $\mathfrak{g}$ are in bijection to $\mathrm{U}(\mathfrak{g})$-module morphisms.

These bijections are, however, not an appropriate language to compare the categories $\mathrm{U}(\mathfrak{g})-\mathrm{mod}$ and $\mathfrak{g}-\mathrm{rep}$ which are bilayered structures consisting of objects and morphisms.

Definition 2.1.20
Let $\mathcal{C}$ and $\mathcal{C}'$ be categories. A <u>*functor*</u> $F : \mathcal{C} \to \mathcal{C}'$ *consists of two maps:*

$$\begin{array}{lcl} F & : & \mathrm{Obj}(\mathcal{C}) \;\; \to \mathrm{Obj}(\mathcal{C}') \\ F & : & \mathrm{Hom}(\mathcal{C}) \to \mathrm{Hom}(\mathcal{C}'), \end{array}$$

which obey the following conditions:

(a) $F(\mathrm{id}_V) = \mathrm{id}_{F(V)}$ *for all objects* $V \in \mathrm{Obj}(\mathcal{C})$

(b) $s(F(f)) = Fs(f)$ *and* $t(F(f)) = Ft(f)$ *for all morphisms* $f \in \mathrm{Hom}\ (\mathcal{C})$

(c) For any pair f, g of composable morphisms, we have

$$F(g \circ f) = F(g) \circ F(f) \ .$$

Two functors

$$\begin{array}{lclcl} F & : & \mathcal{C} & \to & \mathcal{C}' \\ G & : & \mathcal{C}' & \to & \mathcal{C}'' \end{array}$$

can be concatenated to a functor $G \circ F : \mathcal{C} \to \mathcal{C}''$.

We have already encountered examples of functors:

Examples 2.1.21.

1. A functor $*//G \to \mathrm{vect}(\mathbb{K})$ assigns to the single object $*$ a $\mathbb{K}$-vector space M and to any group element $g \in G$ an endomorphism $\rho(g)$ of M. Since functors preserve composition, the map ρ defines a representation of the group G. Thus $\mathbb{K}$-linear representations of G are just functors $*//G \to \mathrm{vect}(\mathbb{K})$.

2. Associating to a vector space V its dual space provides a functor

$$\begin{array}{rcl} \mathrm{vect}(\mathbb{K}) & \to & \mathrm{vect}(\mathbb{K})^{\mathrm{opp}} \\ V & \mapsto & V^* \ . \end{array}$$

Here we have introduced the opposed category $\mathcal{C}^{\mathrm{opp}}$ of a category $\mathcal{C}$. It has the same objects as $\mathcal{C}$, but $\mathrm{Hom}^{\mathrm{opp}}(U, V) := \mathrm{Hom}(V, U)$. The composition is defined in a compatible way. It thus implements the idea of "reversing arrows".

The bidual provides a functor

$$\begin{array}{rcl} \mathrm{Bi} : \mathrm{vect}(\mathbb{K}) & \to & \mathrm{vect}(\mathbb{K}) \\ V & \mapsto & V^{**} \ . \end{array}$$

3. Let $\varphi : A \to A'$ be a morphism of algebras. As in observation 2.1.12, we consider for any A'-module M, $\rho : A' \to \mathrm{End}(M)$ the A-module $\varphi^* M$ that is defined on the same $\mathbb{K}$-vector space M. The $\mathbb{K}$-linear map underlying a morphism $\varphi : M \to M'$ of A'-modules is also a morphism of modules $\varphi^* M \to \varphi^* M'$. We thus obtain a functor
$$\mathrm{Res}_A^{A'} \ : A'\mathrm{-mod} \to A\mathrm{-mod}$$
that is called, by abuse of language, a <u>restriction functor</u>.

4. We have learned that any associative algebra is endowed, by the commutator, with the structure of a Lie algebra. This provides a functor
$$\mathrm{Alg}_{\mathbb{K}} \to \mathrm{Lie}_{\mathbb{K}} \ .$$

5. The universal enveloping algebra provides a functor from the category of Lie algebras to the category of associative algebras,
$$\begin{array}{rcl} \mathrm{U} : \mathrm{Lie}_{\mathbb{K}} & \to & \mathrm{Alg}_{\mathbb{K}} \\ \mathfrak{g} & \mapsto & \mathrm{U}(\mathfrak{g}) \ . \end{array}$$

6. In proposition 2.1.19, we have constructed a functor $\mathfrak{g}\mathrm{-rep} \to \mathrm{U}(\mathfrak{g})\mathrm{-mod}$.

It can be quite important to compare two functors $F, G : \mathcal{C} \to \mathcal{C}'$ between the same categories. We give two motivations:

- We have seen in example 2.1.21.1 that for G a group, a functor $F_\rho : *//G \to \mathrm{vect}(\mathbb{K})$ corresponds to a $\mathbb{K}$-linear representation of the group G. From definition 2.1.7 we know that there are intertwiners between different representations. Given two functors $F_\rho, F_{\rho'} : *//G \to \mathrm{vect}(\mathbb{K})$, we thus need the analogue of an intertwiner.

- To get an idea on how to relate functors, we remark that any vector space V can be embedded into its bidual vector space. This means that for every V there is a linear map
$$\begin{array}{rcl} \iota_V : \mathrm{id}(V) = V & \to & V^{**} = \mathrm{Bi}(V) \\ v & \mapsto & (\beta \mapsto \beta(v)) \end{array}$$
that relates the two functors $\mathrm{id}, \mathrm{Bi} : \mathrm{vect}(\mathbb{K}) \to \mathrm{vect}(\mathbb{K})$.

We formalize this as follows:

Definition 2.1.22

1. *Let $F, G : \mathcal{C} \to \mathcal{C}'$ be functors. A <u>natural transformation</u>*
$$\eta : F \to G$$
is a family of morphisms
$$\eta_V : F(V) \to G(V)$$
in $\mathcal{C}'$, indexed by objects $V \in \mathrm{Obj}(\mathcal{C})$ in the source category such that for any morphism
$$f : V \to W$$
in the source category $\mathcal{C}$ the diagram in $\mathcal{C}'$
$$\begin{tikzcd} F(V) \arrow[r, "\eta_V"] \arrow[d, "F(f)"'] & G(V) \arrow[d, "G(f)"] \\ F(W) \arrow[r, "\eta_W"] & G(W) \end{tikzcd}$$
commutes.

2. *If for each object $V \in \mathrm{Obj}(\mathcal{C})$ the morphism η_V is an isomorphism, then $\eta : F \to G$ is called a natural isomorphism.*

3. *A functor*
$$F : \mathcal{C} \to \mathcal{D}$$
is called an equivalence of categories, if there is a functor
$$G : \mathcal{D} \to \mathcal{C}$$
and natural isomorphisms
$$\begin{array}{rlcl} \eta : & \mathrm{id}_{\mathcal{D}} & \to & FG \\ \theta : & GF & \to & \mathrm{id}_{\mathcal{C}} \ . \end{array}$$

Example 2.1.23.

1. Let G be a finite group, $\mathbb{K}$ a field and consider two functors $F_\rho, F_{\rho'} : *//G \to \mathrm{vect}(\mathbb{K})$. A natural transformation $\eta : F_\rho \to F_{\rho'}$ is a $\mathbb{K}$-linear map $\eta_* : F_\rho(*) \to F_{\rho'}(*)$ which by the commuting diagram in 2.1.22.1 is an intertwiner of G-representations.

2. If the class $\mathrm{Obj}(\mathcal{C})$ is a set, then there is a category $\mathrm{Fun}(\mathcal{C}, \mathcal{C}')$ whose objects are functors $F, G : \mathcal{C} \to \mathcal{C}$ and whose morphisms natural transformations $\eta : F \to G$.

The following lemma is useful to find equivalences of categories:

Lemma 2.1.24.
A functor $F : \mathcal{C} \to \mathcal{D}$ is an equivalence of categories, if and only if

(a) The functor F is essentially surjective, i.e. for any $W \in \mathrm{Obj}(\mathcal{D})$ there is $V \in \mathrm{Obj}(\mathcal{C})$ such that $F(V) \cong W$ in $\mathcal{D}$.

(b) The functor F is fully faithful: for any pair V, V' of objects in $\mathcal{C}$, the map
$$F : \mathrm{Hom}_{\mathcal{C}}(V, V') \to \ \mathrm{Hom}_{\mathcal{D}}(F(V), F(V'))$$
on Hom-spaces is bijective.

Proof: see [Kassel, p. 278]. The proof uses the axiom of choice.

An example for an equivalence of categories is the functor $\mathfrak{g}$-rep $\to$ $\mathrm{U}(\mathfrak{g})$-mod constructed in proposition 2.1.19.

As a further consequence of the universal property of the enveloping algebra $\mathrm{U}(\mathfrak{g})$, we get from maps of Lie algebras maps of unital associative algebras:

$$\begin{array}{lcllrcl}
\mathfrak{g} & \to & \mathbb{K} & \text{gives} & \epsilon : \mathrm{U}(\mathfrak{g}) & \to & \mathbb{K} \\
x & \mapsto & 0 & & & & \\
\mathfrak{g} & \to & \mathfrak{g} \oplus \mathfrak{g} \subset \mathrm{U}(\mathfrak{g} \oplus \mathfrak{g}) & \text{gives} & \Delta : \mathrm{U}(\mathfrak{g}) & \to & \mathrm{U}(\mathfrak{g} \oplus \mathfrak{g}) \cong \mathrm{U}(\mathfrak{g}) \otimes \mathrm{U}(\mathfrak{g}) \\
x & \mapsto & (x, x) & & & & \\
\mathfrak{g} & \to & \mathfrak{g}^{\mathrm{opp}} \subset \mathrm{U}(\mathfrak{g}^{\mathrm{opp}}) & \text{gives} & S : \mathrm{U}(\mathfrak{g}) & \to & \mathrm{U}(\mathfrak{g}^{\mathrm{opp}}) \cong \mathrm{U}(\mathfrak{g})^{\mathrm{opp}} \\
x & \mapsto & -x & & & &
\end{array}$$

These morphisms of algebras are explicitly given by the following expressions on the generators $x \in \mathfrak{g} \subset \mathrm{U}(\mathfrak{g})$

$$\begin{aligned} \epsilon(x) &= 0 \\ \Delta(x) &= 1 \otimes x + x \otimes 1 \\ S(x) &= -x \end{aligned}$$

These maps allow us to endow tensor products of representations of $\mathfrak{g}$, the dual of a vector space underlying a representation of $\mathfrak{g}$ and the ground field $\mathbb{K}$ with the structure of $\mathfrak{g}$-representations.

Observation 2.1.25.

- Let V, W be representations of $\mathfrak{g}$. The $\mathrm{U}(\mathfrak{g})$-module structure on the tensor product $V \otimes W$ is then obtained from

  $$\mathrm{U}(\mathfrak{g}) \xrightarrow{\Delta} \mathrm{U}(\mathfrak{g}) \otimes \mathrm{U}(\mathfrak{g}) \xrightarrow{\rho_V \otimes \rho_W} \mathrm{End}(V) \otimes \mathrm{End}(W) \to \mathrm{End}(V \otimes W) \ .$$

  This is uniquely determined by the condition

  $$(*) \qquad x.(v \otimes w) = x.v \otimes w + v \otimes x.w \quad \text{for all } v \in V, w \in W \text{ and } x \in \mathfrak{g} \ .$$

- The $\mathrm{U}(\mathfrak{g})$-module structure on the ground field $\mathbb{K}$ is obtained from

  $$\mathrm{U}(\mathfrak{g}) \xrightarrow{\epsilon} \mathbb{K} \cong \mathrm{End}_{\mathbb{K}}(\mathbb{K}) \ .$$

  This is uniquely determined by the condition $x.v = 0$ for all $x \in \mathfrak{g}$ and $v \in \mathbb{K}$.

- The $\mathrm{U}(\mathfrak{g})$-module structure on V^* is then obtained via the transpose from

  $$\mathrm{U}(\mathfrak{g}) \xrightarrow{S} \mathrm{U}(\mathfrak{g})^{\mathrm{opp}} \xrightarrow{\rho^t} \mathrm{End}(V^*) \ .$$

- Again, in physics, the representation on $\mathbb{K}$ is used to produce invariant states, and the representation on $V \otimes W$ corresponds to the "coupling of two systems" for symmetries leading by condition $(*)$ to additive quantum numbers.

2.2 Coalgebras and comodules

The maps (Δ, ϵ) in the two examples of a group algebra $\mathbb{K}[G]$ of a group G and the universal enveloping algebra $\mathrm{U}(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ have properties that are best understood by reversing arrows in the definition of an algebra.

Definition 2.2.1

1. *A counital coalgebra over a field $\mathbb{K}$ is a pair (C, Δ), consisting of a $\mathbb{K}$-vector space C and a $\mathbb{K}$-linear map*

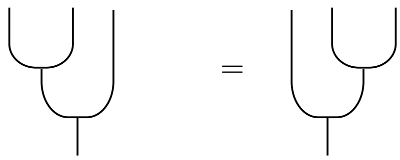
   $$\Delta: \quad C \to C \otimes C \ ,$$

   called the coproduct, such that there is a $\mathbb{K}$-linear map

   $$\epsilon : C \to \mathbb{K} \ ,$$

   called the counit, such that

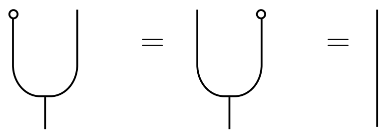
(a) $(\Delta \otimes \mathrm{id}_C) \circ \Delta = (\mathrm{id}_C \otimes \Delta) \circ \Delta$ *(coassociativity)*



In terms of commuting diagrams, we have

$$\begin{tikzcd}
C \otimes C \otimes C & & C \otimes C \arrow[ll, "\Delta\otimes\mathrm{id}_C"'] \\
C \otimes C \arrow[u, "\mathrm{id}_C\otimes\Delta"] & & C \arrow[ll, "\Delta"'] \arrow[u, "\Delta"']
\end{tikzcd}$$

(b) $(\epsilon \otimes \mathrm{id}_C) \circ \Delta = (\mathrm{id}_C \otimes \epsilon) \circ \Delta = \mathrm{id}_C$ *(counitality)*



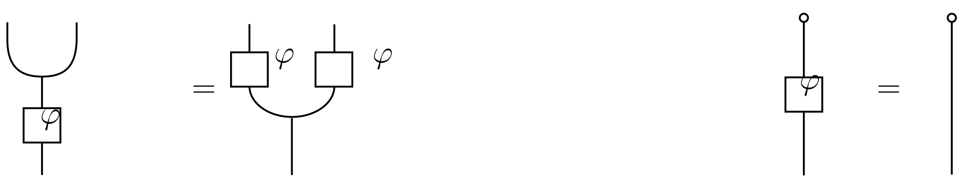
$= \mathrm{id}_C$

In terms of commuting diagrams, we have

$$\begin{tikzcd}
\mathbb{K} \otimes C & C \otimes C \arrow[l, "\epsilon\otimes\mathrm{id}"'] \arrow[r, "\mathrm{id}\otimes\epsilon"] & C \otimes \mathbb{K} \\
C \arrow[u] \arrow[r, equal] & C \arrow[u, "\Delta"] \arrow[r, equal] & C \arrow[u]
\end{tikzcd}$$

2. *Given a coalgebra (C, Δ, ϵ), the coopposed coalgebra C^{copp} is given by $(C, \Delta^{\mathrm{copp}} := \tau_{C,C} \circ \Delta, \epsilon)$.*
   A coalgebra is called cocommutative , if the identity $\Delta^{\mathrm{copp}} = \Delta$ holds. Here τ is again the map flipping the two tensor factors.

3. *A coalgebra map is a linear map*
$$\varphi : C \to C' ,$$
   such that the equations
$$\Delta' \circ \varphi = (\varphi \otimes \varphi) \circ \Delta \quad \textit{and} \quad \epsilon' \circ \varphi = \epsilon$$
   hold. Pictorially,



$= \epsilon$

Examples 2.2.2.

1. Let S be any set and $C = \mathbb{K}[S]$ the free $\mathbb{K}$-vector space with basis S. Then C becomes a coassociative counital coalgebra with coproduct given by the diagonal map $\Delta(s) = s \otimes s$ and $\epsilon(s) = 1$ for all $s \in S$. It is cocommutative.

2. In particular, the group algebra $\mathbb{K}[G]$ for any group G with the maps Δ, ϵ discussed in observation 2.1.12 can be easily shown to be a coalgebra which is cocommutative.

3. The universal enveloping algebra $\mathrm{U}(\mathfrak{g})$ of any Lie algebra with the maps Δ, ϵ discussed before observation 2.1.25 will be shown to be a coalgebra which is cocommutative. (This is easier to do, once we have stated compatibility conditions between product and coproduct.)

Remarks 2.2.3.

1. The counit ϵ is uniquely determined, if it exists.

2. The following notation is due to Heyneman and Sweedler and frequently called Sweedler notation: let (C, Δ, ϵ) be a coalgebra. For any $x \in C$, we can find finitely many elements $x_i' \in C$ and $x_i'' \in C$ such that
$$\Delta(x) = \sum_i x_i' \otimes x_i'' \ .$$
We write this by dropping summation indices as
$$\Delta(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)}$$
and sometimes even omit the sum. In this notation, counitality reads
$$\sum_{(x)} \epsilon(x_{(1)})x_{(2)} = \sum_{(x)} x_{(1)}\epsilon(x_{(2)}) = x \quad \text{for all } x \in C,$$
and cocommutativity
$$\sum_{(x)} x_{(1)} \otimes x_{(2)} = \sum_{(x)} x_{(2)} \otimes x_{(1)} \quad \text{for all } x \in C, \ .$$
Finally, coassociativity reads
$$\sum_{(x)} \left(\sum_{(x_{(1)})} (x_{(1)})_{(1)} \otimes (x_{(1)})_{(2)} \right) \otimes x_{(2)} = \sum_{(x)} x_{(1)} \otimes \left(\sum_{(x_{(2)})} (x_{(2)})_{(1)} \otimes (x_{(2)})_{(2)} \right) .$$
We abbreviate this element also by
$$\sum x_{(1)} \otimes x_{(2)} \otimes x_{(3)} \equiv x_{(1)} \otimes x_{(2)} \otimes x_{(3)} \, .$$

Lemma 2.2.4.

1. If C is a coalgebra, then the dual vector space C^* is an algebra, with multiplication from $m = \Delta^*|_{C^* \otimes C^*}$ and unit $\eta = \epsilon^*$.

   Explicitly,
$$m(f \otimes g)(c) = \Delta^*(f \otimes g)(c) = (f \otimes g)\Delta(c) \quad \text{for all} \quad f, g \in C^* \text{and } c \in C \ .$$

2. If the coalgebra C is cocommutative, then the algebra C^* is commutative.

Proof.
This is shown by dualizing diagrams, together with one additional observation: the dual of the copoduct $\Delta : C \to C \otimes C$ is a map $(C \otimes C)^* \to C^*$. Using the canonical injection $C^* \otimes C^* \subset (C \otimes C)^*$, we can restrict Δ^* to the subspace $C^* \otimes C^*$ to get the multiplication on C^*. □

Remarks 2.2.5.

1. There is a problem when we want to dualize algebras to obtain coalgebras: since $A^* \otimes A^*$ is, in general, a proper subspace, $A^* \otimes A^* \subsetneq (A \otimes A)^*$, e.g. in the case of infinite-dimensional algebras, the image of $m^* : A^* \to (A \otimes A)^*$ may not be contained in the subspace $A^* \otimes A^*$. If A is finite-dimensional, this cannot happen, and A^* is a coalgebra.

2. For this reason, we denote by A^o the finite dual of A:

   $$A^o := \{f \in A^* \mid f(I) = 0 \quad \text{for some ideal } I \subset A \text{ of finite codimension, } \dim A/I < \infty\} \ .$$

   If A is an algebra, then A^o can be shown to be a coalgebra, with coproduct $\Delta = m^*$ and counit ϵ^*. If A is commutative, then A^o is cocommutative.

We can dualize the notion of an ideal to get coalgebra structures on certain quotients:

Definition 2.2.6
Let C be a coalgebra.

1. *A subspace $I \subset C$ is a left coideal, if $\Delta I \subset C \otimes I$.*

2. *A subspace $I \subset C$ is a right coideal, if $\Delta I \subset I \otimes C$.*

3. *A subspace $I \subset C$ is a two-sided coideal, if*

   $$\Delta I \subset I \otimes C + C \otimes I \quad \text{and } \epsilon(I) = 0 \ .$$

Note that any two-sided ideal is, in particular, a left ideal and a right ideal. For coideals, however, an left or right coideal is a two-sided coideal. It is easy to check that a subspace $I \subset C$ is a two-sided coideal, if and only if C/I is a coalgebra with comultiplication induced by Δ.

This raises the question what the algebraic structure on the quotient of C/I with I a left or right ideal is. To this end, we also dualize the notion of a module:

Definition 2.2.7
Let $\mathbb{K}$ be a field and (C, Δ, ϵ) be a $\mathbb{K}$-coalgebra.

1. *A right C-comodule is a pair (M, Δ_M), consisting of a $\mathbb{K}$-vector space M and a $\mathbb{K}$-linear map*

   $$\Delta_M : M \to M \otimes C \ ,$$

   called the coaction such that the two diagrams commute:

$$\begin{tikzcd} M \arrow[r, "\Delta_M"] \arrow[d, "\Delta_M"'] & M \otimes C \arrow[d, "\Delta_M \otimes \mathrm{id}_C"] \\ M \otimes C \arrow[r, "\mathrm{id}_M \otimes \Delta"'] & M \otimes C \otimes C \end{tikzcd}$$

and

$$\begin{tikzcd}
M \arrow[r, "\Delta_M"] \arrow[d, "\cong"] & M \otimes C \arrow[d, "\mathrm{id}_M \otimes \epsilon"] \\
M \otimes \mathbb{K} \arrow[r, equal] & M \otimes \mathbb{K} \ .
\end{tikzcd}$$

2. *A $\mathbb{K}$-linear map $\varphi : M \to N$ between right C-comodules M, N is said to be a* <u>*comodule map*</u>*, if the following diagram commutes*

$$\begin{tikzcd}
M \arrow[r, "\varphi"] \arrow[d, "\Delta_M"'] & N \arrow[d, "\Delta_N"] \\
M \otimes C \arrow[r, "\varphi \otimes \mathrm{id}_C"'] & N \otimes C \ .
\end{tikzcd}$$

3. *We denote the category of right C-comodules by* comod-C*.*

4. *Left comodules and morphisms of left comodules are defined analogously. They form a category, denoted by* $C-$comod*.*

Again, the reader should draw pictures in a graphical notation.

Examples 2.2.8.

1. Let C be a coalgebra and M be a right C-comodule with comodule map
$$\Delta_M(m) = \sum m_0 \otimes m_1 \quad \text{with } m_0 \in M \text{ and } m_1 \in C \ .$$
Here we have adapted Sweedler notation to comodules. By lemma 2.2.4, then C^* is an algebra and M is a left C^*-module, where the action of $f \in C^*$ is defined by
$$f.m = \sum \langle f, m_1 \rangle m_0 \ .$$
Warning: in this way, we do not get all C^*-modules, but only the so-called *rational* modules.

2. Let C be a coalgebra. A subspace $I \subset C$ is a left coideal, if and only if C/I with the natural map
$$\overline{\Delta} : C/I \to C \otimes C/I$$
inherited from the coproduct of C is a left comodule. There is an analogous statement for right coideals.

3. Let S be a set and $C := \mathbb{K}[S]$ the coalgebra described in example 2.2.2.1. Then a $\mathbb{K}$-vector space M has the structure of a C-comodule, if and only if it is S-graded, i.e. if it can be written as a direct sum of subspaces $M_s \subset M$ for $s \in S$:
$$M = \oplus_{s \in S} M_s \ .$$
For an S-graded vector space M, set $\Delta_M(m) := m \otimes s$ for a homogeneous element $m \in M_s$. One directly checks that this is a coaction. Conversely, given a C-comodule M, write $\Delta_M(m) = \sum_{s \in S} m_s \otimes s$. We find
$$(\Delta_M \otimes \mathrm{id}_C) \circ \Delta_M(m) = \sum_{s,t \in S} (m_s)_t \otimes t \otimes s$$

which by coassociativity of the action has to be equal to

$$(\mathrm{id}_M \otimes \Delta) \circ \Delta_M(m) = \sum_{s \in S} m_s \otimes s \otimes s \ .$$

Thus $(m_s)_t = m_s \delta_{s,t}$, which implies $\Delta_M(m_s) = m_s \otimes s$. We introduce the subspaces

$$M_s := \{m_s \,|\, m \in M\} \ .$$

The sum of the subspaces $\oplus M_s$ is direct: $m \in M_s \cap M_t$ for $s \neq t$ implies $m = m'_s = m''_t$ for some $m', m'' \in M$. Then the comparison of

$$\Delta(m) = \Delta(m'_s) = m'_s \otimes s = m \otimes s$$

with the same relation for t shows that $m = 0$. Moreover, counitality of the coaction implies

$$m = \mathrm{id}_M(m) = (\mathrm{id}_M \otimes \epsilon) \circ \Delta_M(m) = \sum_{s \in S} m_s \epsilon(s) = \sum_{s \in S} m_s \ ,$$

so that $M = \oplus_{s \in S} M_s$.

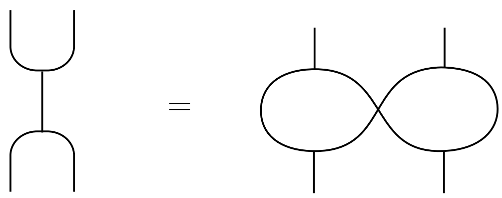
2.3 Bialgebras

Definition 2.3.1

1. *A triple (A, μ, Δ) is called a bialgebra, if*
   - *(A, μ) is an associative algebra, having a unit $\eta : \mathbb{K} \to A$.*
   - *(A, Δ) is a coassociative coalgebra, having a counit $\epsilon : A \to \mathbb{K}$.*
   - *The coproduct $\Delta : A \to A \otimes A$ is a map of unital algebras:*

     $$\Delta(a \cdot b) = \Delta(a) \cdot \Delta(b) \quad \textit{for all } a, b \in A$$

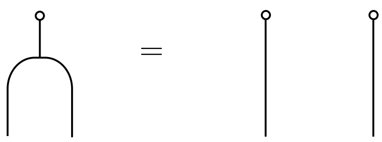
     in pictures

     or in Sweedler notation

     $$\sum_{(ab)} (ab)_{(1)} \otimes (ab)_{(2)} = \sum_{(a)(b)} a_{(1)} b_{(1)} \otimes a_{(2)} b_{(2)} \ .$$

     and $\Delta(1) = 1 \otimes 1$.
   - *The counit $\epsilon : A \to \mathbb{K}$ is a map of unital algebras: $\epsilon(a \cdot b) = \epsilon(a) \cdot \epsilon(b)$. In pictures*

     and $\epsilon(1) = 1$.

2. *A $\mathbb{K}$-linear map is said to be a* <u>*bialgebra map*</u>*, if it is both an algebra and a coalgebra map.*

Examples 2.3.2.

1. The tensor algebra $T(V)$ is a bialgebra with
$$\Delta(v) = v \otimes 1 + 1 \otimes v \quad \text{and} \quad \epsilon(1) = 1 \ \text{ and } \epsilon(v) = 0 \quad \text{for all } v \in V \ .$$
We discuss its structure in more detail. Since ϵ is a morphism of algebras, one has
$$\epsilon(v_1 \otimes \cdots \otimes v_n) = \epsilon(v_1) \cdot \ldots \cdot \epsilon(v_n) = 0 \ .$$
This fixes the counit uniquely. Inductively, one uses the property that Δ is a morphism of algebras to show
$$\Delta(v_1 \ldots v_n) = 1 \otimes (v_1 \ldots v_n) + \sum_{p=1}^{n-1} \sum_{\sigma} (v_{\sigma(1)} \ldots v_{\sigma(p)}) \otimes (v_{\sigma(p+1)} \ldots v_{\sigma(n)}) + (v_1 \ldots v_n) \otimes 1 \, .$$
where the sum is over all $(p, n-p)$-shuffle permutations, i.e. over all permutations $\sigma \in S_n$ for which $\sigma(1) < \sigma(2) < \cdots < \sigma(p)$ and $\sigma(p+1) < \cdots < \sigma(n)$.
Counitality is now a direct consequence of the explicit formulae for coproduct and counit. Similarly, coassociativity can be derived. Alternatively, notice that the coproduct comes from the diagonal map $\delta : v \mapsto v \otimes v$ which obeys $(\delta \otimes \mathrm{id}_V) \circ \delta = (\mathrm{id}_V \otimes \delta) \circ \delta$. Finally, cocommutativity comes from the explicit formula for the coproduct, together with the observation that $(p, n-p)$-shuffles are in bijection to $(n-p, p)$-shuffles via the cyclic permutation in S_n that acts as $(1, 2, \ldots, n) \mapsto (p+1, p+2, \ldots, n, 1, \ldots p)$.

2. A direct calculation shows that the group algebra $\mathbb{K}[G]$ of a group G is a bialgebra. Note that here we do not make use of the inverses in the group G, hence monoid algebras are bialgebras as well.

3. The universal enveloping algebra $\mathrm{U}(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is a bialgebra. In particular, any symmetric algebra over a vector space has a natural bialgebra structure. Since the arguments are similar to the case of the tensor algebra, we do not repeat them.

Remarks 2.3.3.

1. If C and D are coalgebras, the tensor product $C \otimes D$ can be endowed with a natural structure of a coalgebra with comultiplication
$$C \otimes D \xrightarrow{\Delta_C \otimes \Delta_D} C \otimes C \otimes D \otimes D \xrightarrow{\mathrm{id}_C \otimes \tau \otimes \mathrm{id}_D} C \otimes D \otimes C \otimes D$$
and counit
$$C \otimes D \xrightarrow{\epsilon_C \otimes \epsilon_D} \mathbb{K} \otimes \mathbb{K} \cong \mathbb{K} \ .$$

2. In the definition of a bialgebra, the last two axioms of coproduct Δ and counit ϵ being morphisms of algebras can be replaced by the equivalent condition of the product μ and and the unit η being morphisms of counital coalgebras.

3. Since the counit $\epsilon : A \to \mathbb{K}$ is a morphism of algebras, the kernel $A^+ := \ker \epsilon$ is a two-sided ideal of codimension 1, called the <u>augmentation ideal</u>.

4. There is a weakening of the axioms of a bialgebra: one drops the condition of unitality for the coproduct and the counit and replaces them by one of the following equivalent identities

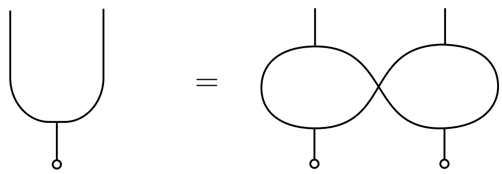
$$\begin{aligned}(\Delta \otimes \mathrm{id}_A) \cdot \Delta(1) &= (\Delta(1) \otimes 1) \cdot (1 \otimes \Delta(1)) \\ &= (1 \otimes \Delta(1)) \cdot (\Delta(1) \otimes 1) \end{aligned} \tag{1}$$

or

$$\begin{aligned}\epsilon(fgh) &= \epsilon(fg_{(1)})\epsilon(g_{(2)}h) \\ &= \epsilon(fg_{(2)})\epsilon(g_{(1)}h) \quad \text{for all } f, g, h \in A\,. \end{aligned} \tag{2}$$

This defines the notion of a <u>weak bialgebra</u>. In a weak bialgebra, we only have the relation

$$\Delta(1) = \Delta(1 \cdot 1) = \Delta(1)\Delta(1)\,,$$



i.e. $\Delta(1)$ is an idempotent in $A \otimes A$.

Remark 2.3.4.
A subspace $I \subset B$ of a bialgebra B is called a <u>biideal</u>, if it is both an ideal and a coideal. In this case, B/I is again a bialgebra.

We again discuss duals:

Lemma 2.3.5.
Let $(A, \mu, \eta, \Delta, \epsilon)$ be a *finite-dimensional* (weak) bialgebra and $A^* = \mathrm{Hom}_{\mathbb{K}}(A, \mathbb{K})$ its linear dual. Then the dual maps

$$\begin{aligned}\Delta^* &: (A \otimes A)^* \cong A^* \otimes A^* \to A^* \\ \epsilon^* &: \mathbb{K} \to A^* \\ \mu^* &: A^* \to (A \otimes A)^* = A^* \otimes A^* \\ \eta^* &: A^* \to \mathbb{K}\end{aligned}$$

define the structure of a (weak) bialgebra $(A^*, \Delta^*, \epsilon^*, \mu^*, \eta^*)$.

Remark 2.3.6.
For any (weak) bialgebra $(A, \mu, \eta, \Delta, \epsilon)$, we have three more (weak) bialgebras:

$$\begin{aligned}A^{\mathrm{opp}} &= (A, \mu^{\mathrm{opp}}, \eta, \Delta, \epsilon) \qquad A^{\mathrm{opp,copp}} = (A, \mu^{\mathrm{opp}}, \eta, \Delta^{\mathrm{copp}}, \epsilon) \\ A^{\mathrm{copp}} &= (A, \mu, \eta, \Delta^{\mathrm{copp}}, \epsilon)\end{aligned}$$

2.4 Tensor categories

We wish to understand the additional structure that is present on the categories of modules over bialgebras. Given two modules V, W over an algebra A, the tensor product has the structure of an $A \otimes A$-module. As in the case of Lie algebras and group algebras, cf. observation 2.1.12, will use for a bialgebra the pullback along the group homomorphism $\Delta : A \to A \otimes A$ to endow the tensor product $V \otimes W$ with the structure of an A-module. This turns a pair of objects (V, W) of the category $A-$mod into an object $V \otimes W$, and a pair of morphisms (f, g) into a morphism $f \otimes g$. We formalize this structure:

Definition 2.4.1
The Cartesian product of two categories $\mathcal{C}, \mathcal{D}$ is defined as the category $\mathcal{C} \times \mathcal{D}$ whose objects are pairs $(V, W) \in \mathrm{Obj}(\mathcal{C}) \times \mathrm{Obj}(\mathcal{D})$ and whose morphism sets are given by the Cartesian product of sets:

$$\mathrm{Hom}_{\mathcal{C}\times\mathcal{D}}((V,W),(V',W')) = \mathrm{Hom}_{\mathcal{C}}(V,V') \times \mathrm{Hom}_{\mathcal{D}}(W,W') \ .$$

We are now ready to discuss the structure induced by the tensor product of modules:

Definition 2.4.2

1. *Let $\mathcal{C}$ be a category and $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ a functor, called a tensor product.*

   Note that this associates to any pair (V, W) of objects an object $V \otimes W$ and to any pair of morphisms (f, g) a morphism $f \otimes g$ with source and target given by the tensor products of the source and target objects. In particular, $\mathrm{id}_{V\otimes W} = \mathrm{id}_V \otimes \mathrm{id}_W$ and for composable morphisms

   $$(f' \otimes g') \circ (f \otimes g) = (f' \circ f) \otimes (g' \circ g) \ .$$

2. *A monoidal category or tensor category consists of a category $(\mathcal{C}, \otimes)$ with tensor product, an object $\mathbb{I} \in \mathcal{C}$, called the tensor unit, and a natural isomorphism, called the associator,*

   $$a : \otimes(\otimes \times \ \mathrm{id}) \to \otimes(\mathrm{id} \times \otimes) \, .$$

   of functors $\mathcal{C} \times \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ and

   $$r : \mathrm{id} \otimes \mathbb{I} \to \mathrm{id} \quad \textit{and} \quad l : \mathbb{I} \otimes \mathrm{id} \to \mathrm{id}$$

   such that the following axioms hold:

   - *The pentagon axiom: for all quadruples of objects $U, V, W, X \in \mathrm{Obj}(\mathcal{C})$ the following diagram commutes*

$$\begin{tikzcd}
 & (U \otimes V) \otimes (W \otimes X) \arrow[dr, "a_{U,V,W\otimes X}"] & \\
((U \otimes V) \otimes W) \otimes X \arrow[ur, "a_{U\otimes V,W,X}"] \arrow[d, "a_{U,V,W}\otimes \mathrm{id}_X"'] & & U \otimes (V \otimes (W \otimes X)) \\
(U \otimes (V \otimes W)) \otimes X \arrow[rr, "a_{U,V\otimes W,X}"'] & & U \otimes ((V \otimes W) \otimes X) \arrow[u, "\mathrm{id}_U \otimes a_{V,W,X}"']
\end{tikzcd}$$

- *The* triangle axiom*: for all pairs of objects $V, W \in \mathrm{Obj}(\mathcal{C})$ the following diagram commutes*

$$\begin{tikzcd}
(V \otimes \mathbb{I}) \otimes W \arrow[rr, "a_{V,\mathbb{I},W}"] \arrow[dr, "r_V \otimes \mathrm{id}_W"'] & & V \otimes (\mathbb{I} \otimes W) \arrow[dl, "\mathrm{id}_V \otimes l_W"] \\
& V \otimes W &
\end{tikzcd}$$

Remarks 2.4.3.

1. A monoidal category can be considered as a higher analogue of an associative, unital monoid, hence the name. The associator a is, however, a *structure*, not a *property*. A property is imposed at the level of natural transformations in the form of the pentagon axiom. For a given category $\mathcal{C}$ and a given tensor product $\otimes$, inequivalent associators can exist. Any associator a gives for any triple U, V, W of objects an isomorphism

$$a_{U,V,W} : (U \otimes V) \otimes W \to U \otimes (V \otimes W)$$

such that all diagrams of the form

$$\begin{tikzcd}
(U \otimes V) \otimes W \arrow[r, "a_{U,V,W}"] \arrow[d, "(f\otimes g)\otimes h"'] & U \otimes (V \otimes W) \arrow[d, "f\otimes(g\otimes h)"] \\
(U' \otimes V') \otimes W' \arrow[r, "a_{U',V',W'}"'] & U \otimes (V \otimes W)
\end{tikzcd}$$

commute.

2. The pentagon axiom can be shown to guarantee that one can change the bracketing of multiple tensor products in a unique way. This is known as Mac Lane's coherence theorem. We refer to [Kassel, XI.5] for details.

3. A tensor category is called strict , if the natural transformations a, l and r are the identity. One can show that any tensor category is equivalent to a strict tensor category.

Examples 2.4.4.

1. The category of vector spaces over a fixed field $\mathbb{K}$ is a tensor category which is not strict. (See the appendix for information about this tensor category.) Tacitly, it is frequently replaced by an equivalent strict tensor category.

2. Let G be a group and $\mathrm{vect}_G(\mathbb{K})$ be the category of G-graded $\mathbb{K}$-vector spaces, i.e. of $\mathbb{K}$-vector spaces with a direct sum decomposition $V = \oplus_{g\in G} V_g$. Then the tensor product $V \otimes W$ is bigraded, $V \otimes W = \oplus_{g,h\in G} V_g \otimes W_h$ and becomes G-graded by the total degree

$$V \otimes W = \oplus_{g\in G} \left(\oplus_{h\in G} V_h \otimes W_{h^{-1}g}\right) .$$

Together with the associativity constraints inherited from $\mathrm{vect}(\mathbb{K})$ and with $\mathbb{K}_e$, i.e. the ground field $\mathbb{K}$ in homogeneous degree $e \in G$, as the tensor unit, this is a monoidal category. For these considerations, inverses in G are not needed and we could consider vector spaces graded by any unital associative monoid.

3. Let $\mathcal{C}$ be a small category. The endofunctors

$$F : \ \mathcal{C} \to \mathcal{C}$$

are the objects of a tensor category $\mathrm{End}(\mathcal{C})$. The morphisms in this category are natural transformations, the tensor product is composition of functors. This tensor category is strict.

4. Let $(G_n)_{n\in\mathbb{N}_0}$ be a family of groups such that $G_0 = \{1\}$.

   Define a category $\mathcal{G}$ whose objects are the natural numbers and whose morphisms are defined by
$$\mathrm{Hom}_{\mathcal{G}}(m,n) = \begin{cases} \emptyset & m \neq n \\ G_n & m = n \end{cases}$$
   Composition is the product in the group, the identity is the neutral element, $\mathrm{id}_n = e \in G_n$.

   Suppose that we are given as further data a group homomorphism for any pair (m,n)
$$\rho_{m,n} : G_m \times G_n \to G_{m+n}$$
   such that for all $m,n,p \in \mathbb{N}$, we have
$$\rho_{m+n,p} \circ (\rho_{m,n} \times \mathrm{id}_{G_p}) = \rho_{m,n+p} \circ (\mathrm{id}_{G_m} \times \rho_{n,p}) \ .$$
   We define a functor
$$\otimes \ : \ \mathcal{G} \times \mathcal{G} \to \mathcal{G}$$
   on objects by $m \otimes n = m+n$ and on morphisms by
$$\begin{array}{rcl} G_m \times G_n & \to & G_{m,n} \\ (f,g) & \mapsto & f \otimes g := \rho_{m,n}(f,g) \ . \end{array}$$
   This turns $\mathcal{G}$ into a strict tensor category.

   Such a structure is provided in particular by the collection $(S_n)_{n\in\mathbb{N}_0}$ of symmetric groups and the collection $(B_n)_{n\in\mathbb{N}}$ of braid groups. Define
$$\begin{array}{rcrcl} \rho_{m,n} & : & B_m \times B_n & \to & B_{m+n} \\ & & (\sigma_i, \sigma_j) & \mapsto & \sigma_i \sigma_{j+m} \ , \end{array}$$
   as the juxtaposition of a braid from B_m to a braid B_n.

Remarks 2.4.5.

1. In any monoidal category, we have a notion of an associative unital algebra (A,μ,η): this is a triple, consisting of an object $A \in \mathcal{C}$, multiplication $\mu : A \otimes A \to A$ and a unit morphism $\eta : \mathbb{I} \to A$ such that associativity identity
$$\mu \circ (\mu \otimes \mathrm{id}_A) = \mu \circ (\mathrm{id}_A \otimes \mu) \circ a_{A,A,A}$$
   for the morphisms $(A\otimes A)\otimes A \to A$ and the unit property
$$\mu \circ (\mathrm{id}_A \otimes \eta) = \mathrm{id}_A \circ r_A \quad \text{and} \quad \mu \circ (\eta \otimes \mathrm{id}_A) = \mathrm{id}_A \circ l_A$$
   hold. Note that the associator enters. For a general monoidal category, we do not have a notion of a commutative algebra.

2. Similarly, we can define the notion of a coalgebra (C,Δ,ϵ) in any monoidal category. For a general monoidal category, we do not have a notion of a cocommutative coalgebra.

3. Similarly, one can define modules and comodules in any monoidal category.

4. A coalgebra in $\mathcal{C}$ gives an algebra in $\mathcal{C}^{\mathrm{opp}}$ and vice versa.

The graphical notation for algebras, coalgebras, modules and comodules in a (strict) monoidal category is introduced in the obvious way.

Tensor categories are categories with some additional structure. It should not come as a surprise that we need also a class of functors and natural transformations that is adapted to this extra structure.

Definition 2.4.6

1. *Let $(\mathcal{C}, \otimes_\mathcal{C}, \mathbb{I}_\mathcal{C}, a_\mathcal{C}, l_\mathcal{C}, r_\mathcal{C})$ and $(\mathcal{D}, \otimes_\mathcal{D}, \mathbb{I}_\mathcal{D}, a_\mathcal{D}, l_\mathcal{D}, r_\mathcal{D})$ be tensor categories. (We will sometimes suppress indices indicating the category to which the data belong.) A tensor functor or monoidal functor from $\mathcal{C}$ to $\mathcal{D}$ is a triple $(F, \varphi_0, \varphi_2)$ consisting of*

$$\begin{array}{rrcl} \text{a functor} & F : \mathcal{C} & \rightarrow & \mathcal{D} \\ \text{an isomorphism} & \varphi_0 : \mathbb{I}_\mathcal{D} & \rightarrow & F(\mathbb{I}_\mathcal{C}) \text{ in the category } \mathcal{D} \\ \text{a natural isomorphism} & \varphi_2 : \otimes_\mathcal{D} \circ (F \times F) & \rightarrow & F \circ \otimes_\mathcal{C} \end{array}$$

   of functors $\mathcal{C} \times \mathcal{C} \to \mathcal{D}$. This includes in particular an isomorphism for any pair of objects $U, V \in \mathcal{C}$

$$\varphi_2(U,V) : F(U) \otimes_\mathcal{D} F(V) \xrightarrow{\sim} F(U \otimes_\mathcal{C} V)\,.$$

   These data have to obey a series of constraints expressed by commuting diagrams:

   - *Compatibility with the associativity constraint:*

$$\begin{tikzcd} (F(U) \otimes F(V)) \otimes F(W) \arrow[r, "a_{F(U),F(V),F(W)}"] \arrow[d, "\varphi_2(U,V)\otimes \mathrm{id}_{F(W)}"'] & F(U) \otimes (F(V) \otimes F(W)) \arrow[d, "\mathrm{id}_{F(U)} \otimes \varphi_2(V,W)"] \\ F(U \otimes V) \otimes F(W) \arrow[d, "\varphi_2(U\otimes V, W)"'] & F(U) \otimes F(V \otimes W) \arrow[d, "\varphi_2(U, V\otimes W)"] \\ F((U \otimes V) \otimes W) \arrow[r, "F(a_{U,V,W})"'] & F(U \otimes (V \otimes W)) \end{tikzcd}$$

   - *Compatibility with the left unit constraint:*

$$\begin{tikzcd} \mathbb{I}_\mathcal{D} \otimes F(U) \arrow[r, "l_{F(U)}"] \arrow[d, "\varphi_0 \otimes \mathrm{id}_{F(U)}"'] & F(U) \\ F(\mathbb{I}_\mathcal{C}) \otimes F(U) \arrow[r, "\varphi_2(\mathbb{I}_\mathcal{C}, U)"'] & F(\mathbb{I}_\mathcal{C} \otimes U) \arrow[u, "F(l_U)"'] \end{tikzcd}$$

   - *Compatibility with the right unit constraint:*

$$\begin{tikzcd} F(U) \otimes \mathbb{I}_\mathcal{D} \arrow[r, "r_{F(U)}"] \arrow[d, "\mathrm{id}_{F(U)} \otimes \varphi_0"'] & F(U) \\ F(U) \otimes F(\mathbb{I}_\mathcal{C}) \arrow[r, "\varphi_2(U, \mathbb{I}_\mathcal{C})"'] & F(U \otimes \mathbb{I}_\mathcal{C}) \arrow[u, "F(r_U)"'] \end{tikzcd}$$

2. *A tensor functor is called strict, if the isomorphism φ_0 and the natural transformation φ_2 are identities in $\mathcal{D}$. In general, the isomorphism and the natural isomorphism is additional structure.*

3. *A* <u>*monoidal natural transformation*</u> *between tensor functors*

$$\eta : (F, \varphi_0, \varphi_2) \to (F', \varphi_0', \varphi_2')$$

is a natural transformation $\eta : F \to F'$ *with the following two properties: such that diagram involving the tensor unit*

$$\begin{tikzcd} & F(\mathbb{I}_\mathcal{C}) \arrow[dd, "\eta_\mathbb{I}"] \\ \mathbb{I}_\mathcal{D} \arrow[ur, "\varphi_0"] \arrow[dr, "\varphi_0'"'] & \\ & F'(\mathbb{I}_\mathcal{C}) \end{tikzcd}$$

commutes, and for all pairs (U, V) *of objects the diagram*

$$\begin{tikzcd} F(U) \otimes F(V) \arrow[r, "\varphi_2(U,V)"] \arrow[d, "\eta_U \otimes \eta_V"'] & F(U \otimes V) \arrow[d, "\eta_{U \otimes V}"] \\ F'(U) \otimes F'(V) \arrow[r, "\varphi_2'(U,V)"'] & F'(U \otimes V) \end{tikzcd}$$

commutes.

4. *One then defines* <u>*monoidal natural isomorphisms*</u> *as invertible monoidal natural transformations. An* <u>*equivalence*</u> *of tensor categories* $\mathcal{C}$,$\mathcal{D}$ *is given by a pair of tensor functors* $F : \mathcal{C} \to \mathcal{D}$ *and* $G : \mathcal{D} \to \mathcal{C}$ *and natural* monoidal *isomorphisms*

$$\eta : \quad \mathrm{id}_\mathcal{D} \to FG \quad \text{and} \quad \theta : \quad GF \to \mathrm{id}_\mathcal{C} \ .$$

We can now characterize algebras whose representation categories are monoidal categories.

Proposition 2.4.7.
Let (A, μ) be a unital associative algebra. Suppose we are given unital algebra maps

$$\Delta : A \to A \otimes A \quad \text{and} \quad \epsilon : A \to \mathbb{K} \ .$$

Use the pullback along the morphism of algebras $\epsilon : A \to \mathbb{K} \cong \mathrm{End}_\mathbb{K}(\mathbb{K})$ to endow the ground field $\mathbb{K}$ with the structure of an A-module $(\mathbb{K}, \epsilon)$, i.e. $a.\lambda := \epsilon(a) \cdot \lambda$ for $a \in A$ and $\lambda \in \mathbb{K}$. Let

$$\otimes : A{-}\mathrm{mod} \times A{-}\mathrm{mod} \to A{-}\mathrm{mod}$$

be the functor which associates to a pair M, N of A-modules their tensor product $M \otimes_\mathbb{K} N$ as vector spaces with the A-module structure given by the morphism of algebras

$$A \xrightarrow{\Delta} A \otimes A \xrightarrow{\rho_M \otimes \rho_N} \mathrm{End}(M) \otimes \mathrm{End}(N) \longrightarrow \mathrm{End}(M \otimes N) \ .$$

Then $(A{-}\mathrm{mod}, \otimes, (\mathbb{K}, \epsilon))$, together with the canonical associativity and unit constraints of the category $\mathrm{vect}(\mathbb{K})$ of $\mathbb{K}$-vector spaces is a monoidal category, if and only if (A, μ, Δ) is a bialgebra with counit ϵ, i.e. if and only if (A, Δ, ϵ) is a coalgebra.

Proof.

- Suppose that (A, μ, Δ) is a bialgebra. We have to show that the canonical isomorphisms of vector spaces
$$\begin{array}{rcl} (U \otimes V) \otimes W & \to & U \otimes (V \otimes W) \\ (u \otimes v) \otimes w & \mapsto & u \otimes (v \otimes w) \end{array}$$
are morphisms of A-modules. Using Sweedler notation, the element $a \in A$ acts on the left hand side by
$$a.(u \otimes v) \otimes w = a_{(1)}.(u \otimes v) \otimes a_{(2)}.w = ((a_{(1)})_{(1)}.u \otimes (a_{(1)})_{(2)}.v) \otimes a_{(2)}.w$$
and on the right hand side
$$a.u \otimes (v \otimes w) = a_{(1)}.u \otimes a_{(2)}.(v \otimes w) = a_{(1)}.u \otimes ((a_{(2)})_{(1)}.v \otimes (a_{(2)})_{(2)}.w)$$
Coassociativity of A implies that the right hand side of the first equation is mapped to the right hand side of the second equation after rebracketing.

  Since the standard associativity constraints in $\mathrm{vect}(\mathbb{K})$ obey the pentagon relation, this relation holds in $A-\mathrm{mod}$, as well. Similarly, we have to show that the two unit constraints
$$\begin{array}{rcl} V \otimes \mathbb{K} & \to & V \\ v \otimes \lambda & \mapsto & \lambda v \end{array} \qquad \text{and} \qquad \begin{array}{rcl} \mathbb{K} \otimes V & \to & V \\ \lambda \otimes v & \mapsto & \lambda v \end{array}$$
are morphisms of A-modules. For the second isomorphism, note that
$$a.(\lambda \otimes v) = \epsilon(a_{(1)})\lambda \otimes a_{(2)}.v \mapsto \epsilon(a_{(1)})a_{(2)}.\lambda v = a.\lambda v$$
where in the last step we used one defining property of the counit. The other unit constraint is dealt with in complete analogy.

- Conversely, suppose that $(A-\mathrm{mod}, \otimes, \mathbb{K})$ is a monoidal category. Then
$$\begin{array}{rcl} (A \otimes A) \otimes A & \to & A \otimes (A \otimes A) \\ (u \otimes v) \otimes w & \mapsto & u \otimes (v \otimes w) \end{array}$$
is an isomorphism of A-modules, which, upon setting $u = v = w = 1_A$, implies by the identities above that the given unital algebra map Δ is coassociative. Similarly, we conclude from the fact that the canonical maps $\mathbb{K} \otimes A \to A$ and $A \otimes \mathbb{K} \to A$ are isomorphisms of A-modules that ϵ is a counit.

□

Remark 2.4.8.
Let (A, μ, Δ) again be a bialgebra. Then the category comod-A of right A-comodules is a tensor category as well. Given two comodules (M, Δ_M) and (N, Δ_N), the coaction on the tensor product $M \otimes N$ is defined using the multiplication:
$$\Delta_{M\otimes N}: \quad M \otimes N \xrightarrow{\Delta_M \otimes \Delta_N} M \otimes A \otimes N \otimes A \xrightarrow{\mathrm{id}_M \otimes \tau \otimes \mathrm{id}_A} M \otimes N \otimes A \otimes A \xrightarrow{\mathrm{id}_{M\otimes N} \otimes \mu} M \otimes N \otimes A\,.$$
It is straightforward to dualize all statements we made earlier.

In particular, the tensor unit is the trivial comodule which is the ground field $\mathbb{K}$ with a coaction that is given by the unit $\eta : \mathbb{K} \to A$:
$$\mathbb{K} \xrightarrow{\eta} A \cong \mathbb{K} \otimes A\,.$$
Again, the associativity and unit constraints of comodules are inherited from the constraints for vector spaces:
$$\begin{array}{rcl} (M \otimes N) \otimes P & \cong & M \otimes (N \otimes P) \\ \mathbb{K} \otimes M \cong & M & \cong M \otimes \mathbb{K}\,. \end{array}$$

2.5 Hopf algebras

Observation 2.5.1.
Let (A, μ) be a unital algebra and (C, Δ) a counital coalgebra over the same field $\mathbb{K}$. We can then define on the $\mathbb{K}$-vector space of $\mathbb{K}$-linear maps $\mathrm{Hom}(C, A)$ a product, called convolution. For $f, g \in \mathrm{Hom}(C, A)$, this is the $\mathbb{K}$-linear map

$$f * g : C \xrightarrow{\Delta} C \otimes C \xrightarrow{f \otimes g} A \otimes A \xrightarrow{\mu} A\,.$$

This product is $\mathbb{K}$-bilinear and associative. In Sweedler notation

$$(f * g)(x) = f(x_{(1)}) \cdot g(x_{(2)})\,.$$

The linear map

$$C \xrightarrow{\epsilon} \mathbb{K} \xrightarrow{\eta} A$$

is a unit for this product.

This endows in particular the space $\mathrm{End}_{\mathbb{K}}(A)$ of endomorphisms of a bialgebra A with the structure of a unital associative $\mathbb{K}$-algebra. Its unit is *not* the identity $\mathrm{id}_A \in \mathrm{End}_{\mathbb{K}}(A)$. It is, however, not clear whether in this case the identity id_A has the property of being an invertible element of the convolution algebra.

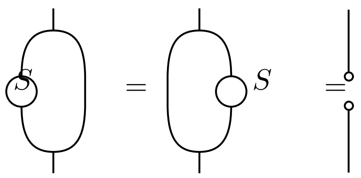
Definition 2.5.2
We say that a bialgebra (H, μ, Δ) is a Hopf algebra, if the identity id_H has a two-sided inverse $S : H \to H$ under the convolution product. This inverse is then called the antipode of the Hopf algebra.

Remarks 2.5.3.

1. The defining identity of the antipode

$$S * \mathrm{id}_H = \mathrm{id}_H * S = \eta\epsilon$$

reads in graphical notation



and in Sweedler notation

$$x_{(1)} \cdot S(x_{(2)}) = \epsilon(x) \cdot 1 = S(x_{(1)}) \cdot x_{(2)}\,.$$

2. If an antipode exists, it is, as a two-sided inverse for an associative product, uniquely determined:

$$\begin{aligned} S &= S * (\eta\epsilon) = S * (\mathrm{id}_H * S') = (S * \mathrm{id}_H) * S' \\ &= \eta\epsilon * S' = S'\,. \end{aligned}$$

Thus, for a bialgebra, being a Hopf algebra is a property rather than a structure.

3. With $H = (A, \mu, \eta, \Delta, \epsilon, S)$ a finite-dimensional Hopf algebra, its dual $H^* = (A^*, \Delta^*, \epsilon^*, \mu^*, \eta^*, S^*)$ is a Hopf algebra as well.
4. We will see in corollary 2.5.9 that any morphism $f : H \to K$ of bialgebras between Hopf algebras respects the antipode, $f(S_H h) = S_K f(h)$ for all $h \in H$. It is thus a morphism of Hopf algebras.
5. A subspace $I \subset H$ of a Hopf algebra H is a Hopf ideal, if it is a biideal, cf. remark 2.3.4 and if $S(I) \subset I$. In this case, H/I with the structure induced from H is a Hopf algebra.

Example 2.5.4.
If G is a group, the group algebra $\mathbb{K}[G]$ is a Hopf algebra with antipode

$$S(g) = g^{-1} \quad \text{for all} \quad g \in G \ .$$

Indeed, we have for $g \in G$:

$$\mu \circ (S \otimes \mathrm{id}) \circ \Delta(g) = g \cdot g^{-1} = \epsilon(g)1 \ .$$

Before giving another class of examples, we need a fundamental property of the antipode. If (A, μ_A) and (B, μ_B) are algebras, a map $f : A \to B$ is called an antialgebra map, if it is a map of unital algebras $f : A \to B^{\mathrm{opp}}$, i.e. if $f(a \cdot a') = f(a') \cdot f(a)$ for all $a, a' \in A$ and $f(1_A) = 1_B$.

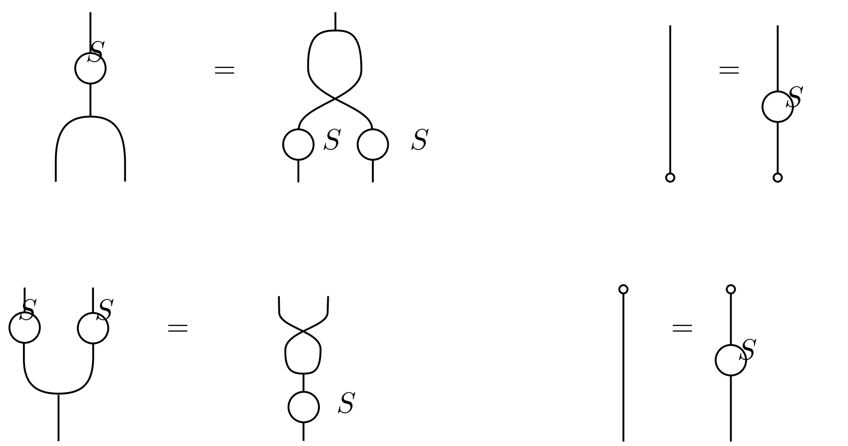
Similarly, if (C, Δ_C) and (D, Δ_D) are coalgebras, a map $g : C \to D$ is called an anticoalgebra map, if it is a counital coalgebra map $g : C \to C^{\mathrm{copp}}$, i.e. if $\epsilon_D \circ g = \epsilon_C$ and

$$g(c)_{(2)} \otimes g(c)_{(1)} = g(c_{(1)}) \otimes g(c_{(2)}) \ .$$

Proposition 2.5.5.
Let H be a Hopf algebra. Then the antipode S is a morphism of bialgebras $S : H \to H^{\mathrm{opp,copp}}$, i.e. an antialgebra and anticoalgebra map: we have for all $x, y \in H$

$$\begin{array}{rcl} S(xy) & = & S(y)S(x) \qquad S(1) = 1 \\ (S \otimes S) \circ \Delta & = & \Delta^{\mathrm{copp}} \circ S \qquad \epsilon \circ S = \epsilon \,. \end{array}$$

Graphically,



Proof.

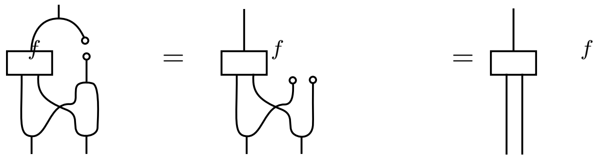
Since $H \otimes H$ is in particular a coalgebra and H an algebra, we can endow the vector space $B :=$ Hom $(H \otimes H, H)$ with bilinear product given by the convolution product: the product $\nu, \rho \in$ Hom $(H \otimes H, H)$ is by definition

$$\nu * \rho : H \otimes H \overset{(id \otimes \tau \otimes id)\circ(\Delta\otimes\Delta)}{\longrightarrow} H^{\otimes 4} \overset{\nu\otimes\rho}{\longrightarrow} H^{\otimes 2} \overset{\mu}{\longrightarrow} H \,.$$

As any convolution product, this product is associative. The unit is

$$1_B := \eta \circ \epsilon \circ \mu \quad : \quad H \otimes H \overset{\mu}{\longrightarrow} H \overset{\epsilon}{\longrightarrow} \mathbb{K} \overset{\eta}{\longrightarrow} H$$

as can be seen graphically: for any $f \in \mathrm{Hom}_{\mathbb{K}}(H \otimes H, H)$, we have



Here we used that the counit ϵ of a bialgebra is a morphism of algebras and then we used the counit property twice. Recall that, as for any associative product, two-sided inverses are unique: given $\mu \in B$, for any $\rho, \nu \in B$, the relation

$$\rho * \mu = \mu * \nu = 1_B$$

implies

$$\nu = 1 * \nu = (\rho * \mu) * \nu = \rho * (\mu * \nu) = \rho * 1 = \rho \,.$$

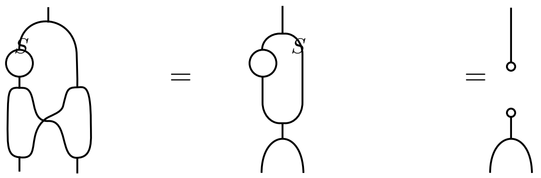
We apply this to the two elements in B

$$\begin{array}{rrcl} & H \otimes H & \to & H \\ \nu : & x \otimes y & \mapsto & S(y) \cdot S(x) \\ \rho : & x \otimes y & \mapsto & S(x \cdot y) \end{array}$$

We compute:

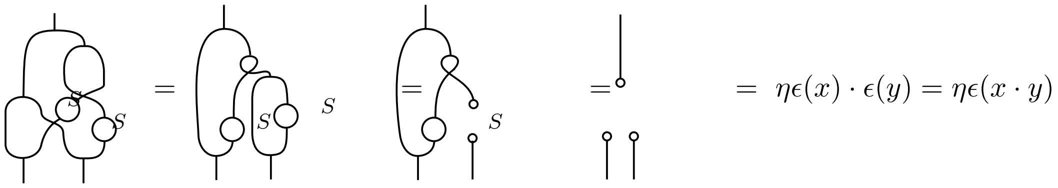
$$\begin{aligned} (\rho * \mu)(x \otimes y) &= \sum_{x\otimes y} \rho((x \otimes y)_{(1)}) \cdot \mu((x \otimes y)_{(2)}) \quad \text{[defn. of the convolution } *] \\ &= \sum \rho(x_{(1)} \otimes y_{(1)}) \mu(x_{(2)} \otimes y_{(2)}) \quad \text{[defn. of the coproduct of } H \otimes H] \\ &= \sum S(x_{(1)} y_{(1)}) x_{(2)} y_{(2)} \qquad \text{[defn. of } \rho \text{ and } \mu] \\ &= \sum_{(xy)} S((xy)_{(1)})(xy)_{(2)} \qquad [\Delta \text{ is multiplicative}] \\ &= \eta\epsilon(xy) \qquad \text{[defn. of the antipode]} \\ &= 1_B(x \otimes y) \end{aligned}$$

It is instructive to do such a calculation graphically:



The first equality is the multiplicativity of the coproduct in a bialgebra, the second is the definition of the antipode.

On the other hand, we compute $\mu * \nu$:



where in the first step we used associativity and in last step we used that the counit ϵ is a map of algebras.

Finally, the equality defining the antipode

$$\mathrm{id} \; * S = \eta\epsilon \, ,$$

can be applied to 1_H and then yields

$$1_H \cdot S(1_H) = \mathrm{id} * S(1_H) = \eta\epsilon(1_H) = 1_H \, ,$$

where the first equality is unitality of the coproduct Δ and the last identity is the unitality of the counit ϵ. This identity in H implies $S(1_H) = 1_H$. The assertions about the coproduct are proven in an analogous way: use the identity

$$\Delta \circ S = (S \otimes S) \circ \Delta^{\mathrm{copp}} \quad \text{in} \quad \mathrm{Hom}(H, H \otimes H) \; .$$

Finally, apply ϵ to the equality

$$\epsilon(x)1 = S(x_{(1)}) \cdot x_{(2)}$$

to get

$$\epsilon(x) = \epsilon(x)\epsilon(1) = \epsilon(S(x_{(1)})\epsilon(x_{(2)}) = \epsilon \circ S(x) \quad \text{for all } x \in H \; .$$

□

We now present another class of examples of Hopf algebras

Example 2.5.6.
The universal enveloping algebra $\mathrm{U}(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is a Hopf algebra with antipode

$$S(x) = -x \quad \text{for all} \quad x \in \mathfrak{g} \; .$$

Indeed, we have for $x \in \mathfrak{g}$:

$$\mu \circ (S \otimes \mathrm{id}) \circ \Delta(x) = \mu(-x \otimes 1 + 1 \otimes x) = -x + x = 0 = 1\epsilon(x) \; .$$

We extend this to all of $\mathrm{U}(\mathfrak{g})$ by the following observation: let H be a bialgebra that is generated, as an algebra, by a subset $X \subset H$. Suppose that the defining relation for an antipode holds for all $x \in X$, i.e.

$$S * \mathrm{id}_H(x) = \mathrm{id}_H * S(x) = \eta\epsilon(x) \quad \text{for all } x \in X \; .$$

Then S is an antipode for H. In fact, it is enought to check that the relation holds for products xy with $x, y \in X$. Then

$$\begin{array}{rcll} (xy)_{(1)}S((xy)_{(2)} & = & x_{(1)}y_{(1)}S(x_{(2)}y_{2)}) & \text{[bialgebra]} \\ & = & x_{(1)}y_{(1)}S(y_{(2)})S(x_{2)}) & \text{[antialgebra morphism]} \\ & = & \epsilon(x)\epsilon(y) = \epsilon(xy) & \text{[relation for } x, y \text{ and } \epsilon \text{ algebra morphism.} \end{array}$$

The other relation follows analogously.

In particular, the symmetric algebra over a vector space V is a Hopf algebra, since it is the universal enveloping algebra of the abelian Lie algebra on the vector space V. Similarly, the tensor algebra TV over a vector space V is a Hopf algebra, since it can be considered as the enveloping algebra of the free Lie algebra on V.

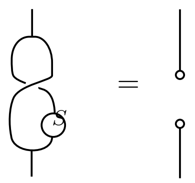
Proposition 2.5.7.
Let H be a Hopf algebra. Then the following identities are equivalent:

(a) $S^2 = \mathrm{id}_H$

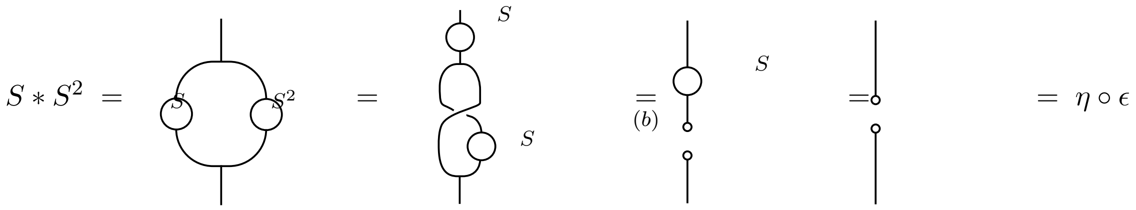
(b) $\sum_x S(x_{(2)})x_{(1)} = \epsilon(x)1_H$ for all $x \in H$.

(c) $\sum_x x_{(2)}S(x_{(1)}) = \epsilon(x)1_H$ for all $x \in H$.

Proof.
We show (b) $\Rightarrow$ (a) by first showing from (b) that $S * S^2$ is the unit for the convolution product. In graphical notation, (b) reads



Thus

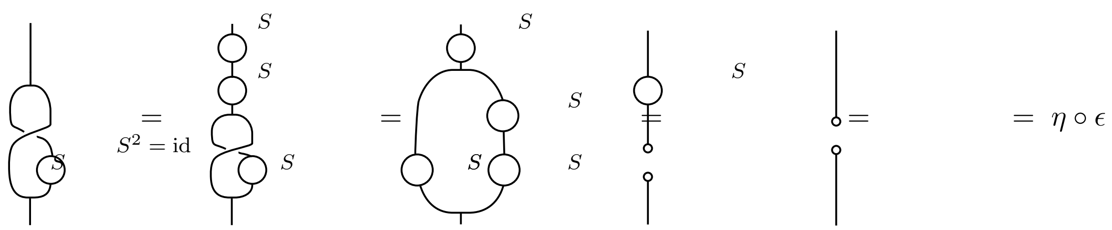


For comparison, we also compute in equations:

$$\begin{aligned} S * S^2(x) &= \sum_{(x)} S(x_{(1)})S^2(x_{(2)}) = S\Big(\sum_{(x)} S(x_{(2)})x_{(1)}\Big) \\ &\overset{(b)}{=} S(\epsilon(x)1) = \epsilon(x)S(1) = \epsilon(x)1\,. \end{aligned}$$

Then conclude $S^2 = \mathrm{id}$ by the uniqueness of the inverse of S with respect to the convolution product.

Conversely, assume $S^2 = \mathrm{id}_H$



where we used $S^2 = \mathrm{id}$, the fact that S is an anticoalgebra map, again $S^2 = \mathrm{id}$ and then the defining property of the antipode S. The equivalence of (c) and (a) is proven in complete analogy. □

The following simple lemma will be useful in many places:

Lemma 2.5.8.
Let H be a Hopf algebra with invertible antipode. Then

$$S^{-1}(a_{(2)}) \cdot a_{(1)} = a_{(2)} \cdot S^{-1}(a_{(1)}) = 1_H \epsilon(a) \quad \text{for all } a \in H \ .$$

Proof.
The following calculation shows the claim:

$$\begin{aligned} S^{-1}(a_{(2)}) \cdot a_{(1)} &= S^{-1} \circ S \left(S^{-1}(a_{(2)}) \cdot a_{(1)} \right) \\ &= S^{-1} \left(S(a_{(1)}) \cdot a_{(2)} \right) \quad [\ S \text{ is antialgebra morphism}] \\ &= S^{-1}(1_H)\epsilon(a) = 1_H \epsilon(a) \end{aligned}$$

The other identity is proven analogously. □

Remark 2.5.9.
Let H be a bialgebra. An endomorphism $\tilde{S} : H \to H$ such that

$$\sum_x \tilde{S}(x_{(2)}) x_{(1)} = \sum_x x_{(2)} \tilde{S}(x_{(1)}) = \epsilon(x) 1_H \quad \text{for all} \quad x \in H$$

is also called a skew antipode. We will usually avoid requiring the existence of an antipode and of a skew antipode and impose instead the stronger condition on the antipode of being invertible.

Corollary 2.5.10.
1. If H is either commutative or cocommutative, then the identity $S^2 = \mathrm{id}_H$ holds.
2. If H and K are Hopf algebras with antipodes S_H and S_K, respectively, then any bialgebra map $\varphi : H \to K$ is a Hopf algebra map, i.e. $\varphi \circ S_H = S_K \circ \varphi$.

Proof.
1. If H is commutative, then
$$x_{(2)} \cdot S(x_{(1)}) = S(x_{(1)}) \cdot x_{(2)} \stackrel{\text{defn. of } S}{=} \epsilon(x) 1_H \ .$$
From the previous proposition, we conclude that $S^2 = \mathrm{id}_H$. If H is cocommutative, then
$$x_{(2)} \cdot S(x_{(1)}) = x_{(1)} \cdot S(x_{(2)}) \stackrel{\text{defn. of } S}{=} \epsilon(x) 1_H \ .$$
Again we conclude that $S^2 = \mathrm{id}_H$.
2. Use again a convolution product to endow $B := \mathrm{Hom}(H, K)$ with the structure of an associative unital algebra. Then compute
$$(\varphi \circ S_H) * \varphi = \mu_K \circ (\varphi \otimes \varphi) \circ (S_H \otimes \mathrm{id}_H) \circ \Delta_H = \varphi \circ \mu_H (S_H \otimes \mathrm{id}_H) \circ \Delta_H = 1_K \epsilon_H$$
and
$$\varphi * (S_K \circ \varphi) = \mu_K \circ (\mathrm{id} \otimes S_K) \circ \Delta_K \circ \varphi = 1_K \epsilon_K \circ \varphi = 1_K \epsilon_H$$
The uniqueness of the inverse of φ under the convolution product shows the claim.

$\square$

We use the antipode to endow the category of left modules over a Hopf algebra H with a structure that generalizes contragredient or dual representations. We first state a more general fact:

Proposition 2.5.11.

1. Let A be a $\mathbb{K}$-algebra and U, V objects in $A-$mod. Then the $\mathbb{K}$-vector space $\mathrm{Hom}_{\mathbb{K}}(U,V)$ is an $A \otimes A^{\mathrm{opp}}$-module by
$$\Big[(a \otimes a').f\Big](u) := a.f(a'.u) \ .$$

2. If H is a Hopf algebra, then $\mathrm{Hom}_{\mathbb{K}}(U,V)$ is an H-module by
$$(af)(u) = \sum_{(a)} a_{(1)} f(S(a_{(2)})u) \, .$$
In the special case of the trivial module, $V = \mathbb{K}$, the dual vector space $U^* = \mathrm{Hom}_{\mathbb{K}}(U, \mathbb{K})$ becomes an H-module by
$$(af)u = f(S(a)u) \ .$$

3. Similarly, if H is a Hopf algebra and if the antipode S of H is an invertible endomorphism of H, [1] then the $\mathbb{K}$-vector space $\mathrm{Hom}_{\mathbb{K}}(U,V)$ is also an H-module by
$$(af)(u) = \sum_{(a)} a_{(1)} f(S^{-1}(a_{(2)})u) \, .$$
In the special case $V = \mathbb{K}$, the dual vector space $U^* = \mathrm{Hom}_{\mathbb{K}}(U, \mathbb{K})$ becomes an H-module by
$$(af)u = f(S^{-1}(a)u) \ .$$

Proof.
We compute with $a, b \in A$ and $a', b' \in A^{\mathrm{opp}}$:
$$\begin{aligned} \big((a \otimes a')(b \otimes b')\big) f(u) &= (ab \otimes b'a') f(u) \\ &= abf(b'a'u) \\ &= a\Big((b \otimes b')f\Big)(a'u) \\ &= (a \otimes a')\big((b \otimes b')f(u)\big) \end{aligned}$$
Then note that
$$A \xrightarrow{\Delta} A \otimes A \xrightarrow{\mathrm{id}_A \otimes S} A \otimes A^{\mathrm{opp}}$$
and, if S is invertible, also
$$A \xrightarrow{\Delta} A \otimes A \xrightarrow{\mathrm{id}_A \otimes S^{-1}} A \otimes A^{\mathrm{opp}}$$
are morphisms of algebras.

[1] As we will see, a theorem of Larson and Sweedler asserts that for any finite-dimensional Hopf algebra the antipode is invertible. Alternatively, we could have required here the existence of a skew antipode.

In the specific case of the trivial module, $V = \mathbb{K}$, we find

$$(af)(u) = \sum_{(a)} \epsilon(a_{(1)}) f(S(a_{(2)})u) = \sum_{(a)} f\Big(S(\epsilon(a_{(1)})a_{(2)})u\Big) = f(S(a)u)$$

where the second equality holds since f is $\mathbb{K}$-linear and the last equality holds by counitality. □

We recall the following maps relating a $\mathbb{K}$-vector space X and its dual $X^* = \mathrm{Hom}_{\mathbb{K}}(X, \mathbb{K})$: we have two evaluation maps

$$\begin{array}{rcl} d_X : X^* \otimes X & \to & \mathbb{K} \\ \beta \otimes x & \mapsto & \beta(x) \\ \tilde{d}_X : X \otimes X^* & \to & \mathbb{K} \\ x \otimes \beta & \mapsto & \beta(x) \end{array}$$

We call d_X a right evaluation and $\tilde{d}_X$ a left evaluation. If the $\mathbb{K}$-vector space X is *finite-dimensional*, consider a basis $\{x_i\}_{i\in I}$ of X and a dual basis $\{x^i\}_{i\in I}$ of X^*. We then have two coevaluation maps:

$$\begin{array}{rcl} b_X : \mathbb{K} & \to & X \otimes X^* \\ \lambda & \mapsto & \lambda \sum_{i\in I} x_i \otimes x^i \\ \tilde{b}_X : \mathbb{K} & \to & X^* \otimes X \\ \lambda & \mapsto & \lambda \sum_{i\in I} x^i \otimes x_i \end{array}$$

The two maps b_X and $\tilde{b}_X$ are in fact independent of the choice of basis. For example,

$$\begin{array}{rcl} b_X : \mathbb{K} & \to & \mathrm{End}_{\mathbb{K}}(X) \cong X \otimes X^* \\ \lambda & \mapsto & \lambda \mathrm{id}_X \end{array}$$

We call b_X a right coevaluation and $\tilde{b}_X$ a left coevaluation.

Definition 2.5.12

1. *Let $\mathcal{C}$ be a tensor category. An object V of $\mathcal{C}$ is called right dualizable, if there exists an object $V^\vee \in \mathcal{C}$ and morphisms*

   $$b_V : \mathbb{I} \to V \otimes V^\vee \quad \text{and} \quad d_V : V^\vee \otimes V \to \mathbb{I}$$

   such that

   $$\begin{array}{rcl} r_V \circ (\mathrm{id}_V \otimes d_V) \circ a_{V,V^\vee,V} \circ (b_V \otimes \mathrm{id}_V) \circ l_V^{-1} & = & \mathrm{id}_V \\ l_V \circ (d_V \otimes \mathrm{id}_{V^\vee}) \circ a^{-1}_{V^\vee,V,V^\vee} \circ (\mathrm{id}_{V^\vee} \otimes b_V) \circ r_V^{-1} & = & \mathrm{id}_{V^\vee} \end{array}$$

   Such an object $V^\vee$ is called a right dual to V.

   The morphism d_V is called an evaluation, the morphism b_V a coevaluation.

2. *A monoidal category is called right-rigid or right-autonomous, if every object has a right dual.*

3. *A left dual to V is an object ${}^\vee V$ of $\mathcal{C}$, together with two morphisms*

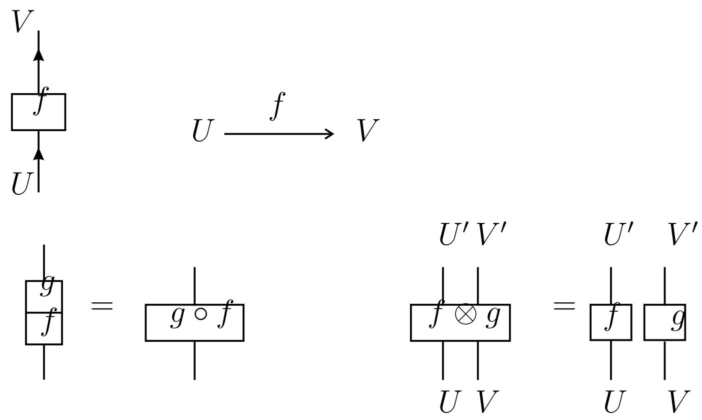
   $$\tilde{b}_V : \mathbb{I} \to {}^\vee V \otimes V \quad \text{and} \quad \tilde{d}_V : V \otimes {}^\vee V \to \mathbb{I}$$

   such that analogous equations hold. A left-rigid or left autonomous category is a monoidal category in which every object has a left dual.

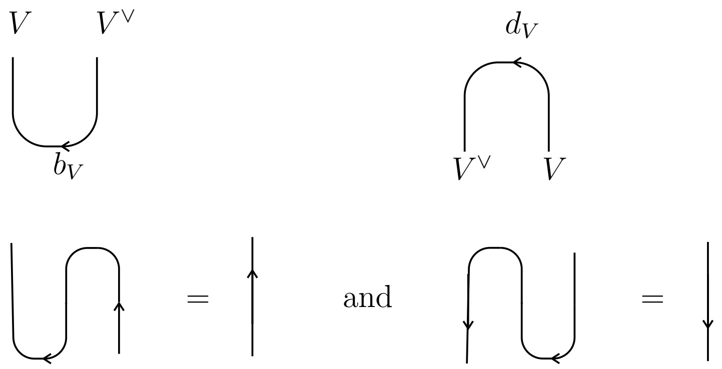
4. *A monoidal category is* rigid *or* autonomous*, if it is both left and right rigid or autonomous.*

Remarks 2.5.13.

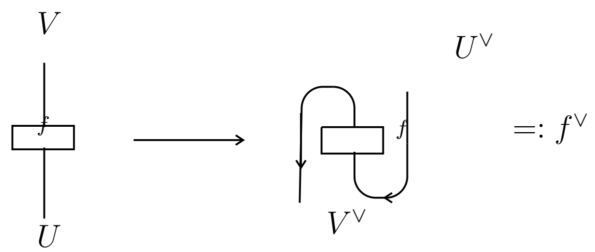
1. A $\mathbb{K}$-vector space has a left dual (or a right dual), if and only if it is finite-dimensional.
2. In any *strict* tensor category, we have a graphical calculus. Morphisms are to be read from below to above. Composition of morphisms is by joining vertically superposed boxes. The tensor product of morphisms is described by horizontally juxtaposed boxes.



We represent coevaluation and evaluation of a right duality and their defining properties as follows.



3. By definition, a right duality in a rigid tensor category associates to every object V another object $V^\vee$. We also define its action on morphisms:



One checks graphically that this gives a functor $?^\vee : \mathcal{C} \to \mathcal{C}^{\mathrm{opp}}$, i.e. a contravariant functor.

Similarly, we get from the left duality a functor ${}^\vee ? : \mathcal{C} \to \mathcal{C}^{\mathrm{opp}}$. There is no reason, in general, for these functors to be isomorphic.

Proposition 2.5.14.
Let H be a Hopf algebra. Let V be an H-module. We denote by $V^\vee$ the H-module defined on the dual vector space $V^* = \mathrm{Hom}_{\mathbb{K}}(V, \mathbb{K})$ with the action given by pullback of the transpose along S. If the antipode has an inverse $S^{-1} \in \mathrm{End}(H)$, then denote by ${}^\vee V$ the H-module defined on the same vector space $V^* = \mathrm{Hom}_{\mathbb{K}}(V, \mathbb{K})$ with the action given by pullback of the transpose along S^{-1}.

1. The right evaluation
$$\begin{array}{rcl} d_V : V^\vee \otimes V & \to & \mathbb{K} \\ \alpha \otimes v & \mapsto & \alpha(v) \end{array}$$
is a map of H-modules.

2. If the antipode S of H is invertible, the left evaluation
$$\begin{array}{rcl} \tilde{d}_V : V \otimes {}^\vee V & \to & \mathbb{K} \\ v \otimes \alpha & \mapsto & \alpha(v) \end{array}$$
is a map of H-modules.

3. If V is finite-dimensional, then the right coevaluation
$$b_V : \mathbb{K} \to V \otimes V^\vee$$
is a map of H-modules.

4. If V is finite-dimensional and if the antipode S of H is invertible, then the left coevaluation is a map of H-modules.

Proof.

1. Let $a \in H, v \in V$ and $\alpha \in V^*$. Then we compute
$$\begin{array}{rcl} d_V(a.(\alpha \otimes v)) & = & \sum_{(a)} d_V(a_{(1)}.\alpha \otimes a_{(2)}.v) \\ & = & \sum_{(a)} (a_{(1)}.\alpha)(a_{(2)}.v) \quad \text{[defn. of } d_V\text{]} \\ & = & \alpha\Big(\sum_{(a)} S(a_{(1)})a_{(2)}.v\Big) \quad \text{[defn. of action on } V^\vee\text{]} \\ & = & \alpha(\epsilon(a)v) = \epsilon(a)\alpha(v) = a.d_V(\alpha \otimes v)\,. \end{array}$$
In the last line, we used the defining property of the antipode, linearity of α and the definition of the H-action on $\mathbb{K}$.

2. Similarly, we use the identity
$$S^{-1}(a_{(2)}) \cdot a_{(1)} = 1_H \epsilon(a)$$
from lemma 2.5.7 to compute
$$\begin{array}{rcl} \tilde{d}_V(a.(v \otimes \alpha) & = & \tilde{d}_V(a_{(1)}.v \otimes a_{(2)}.\alpha) \\ & = & \alpha(S^{-1}(a_{(2)})a_{(1)}.v) \\ & = & \alpha(\epsilon(a)v) = a.\alpha(v) \end{array}$$

3. As a final example, we discuss the left coevaluation. We have to compare linear maps $\mathbb{K} \to V^* \otimes V \cong \mathrm{End}_{\mathbb{K}}(V)$. We compute for $\lambda \in K$ and $v \in V$

$$\begin{aligned} a.\tilde{b}_V(\lambda)v &= \lambda \textstyle\sum_i x^i(S^{-1}(a_{(1)}).v) \otimes a_{(2)}.x_i \\ &= \lambda a_{(2)}(\textstyle\sum_i x^i \otimes x_i)(S^{-1}(a_{(1)}).v) \\ &= \lambda \left(a_{(2)} \cdot S^{-1}(a_{(1)})\right).v = \epsilon(a)v = \tilde{b}_V(a.\lambda) \end{aligned}$$

$\square$

We conclude:

Corollary 2.5.15.
The category H-$\mathrm{mod}_{\mathrm{fd}}$ of finite-dimensional modules over any Hopf algebra is right rigid. If the antipode S of H is an invertible element of $\mathrm{End}_{\mathbb{K}}(H)$, the category H-$\mathrm{mod}_{\mathrm{fd}}$ is rigid.

We construct another example of a monoidal category.

Definition 2.5.16

1. *Let n be any positive integer. We define a category $\mathrm{Cob}(n)$ of n-dimensional <u>cobordisms</u> as follows:*

   (a) *An object of $\mathrm{Cob}(n)$ is a closed oriented $(n-1)$-dimensional smooth oriented manifold.*

   (b) *Given a pair of objects $M, N \in \mathrm{Cob}(n)$, a morphism $M \to N$ is a class of bordisms from M to N. A bordism is an oriented, n-dimensional smooth oriented manifold B with boundary, together with an orientation preserving diffeomorphism*

   $$\phi_B : \quad \overline{M} \coprod N \xrightarrow{\sim} \partial B \ .$$

   Here $\overline{M}$ denotes the same manifold with opposite orientation.
   Two bordisms B, B' give the same morphism, if there is an orientation-preserving diffeomorphism $\phi : B \to B'$ which restricts to the evident diffeomorphism

   $$\partial B \xrightarrow{\phi_{B'}^{-1}} \overline{M} \coprod N \xrightarrow{\phi_{B'}} \partial B' \ ,$$

   i.e. the following diagram commutes:

   $$\begin{tikzcd} B \arrow[rr, "\phi"] & & B' \\ & \overline{M} \coprod N \arrow[ul, "\phi_B"] \arrow[ur, "\phi'_B"'] & \end{tikzcd}$$

   (c) *For any object $M \in \mathrm{Cob}(n)$, the identity map is represented by the product bordism $B = M \times [0,1]$, i.e. the so-called cylinder over M.*

   (d) *Composition of morphisms in $\mathrm{Cob}(n)$ is given by gluing bordisms together.: given objects $M, M', M'' \in \mathrm{Cob}(n)$, and bordisms $B : M \to M'$ and $B' : M' \to M''$, the composition is defined to be the morphism represented by the manifold $B \coprod_{M'} B'$. (To get a smooth structure on this manifold, choices like collars are necessary. They lead to diffeomorphic glued bordisms, however.)*

2. *For each n, the category* $\mathrm{Cob}(n)$ *can be endowed with the structure of a tensor category. The tensor product*
$$\otimes : \mathrm{Cob}(n) \times \mathrm{Cob}(n) \to \mathrm{Cob}(n)$$
is given by disjoint union. The unit object of $\mathrm{Cob}(n)$ *is the empty set, regarded as a smooth manifold of dimension $n-1$.*

Example 2.5.17.
The objects of $\mathrm{Cob}(1)$ are finitely many oriented points. Thus objects are finite unions of $(\bullet, +)$ and $(\bullet, -)$.

The morphisms are oriented one-dimensional manifolds, possibly with boundary, i.e. unions of intervals and circles.

An isomorphism class of objects is characterized by the numbers (n_+, n_-) of points with positive and negative orientation. Sometimes, one also considers another equivalence relation on objects: two $d-1$-dimensional closed manifolds M and N are called <u>cobordant</u>, if there is a cobordism $B : N \to N$. Since there is a cobordism $(\bullet, +) \coprod (\bullet, -) \to \emptyset$, the objects (n_+, n_-) and (n'_+, n'_-) are cobordant, if and only if $n_+ - n_- = n'_+ - n'_-$.

One can also define a category of unoriented cobordisms. In this case, simple objects are disjoint unions of points, isomorphism classes are in bijection to the number of points. Since a pair of points can annihilate, there are only two cobordism classes, containing the set with an even and and odd number of points, respectively.

We next comment on the rigidity of the category $\mathrm{Cob}(n)$:

Observation 2.5.18.
Let M be a closed oriented $n-1$-dimensional manifold. Then the oriented n-dimensional manifold $B := M \times [0,1]$, the cylinder over M, has boundary $M \coprod \overline{M}$. The manifold B can be considered as a cobordism in six different ways, corresponding to decomposition of its boundary:

- As a bordism $M \to M$. This represents the identity on M.
- As a bordism $\overline{M} \to \overline{M}$. This represents the identity on $\overline{M}$.
- As a morphism $d_M : \overline{M} \coprod M \to \emptyset$ or, alternatively, as a morphism $\tilde{d}_M : M \coprod \overline{M} \to \emptyset$.
- As a morphism $\tilde{b}_M : \emptyset \to \overline{M} \coprod M$ or, alternatively, as a morphism $b_M : \emptyset \to M \coprod \overline{M}$.

One checks that the axioms of a left and a right duality hold. We conclude that the category $\mathrm{Cob}(n)$ is rigid.

We discuss a final example.

Example 2.5.19.
We have seen that for any small category $\mathcal{C}$, the endofunctors of $\mathcal{C}$, together with natural transformations, form a monoidal category. In this case, a left dual of an object, i.e. of a functor, is also called its left adjoint functor. Indeed, the following generalization beyond endofunctors is natural.

Definition 2.5.20

1. *Let $\mathcal{C}$ and $\mathcal{D}$ be any categories. A functor $F : \mathcal{C} \to \mathcal{D}$ is called left adjoint to a functor $G : \mathcal{D} \to \mathcal{C}$, if for any two objects c in $\mathcal{C}$ and d in $\mathcal{D}$ there is an isomorphism of Hom-spaces*
$$\Phi_{c,d} : \ \mathrm{Hom}_{\mathcal{C}}(c, Gd) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}}(Fc, d)$$
with the following natural property:

For any homomorphism $c' \xrightarrow{f} c$ in $\mathcal{C}$ and $d \xrightarrow{g} d'$ in $\mathcal{D}$ consider for $\varphi \in \mathrm{Hom}_{\mathcal{D}}(Fc, d)$ the morphism
$$\mathrm{Hom}(Ff, g)(\varphi) := Fc' \xrightarrow{Ff} Fc \xrightarrow{\varphi} d \xrightarrow{g} d' \in \mathrm{Hom}_{\mathcal{D}}(Fc', d')$$
and for $\varphi \in \mathrm{Hom}_{\mathcal{C}}(c, Gd)$ the morphism
$$\mathrm{Hom}(f, Gg)(\varphi) := c' \xrightarrow{f} c \xrightarrow{\varphi} Gd \xrightarrow{Gg} Gd' \in \mathrm{Hom}_{\mathcal{C}}(c', Gd') \ .$$
The naturality requirement is then the requirement that the diagram
$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(c, Gd) \arrow[r, "\mathrm{Hom}(f,Gg)"'] \arrow[d, "\Phi_{c,d}"] & \mathrm{Hom}_{\mathcal{C}}(c', Gd') \arrow[d, "\Phi_{c',d'}"'] \\
\mathrm{Hom}_{\mathcal{D}}(Fc, d) \arrow[r, "\mathrm{Hom}(Ff,g)"'] & \mathrm{Hom}_{\mathcal{D}}(Fc', d')
\end{tikzcd}$$
commutes for all morphisms f, g.

2. *We write $F \dashv G$ and also say that the functor G is a right adjoint to F.*

Examples 2.5.21.

1. As explained in the appendix, the forgetful functor
$$U : \ \mathrm{vect}(\mathbb{K}) \to \mathrm{Set} \ ,$$
which assigns to any $\mathbb{K}$-vector space the underlying set has as a left adjoint, the freely generated vector space on a set:
$$F : \ \mathrm{Set} \to \mathrm{vect}(\mathbb{K}) \, ,$$
Indeed, we have for any set M and any $\mathbb{K}$-vector space V an isomorphism
$$\begin{array}{rlcl} \Phi_{M,V} : & \mathrm{Hom}_{\mathrm{Set}}(M, U(V)) & \to & \mathrm{Hom}_K(F(M), V) \\ & \varphi & \mapsto & \Phi_{M,V}(\varphi) \end{array}$$
where $\Phi_{M,V}(\varphi)$ is the $\mathbb{K}$-linear map defined by prescribing values in V on the basis of $F(M)$ using φ and extending linearly:
$$\Phi_{M,V}(\varphi)(\sum_{m \in M} \lambda_m m) := \sum_{m \in M} \lambda_m \varphi(m) \ .$$
In particular, we find the isomorphism of sets $\mathrm{Hom}_{\mathrm{Set}}(\emptyset, U(V)) \cong \mathrm{Hom}_{\mathbb{K}}(F(\emptyset), V)$ for all $\mathbb{K}$-vector spaces V. Thus $\mathrm{Hom}_{\mathbb{K}}(F(\emptyset), V)$ has exactly one element for any vector space V. This shows $F(\emptyset) = \{0\}$, i.e. the vector space freely generated by the empty set is the zero-dimensional vector space.

2. In general, freely generated objects are obtained as images under left adjoints of forgetful functors. It is, however, not true that any forgetful functor has a left adjoint. As a counterexample, take the forgetful functor U from fields to sets. Suppose a left adjoint exists and study the image K of the empty set under it. Then K is a field such that for any other field L, we have a bijection

$$\mathrm{Hom}_{Field}(K,L) \cong \mathrm{Hom}_{Set}(\emptyset, U(L)) \cong \star \ .$$

Since morphisms of fields are injective, such a field K would be a subfield of any field L. Such a field does not exist.

To make contact with the notion of duality, the following reformulation is needed:

Observation 2.5.22.

1. Let $F \dashv G$ be adjoint functors. From the definition, we get isomorphisms

$$\mathrm{Hom}_{\mathcal{C}}(G(d), G(d)) \cong \mathrm{Hom}_{\mathcal{D}}(F(G(d)), d)$$

and

$$\mathrm{Hom}_{\mathcal{D}}(F(c), F(c)) \cong \mathrm{Hom}_{\mathcal{C}}(c, G(F(c))) \ .$$

The images of the identity on $G(d)$ and $F(c)$ respectively form together natural transformations

$$\epsilon : F \circ G \to \mathrm{id}_{\mathcal{D}} \quad \text{and} \quad \eta : \mathrm{id}_{\mathcal{C}} \to G \circ F \ .$$

Note the different order of the functors F, G in the composition and compare to the definition of a duality.

These natural transformations have the property that for all objects c in $\mathcal{C}$ and d in $\mathcal{D}$ the morphisms

$$G(d) \xrightarrow{\eta_{G(d)}} (GF)\, G(d) = \ G(FG)(d) \xrightarrow{G(\epsilon_d)} G(d)$$

and

$$F(c) \ \xrightarrow{F(\eta_c)} F(GF)(c) = (FG)F(c) \xrightarrow{\epsilon_{F(c)}} F(c)$$

are identities. Again compare with the properties of dualities. In particular, the left adjoint of an endofunctor is its left dual in the monoidal category of endofunctors with monoidal product $F \otimes G = G \circ F$. For proofs, we refer to [McL, Chapter IV]

2. Conversely, we can recover the adjunction isomorphisms $\Phi_{c,d}$ from the natural transformations ϵ and η by

$$\mathrm{Hom}_{\mathcal{C}}(c, G(d)) \xrightarrow{F} \mathrm{Hom}_{\mathcal{D}}(F(c), F(G(d)) \xrightarrow{(\epsilon_d)_*} \mathrm{Hom}_{\mathcal{D}}(F(c), d)$$

and their inverses by

$$\mathrm{Hom}_{\mathcal{D}}(F(c), d) \xrightarrow{G} \mathrm{Hom}_{\mathcal{C}}(G(F(c)), G(d)) \xrightarrow{\eta_d^*} \mathrm{Hom}_{\mathcal{C}}(c, G(d))\,.$$

3. Note that a pair of adjoint functors $F \dashv G$ is an equivalence of categories, if and only if ϵ and η are natural isomorphisms of functors.

Example 2.5.23.
Let $\mathcal{C}$ be a rigid tensor category. Then for any triple U, V, W of objects of $\mathcal{C}$, we have natural bijections

$$\begin{array}{ccc} \mathrm{Hom}(U \otimes V, W) & \cong & \mathrm{Hom}(U, W \otimes V^\vee) \\ \lambda & \mapsto & (\lambda \otimes \mathrm{id}_{V^\vee}) \circ (\mathrm{id}_U \otimes b_V) \\ \mathrm{Hom}(U \otimes V, W) & \cong & \mathrm{Hom}(V, {}^\vee U \otimes W) \\ \lambda & \mapsto & (\mathrm{id}_{{}^\vee U} \otimes \lambda) \circ (\tilde{b}_U \otimes \mathrm{id}_W) \end{array}$$

We have thus the following adjunctions of functors:

$$(? \otimes V) \dashv (? \otimes V^\vee) \quad \text{and} \quad (U \otimes ?) \dashv ({}^\vee U \otimes ?) \ .$$

2.6 Examples of Hopf algebras

We will now consider several examples of Hopf algebras that are neither group algebras nor universal enveloping algebras.

Observation 2.6.1.
The following example is due to Taft. Let $\mathbb{K}$ be a field and N a natural number. Assume that there exists a primitive N-th root of unity ζ in $\mathbb{K}$. Consider the algebra H generated over $\mathbb{K}$ by two elements g and x, subject to the relations

$$g^N = 1, \quad x^N = 0, \quad xg = \zeta gx \ .$$

We say that the elements x and g ζ-commute. We claim that there are algebra maps

$$\Delta : H \to H \otimes H, \quad S : H \to H^{\mathrm{opp}} \quad \text{and } \epsilon : H \to \mathbb{K}$$

uniquely determined on the generators g, x by

$$\begin{gathered} \Delta(g) = g \otimes g \text{ and } \Delta(x) = 1 \otimes x + x \otimes g \\ \epsilon(x) = 0 \text{ and } \epsilon(g) = 1 \\ S(g) = g^{-1} \text{ and } S(x) = -xg^{-1} \end{gathered}$$

The special case $\zeta = -1$, i.e. $N = 2$, is also known as Sweedler's Hopf algebra H_4.

We work out the coproduct Δ in detail and leave the discussion of the counit ϵ and the antipode S to the reader. We have to show that the map Δ is well-defined, i.e. compatible with the three defining relations. Compatibility with the relation $g^N = 1$ follows from

$$\Delta(g^n) = \Delta(g)^n = g^n \otimes g^n \ ,$$

which equals $1 \otimes 1 = \Delta(1)$. To show compatibility with the relation $xg = \zeta gx$, compare

$$\Delta(x) \cdot \Delta(g) = (1 \otimes x + x \otimes g) \cdot (g \otimes g) = g \otimes xg + xg \otimes g^2$$

and

$$\zeta\Delta(g) \cdot \Delta(x) = \zeta(g \otimes g) \cdot (1 \otimes x + x \otimes g) = \zeta g \otimes gx + \zeta gx \otimes g^2$$

which implies $\Delta(xg) = \Delta(\zeta gx)$.

For the remaining relation, we need a few more relations:

Observation 2.6.2.

1. Define in the polynomial algebra $\mathbb{Z}[\mathbf{q}]$ for $n \in \mathbb{N}$:
$$(n)_{\mathbf{q}} := 1 + \mathbf{q} + \ldots + \mathbf{q}^{n-1} \ .$$
Define
$$(n)!_{\mathbf{q}} := (n)_{\mathbf{q}} \cdots (2)_{\mathbf{q}}(1)_{\mathbf{q}} \in \mathbb{Z}[\mathbf{q}] \ .$$
Finally, define for $0 \leq i \leq n$ in the field of fractions
$$\left(\begin{array}{c} n \\ i \end{array}\right)_{\mathbf{q}} := \frac{(n)!_{\mathbf{q}}}{(n-i)!_{\mathbf{q}}(i)!_{\mathbf{q}}} \ .$$

2. We note that
$$\mathbf{q}^k(n+1-k)_{\mathbf{q}} + (k)_{\mathbf{q}} = (n+1)_{\mathbf{q}}$$
and thus deduce
$$\begin{array}{rcl} \mathbf{q}^k \left(\begin{array}{c} n \\ k \end{array}\right)_{\mathbf{q}} + \left(\begin{array}{c} n \\ k-1 \end{array}\right)_{\mathbf{q}} & = & \frac{(n)!_{\mathbf{q}}}{(n+1-k)!_{\mathbf{q}}(k)!_{\mathbf{q}}} \cdot \left(\mathbf{q}^k(n+1-k)_{\mathbf{q}} + (k)_{\mathbf{q}}\right) \\ & = & \left(\begin{array}{c} n+1 \\ k \end{array}\right)_{\mathbf{q}} \end{array}$$
from which we conclude by induction on n that $\left(\begin{array}{c} n \\ k \end{array}\right)_{\mathbf{q}} \in \mathbb{Z}[\mathbf{q}]$.

For any associative algebra A over a field $\mathbb{K}$, we can then specialize for $q \in \mathbb{K}$ the values of $(n)_{\mathbf{q}}$, $(n)!_{\mathbf{q}}$ and the of the $\mathbf{q}$-binomials and denote them by $(n)_q$, $(n)!_q$ and $\left(\begin{array}{c} n \\ k \end{array}\right)_q$. Note that $(n)_1 = n$. If q is an N-th root of unity different from 1, then
$$(N)_q = 1 + q + \ldots + q^{N-1} = \frac{1-q^N}{1-q} = 0 \ .$$
In a field of characteristic p with p a divisor of q, the quantity $N = 1+\ldots+1$ also vanishes. There are similarities between q-deformed situations and situations in fields of prime characteristic. As a further consequence,
$$\left(\begin{array}{c} N \\ k \end{array}\right)_q = 0 \quad \text{for all} \quad 0 < k < N \ .$$

Lemma 2.6.3.
Let A be an associative algebra over a field $\mathbb{K}$ and $q \in \mathbb{K}$. Let $x, y \in A$ be two elements that q-commute, i.e. $xy = qyx$. Then the quantum binomial formula holds for all $n \in \mathbb{N}$:
$$(x+y)^n = \sum_{i=0}^{n} \left(\begin{array}{c} n \\ i \end{array}\right)_q y^i x^{n-i} \ .$$

Proof.
By induction on n, using the relation we just proved. □

We then conclude, since for the Taft-Hopf algebra $1 \otimes x$ and $x \otimes g$ ζ-commute, we have
$$\begin{array}{rcl} \Delta(x)^N & = & (1 \otimes x + x \otimes g)^N = \sum_{i=0}^{N} \left(\begin{array}{c} N \\ i \end{array}\right)_\zeta (x \otimes g)^i (1 \otimes x)^{N-i} \\ & = & (x \otimes g)^N + (1 \otimes x)^N = x^N \otimes g^N + 1 \otimes x^N = 0 \end{array}$$

In the second identity, we used that the binomial coefficients vanish, except for $i = 0, N$. This shows that the Taft Hopf algebra is well-defined. It can be shown to have finite dimension N^2 and a basis $g^i x^j$ with $0 \leq i, j \leq N-1$.

We remark that for the square of the antipode, we have

$$S^2(g) = S(g^{-1}) = g \quad \text{and } S^2(x) = S(-xg^{-1}) = -S(g^{-1})S(x) = gxg^{-1}$$

which is a so-called inner automorphism of order N. Thus there exist finite-dimensional Hopf algebras with antipode S of any even order.

Note that the Taft algebra is, in general, not cocommutative. Indeed, one can show that over an algebraically closed field $\mathbb{K}$ of characteristic zero, all finite-dimensional cocommutative Hopf algebras are group algebras of some finite group. This is not true in finite characteristic. To provide a counterexample, we need a class of Lie algebras with extra structure: restricted Lie algebras.

Observation 2.6.4.
Let $\mathbb{K}$ be a field of prime characteristic, $\mathrm{char}\mathbb{K} = p$. Let A be any $\mathbb{K}$-algebra. The algebra A might even be non-associative. Then the derivations $\mathrm{Der}(A)$ form a Lie subalgebra of the Lie algebra $\mathrm{End}_{\mathbb{K}}(A)$. Moreover, if $D : A \to A$ is a derivation, then because of

$$D^p(a \cdot b) = \sum_{i=0}^{p} \binom{p}{i} D^i(a) \cdot D^{p-i}(b) = D^p(a) \cdot b + a \cdot D^p(b)$$

$D^p : A \to A$ is a derivation as well. Thus the Lie algebra $\mathrm{Der}(A)$ has more structure: the structure of a restricted Lie algebra.

Definition 2.6.5

1. *Let $\mathbb{K}$ be a field of characteristic $p > 0$. A restricted Lie algebra L over $\mathbb{K}$ is a Lie algebra, together with a map*
$$\begin{array}{ccc} L & \to & L \\ a & \mapsto & a^{[p]} \end{array}$$
such that for all $a, b \in L$ and $\lambda \in \mathbb{K}$
$$\begin{array}{rcl} (\lambda a)^{[p]} & = & \lambda^p a^{[p]} \\ \mathrm{ad}(b^{[p]}) & = & (\mathrm{ad} b)^p \\ (a+b)^{[p]} & = & a^{[p]} + b^{[p]} + \sum_{i=1}^{p-1} s_i(a,b) \end{array}$$
Here $\mathrm{ad}(a) : L \to L$ denotes the adjoint representation with $\mathrm{ad}(a)(b) = [a, b]$. Moreover, $s_i(a, b)$ is the coefficient of λ^{i-1} in $\mathrm{ad}(\lambda a + b)^{p-1}(a)$.

2. *A morphism of restricted Lie algebras $f : L \to L'$ is a morphism of Lie algebras such that $f(a^{[p]}) = f(a)^{[p]}$ for all $a \in L$.*

Example 2.6.6.
If A is an associative $\mathbb{K}$-algebra with $\mathbb{K}$ a field of prime characteristic, $\mathrm{char}(\mathbb{K}) = p$, then the commutator and the map $a \mapsto a^p$ turns it into a restricted Lie algebra.

Observation 2.6.7.

1. Let L be a restricted Lie algebra, U its universal enveloping algebra. Denote by B the two-sided ideal in U generated by $a^p - a^{[p]}$ for all $a \in L$. Denote by $\mathcal{U}$ the quotient algebra $\mathcal{U} := U/B$. It is a restricted Lie algebra with $a^{[p]}$ given by the p-th power.

2. Then the canonical quotient map $\pi : L \to \mathcal{U}$ is a morphism of restricted Lie algebras. It is universal in the following sense: if A is any associative algebra over $\mathbb{K}$ and $f : L \to A$ a morphism of restricted Lie algebras, then there exists a unique algebra map $F : \mathcal{U} \to A$ such that $f = F \circ \pi$:
$$\begin{tikzcd} L \arrow[r, "\pi"] \arrow[dr, "f"'] & \mathcal{U} \arrow[d, dashed, "\exists! F"] \\ & A \end{tikzcd}$$

3. By the universal property, the restricted morphisms
$$\begin{array}{rcl} L & \to & \mathbb{K} \\ a & \mapsto & 0 \\ L & \to & L \times L \\ a & \mapsto & (a, a) \\ L & \to & L^{\text{opp}} \\ a & \mapsto & -a \end{array}$$
define algebra maps
$$\epsilon : \mathcal{U} \to \mathbb{K} \,, \quad \Delta : \mathcal{U} \to \mathcal{U} \otimes \mathcal{U} \quad \text{and} \quad S : \mathcal{U} \to \mathcal{U}^{\text{opp}}$$
that are uniquely determined by
$$\begin{array}{rcl} \epsilon(\pi(a)) & = & 0 \\ \Delta(\pi(a)) & = & 1 \otimes \pi(a) + \pi(a) \otimes 1 \\ S(\pi(a)) & = & -\pi(a) \end{array}$$
for $a \in L$ that turn $\mathcal{U}$ into a cocommutative Hopf algebra. It is called the u-algebra of the restricted Lie algebra L.

4. One has the following analogue of the Poincaré-Birkhoff-Witt theorem: the natural map $\iota_L : L \to \mathcal{U}$ is injective. If $(u_i)_{i \in I}$ is an ordered basis for L, then
$$u_{i_1}^{k_1} \cdot u_{i_2}^{k_2} \ldots u_{i_r}^{k_r} \quad \text{with} \quad i_1 \le i_2 \le \ldots i_r \quad \text{and} \quad 0 \le k_j \le p-1$$
is a basis of $\mathcal{U}$.

5. Thus if L has finite-dimension, $\dim_{\mathbb{K}} L = n$, then $\mathcal{U}$ is finite-dimensional of dimension $\dim \mathcal{U} = p^n$. Thus $\mathcal{U}$ is a cocommutative finite-dimensional Hopf algebra. We next show that it is not isomorphic to the group algebra of any finite group.

Definition 2.6.8

1. *An element $h \in H \setminus \{0\}$ of a Hopf algebra H is called* group-like *, if $\Delta(h) = h \otimes h$. The set of group-like elements of a Hopf algebra H is denoted by $G(H)$.*

2. *An element $h \in H$ of a bialgebra H is called a* primitive element*, if $\Delta(h) = 1 \otimes h + h \otimes 1$. The set of primitive elements of a bialgebra H is denoted by $P(H)$.*

3. *More generally, if $g_1, g_2 \in G(H)$ are group-like elements, an element $h \in H$ is called* <u>*g_1, g_2-primitive*</u>*, if $\Delta(h) = h \otimes g_1 + g_2 \otimes h$.*

Remark 2.6.9.

1. Consider group-like elements in the dual Hopf algebra H^*. These are $\mathbb{K}$-linear maps $\beta : H \to \mathbb{K}$ such that for $h_1, h_2 \in H$

$$\beta(h_1 \cdot h_2) = \Delta(\beta)(h_1 \otimes h_2) = (\beta \otimes \beta)(h_1 \otimes h_2) = \beta(h_1) \cdot \beta(h_2) \; .$$

   Thus the group-like elements in the dual Hopf algebra H^* are the algebra maps $H \to \mathbb{K}$ which are also called <u>characters</u>.

2. The primitive elements in the dual Hopf algebra H^* are the derivations of H, i.e. the linear maps $D : H \to \mathbb{K}$ such that for all $h_1, h_2 \in H$

$$D(h_1 \cdot h_2) = (1 \otimes D + D \otimes 1)(h_1 \otimes h_2) = \epsilon(h_1) \cdot D(h_2) + D(h_1) \cdot \epsilon(h_2) \; .$$

We need the following

Lemma 2.6.10 (Artin)**.**
Let $\chi_1, \ldots \chi_n$ be pairwise different characters $\chi_i : M \to \mathbb{K}^\times$, i.e. group homomorphisms of a monoid M with values in the multiplicative group $\mathbb{K}^\times$ of a field $\mathbb{K}$. Then these characters are linearly independent as $\mathbb{K}$-valued functions on M.

Proof.
By induction on n. The assertion holds for $n = 1$, since for a character $\chi(M) \subseteq \mathbb{K}^\times$ so that a single character is linearly independent.

Thus assume $n > 1$ and consider a non-trivial relation

$$a_1\chi_1 + \cdots + a_m\chi_m = 0 \tag{3}$$

of minimal length m in which all coefficients are non-zero, $a_m \neq 0$. Thus $2 \leq m \leq n$.

From $\chi_1 \neq \chi_2$ we deduce that there is $z \in M$ such that $\chi_1(z) \neq \chi_2(z)$. Using the multiplicativity of characters, we find for all $x \in M$:

$$\begin{aligned} 0 &= a_1\chi_1(zx) + \cdots + a_m\chi_m(zx) \\ &= a_1\chi_1(z)\chi_1(x) + \cdots + a_m\chi_m(z)\chi_m(x). \end{aligned}$$

and thus a different non-trivial linear relation of the characters:

$$\sum_{i=1}^{m} a_i\chi_i(z)\,\chi_i = 0\,.$$

Dividing this relation by $\chi_1(z)$ and subtracting it from (3), we find

$$a_2\Bigg(\underbrace{\frac{\chi_2(z)}{\chi_1(z)} - 1}_{\neq 0}\Bigg)\chi_2 + \cdots + a_m\left(\frac{\chi_m(z)}{\chi_1(z)} - 1\right)\chi_m = 0\,.$$

and thus a shorter non-trivial relation. □

Proposition 2.6.11.
Let H be a Hopf algebra.

1. We have $\epsilon(x) = 1$ for any group-like element $x \in H$.

2. The set of group-like elements $G(H)$ is a subgroup of the set of units of H. The inverse of $x \in G(H)$ is $S(x)$.

3. Distinct group-like elements are linearly independent. In particular, the set of group-like elements of a group algebra $\mathbb{K}[G]$ is precisely G.

Proof.

1. We note that by definition of the counit ϵ,
$$x = (\epsilon \otimes \mathrm{id}) \circ \Delta(x) = \epsilon(x)x \ .$$
Since by definition for a group-like element x, we have $x \neq 0$, this implies over a field $\epsilon(x) = 1$.

2. Using the fact that S is a coalgebra antihomomorphism, we find for a group-like element $x \in H$
$$\Delta(S(x)) = (S \otimes S) \circ \Delta^{\mathrm{copp}}(x) = (S \otimes S)(x \otimes x) = S(x) \otimes S(x)$$
so that $S(x)$ is group-like, provided that $S(x) \neq 0$. The defining identity of the antipode, applied to a group-like element x shows
$$xS(x) = (\mathrm{id} * S)(x) = 1\epsilon(x) = 1$$
so that $S(x)$ is the multiplicative inverse of x. In particular, $S(x) \neq 0$.

3. Using the embedding $H \hookrightarrow H^{**}$, group-like elements of H are characters on the monoid H^* with values in the field $\mathbb{K}$. By lemma 2.6.10, characters are linearly independent.

□

Proposition 2.6.12.

1. For any primitive element x in a bialgebra H, we have $\epsilon(x) = 0$.

2. The commutator
$$[x, y] = xy - yx$$
of two primitive elements x, y of a bialgebra H is again primitive.

Proof.

1. The equation
$$x = (\epsilon \otimes \mathrm{id}) \circ \Delta(x) = (\epsilon \otimes \mathrm{id})(x \otimes 1 + 1 \otimes x) = \epsilon(x)1 + \epsilon(1)x$$
for x primitive implies $\epsilon(x) = 0$.

2. We compute for primitive elements $x, y \in H$

$$\begin{aligned}\Delta(x \cdot y) &= \Delta(x)\Delta(y) = (1 \otimes x + x \otimes 1)(1 \otimes y + y \otimes 1) \\ &= 1 \otimes xy + x \otimes y + y \otimes x + xy \otimes 1\end{aligned}$$

Combining this with the corresponding equation for $\Delta(yx)$, we find

$$\Delta([x, y]) = 1 \otimes [x, y] + [x, y] \otimes 1 .$$

□

The following proposition applies in particular to universal enveloping algebras of Lie algebras and u-algebras of restricted Lie algebras.

Lemma 2.6.13.
Let $\mathbb{K}$ be a field. If H is a Hopf algebra over $\mathbb{K}$ which is generated as an algebra by primitive elements, then the set of group-like elements of H is trivial, $G(H) = \{1_H\}$.

Proof.
Let $\{x_i\}_{i \in I}$ denote the family of non-zero primitive elements of H. For $n > 0$, denote by A_n the linear span in H of elements of the form $x_{i_1}^{k_1} \ldots x_{i_m}^{k_m}$ such that $k_j \in \mathbb{Z}_{\geq 0}$ with $k_1 + k_2 + \ldots k_m \leq n$. Then

- $A_n \subset A_{n+1}$.
- Since H is generated, as an algebra, by primitive elements, $\cup_{n \geq 0} A_n = H$.
- By multiplicativity of the coproduct, $\Delta(A_n) \subset \sum_{i=0}^{n} A_i \otimes A_{n-i}$

Let $g \neq 1$ be group-like. Then $g \in A_m$ for some m. Choose m to be minimal. Since g is non-trivial, $g \notin \mathbb{K} = A_0$. Then find $f \in H^*$ such that $f(A_0) = 0$ and $f(g) = 1$.

Now $g \in A_m$ implies

$$\Delta(g) = \sum_{i=0}^{m} a_i \otimes a'_{m-i}$$

with $a_j, a'_j \in A_j$ which in turn implies

$$g = \langle \mathrm{id} \otimes f, \Delta(g) \rangle = \sum_{i=0}^{m-1} a_i f(a'_{m-i}) \in A_{m-1} \quad ,$$

contradicting the minimality of m.

□

The lemma implies that the u-algebra of a non-trivial restricted Lie algebra cannot be isomorphic, as a Hopf algebra, to a group algebra, since it contains no non-trivial group-like elements.

Over fields of characteristic zero, we can recover a Lie algebra from the primitive elements in its universal enveloping algebra:

Proposition 2.6.14.
Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{K}$ of characteristic zero with an ordered basis and $\iota_{\mathfrak{g}} : \mathfrak{g} \to \mathrm{U}(\mathfrak{g})$ its universal enveloping algebra. Then the primitive elements of $\mathrm{U}(\mathfrak{g})$ are given by the image of $\mathfrak{g}$,

$$P(\mathrm{U}(\mathfrak{g})) = \iota_{\mathfrak{g}}(\mathfrak{g}) .$$

If $\mathrm{char}(\mathbb{K}) = p$, then the subspace of primitive elements of $\mathrm{U}(\mathfrak{g})$ is the span of all x^{p^k} with $x \in \mathfrak{g}$ and $k \geq 0$. It is a restricted Lie algebra.

Proof.
Define

$$\mathrm{U}^n(\mathfrak{g}) := \mathrm{span}_{\mathbb{K}}\{x^n | x \in \mathfrak{g}\}$$

and consider the direct sum:

$$\mathrm{U}(\mathfrak{g}) \supset \bigoplus_{n=0}^{\infty} \mathrm{U}^n(\mathfrak{g}) \, .$$

Since $x \in \mathfrak{g}$ is primitive in the Hopf algebra $\mathrm{U}(\mathfrak{g})$, we find

$$\Delta(x^n) = \sum_{k=0}^{n} \binom{n}{k} x^k \otimes x^{n-k} \quad .$$

Thus the direct sum is a subcoalgebra of $\mathrm{U}(\mathfrak{g})$ and the coproduct

$$\Delta : U(\mathfrak{g}) \to \mathrm{U}(\mathfrak{g}) \otimes \mathrm{U}(\mathfrak{g})$$

preserves the degree where the right hand side is endowed with the total degree. One checks inductively using the Poincaré-Birkhoff-Witt theorem, that the direct sum is closed under multiplication as well (the multiplication is not homogeneous, though). Since the elements $x \in \mathfrak{g}$ generate $\mathrm{U}(\mathfrak{g})$ as an algebra, we conclude $\mathrm{U}(\mathfrak{g}) = \bigoplus_{n=0}^{\infty} \mathrm{U}^n(\mathfrak{g})$.

We can restrict to homogenous elements

$$x = \sum_{j=1}^{l} \lambda_j (x_j)^n \in U^n(\mathfrak{g})$$

with $n \geq 2$ to investigate the coproduct:

$$\Delta(x) = \sum_{j=1}^{l} \lambda_j \sum_{k=0}^{n} \binom{n}{k} (x_j)^k \otimes (x_j)^{n-k}$$

Then x is primitive, if and only if all components with bigrade $(k, n-k)$ and $1 \leq k \leq n-1$ vanish. Applying multiplication, we find

$$\binom{n}{k} \sum_{j=1}^{l} \lambda_j (x_j)^n = 0 \quad \text{for all} \quad k = 1, \ldots, n-1 \, .$$

Over a field of characteristic zero, this implies $x = 0$. □

3 Finite-dimensional Hopf algebras

3.1 Hopf modules and integrals

The goal of this subsection is to introduce the notion of an integral on a Hopf algebra that is fundamental for representation theory and some applications to topological field theory. Hopf modules are an essential tool to show the existence of integrals.

Definition 3.1.1

1. *Let H be a $\mathbb{K}$-Hopf algebra. A $\mathbb{K}$-vector space V is called a right Hopf module, if*
   - *It has the structure of a right (unital) H-module.*
   - *It has the structure of a right (counital) H-comodule with right coaction $\Delta_V : V \to V \otimes H$.*
   - *Δ_V is a morphism of right H-modules.*
2. *If V and W are Hopf modules, a $\mathbb{K}$-linear map $f : V \to W$ is a map of Hopf modules, if it is both a module and a comodule map.*
3. *We denote by $\mathcal{M}_H^H$ the category of right Hopf modules. The categories ${}_H\mathcal{M}^H$, ${}^H\mathcal{M}_H$ and ${}_H^H\mathcal{M}$ are defined analogously.*

Remarks 3.1.2.

1. We have in Sweedler notation with $\Delta_V(v) = v_{(0)} \otimes v_{(1)}$ where $v_{(0)} \in V$ and $v_{(1)} \in H$
$$\Delta_V(v.x) = v_{(0)}.x_{(1)} \otimes v_{(1)} \cdot x_{(2)} \quad \text{for all} \quad x \in H, v \in V \ .$$
2. Any Hopf algebra H is a Hopf module over itself with action given by multiplication and coaction given by the coproduct.
3. More generally, let $K \subset H$ be a Hopf subalgebra. We may consider the restriction of the right action to K, but the coaction of all of H to get the category of right (H, K)-Hopf modules $\mathcal{M}_K^H$.
4. Given any H-module M, the tensor product $M \otimes H$ is a right H-module, where H is seen as a regular right H-module. Using $\Delta_{M\otimes H} := \mathrm{id}_M \otimes \Delta$ as a coaction, one checks that it becomes a Hopf module.
5. In particular, let M be a trivial H-module, i.e. a $\mathbb{K}$-vector space M with H-action defined by $m.h = \epsilon(h) \cdot m$ for all $h \in H$ and $m \in M$. In this case, the right action on $M \otimes H$ is $(m \otimes k).h = m \otimes k \cdot h$. Such a module is called a trivial Hopf module.

We also need the notion of invariants and coinvariants:

Definition 3.1.3
Let H be a Hopf algebra.

1. *Let M be a left H-module. The invariants of H on M are defined as the $\mathbb{K}$-vector subspace*
$$M^H := \{m \in M \mid h.m = \epsilon(h)m \quad \text{for all } h \in H\}$$
of M. This defines a functor $H\text{−mod} \to \mathrm{vect}(\mathbb{K})$. For invariants of left modules, the notation ${}^H M$ would be more logical, but is not common. Similarly, invariants are defined for right H-modules.
2. *Let (M, Δ_M) be a right H-comodule. The coinvariants of H on M are defined as the $\mathbb{K}$-vector space*
$$M^{\mathrm{co}H} := \{m \in M \mid \Delta_M(m) = m \otimes 1\} \ .$$

Examples 3.1.4.

1. If M is a right H-comodule, it can be considered as a left H^*-module. Then $M^{H^*} = M^{\mathrm{co}H}$.
2. Consider a group algebra, $H = \mathbb{K}[G]$. For a left $\mathbb{K}[G]$-module
$$M^{\mathbb{K}[G]} = \{m \in M \mid g.m = m \quad \text{for all } g \in G\} \ .$$
For a $\mathbb{K}[G]$-comodule the coinvariants
$$M^{\mathrm{co}\mathbb{K}[G]} = M_e$$
are the identity component of the G-graded vector space underlying according to example 2.2.8.3 the comodule.
3. For a module M over the universal enveloping algebra $H = \mathrm{U}(\mathfrak{g})$,
$$M^{\mathrm{U}(\mathfrak{g})} = \{m \in M \,|\, x.m = 0 \quad \text{for all} \quad x \in \mathfrak{g}\}.$$

The category of Hopf modules in itself is not of particular interest, but the equivalence to be stated next provides a powerful tool:

Theorem 3.1.5.
Let M be a right H-Hopf module. Then the multiplication map:
$$\begin{array}{rrcl} \rho: & M^{\mathrm{co}H} \otimes H & \to & M \\ & m \otimes h & \mapsto & m.h \end{array}$$
is an isomorphism of Hopf modules, where the left hand side has the structure of a trivial Hopf module.

In particular, any Hopf module M is equivalent to a trivial Hopf module and thus a free right H-module of rank $\dim_{\mathbb{K}} M^{\mathrm{co}H}$.

Proof.
We perform the proof graphically, see separate file. □

Example 3.1.6.
Consider a Hopf module M over a group algebra $\mathbb{K}[G]$. Since it is a comodule, M has the structure of a G-graded vector space
$$M = \oplus_{g \in G} M_g$$
with coaction $\Delta_M(m_g) = m_g \otimes g$ for $m_g \in M_g$. Moreover, G acts on M. Since we have a Hopf module, G acts such that $\Delta_M(m.g) = \Delta_M(m).g$. Thus for $m_g \in M_g$ and $h \in G$, we have $\Delta(m_g.h) = m_g.h \otimes gh$. Thus $M_g.h \subset M_{gh}$. Using the action of h^{-1}, we find the equality of subspaces, $M_g.h = M_{gh}$. Thus the G-action permutes the homogeneous components and
$$M_g = M_1.g = M^{\mathrm{co}\mathbb{K}[G]}.g \ .$$
This is exactly the statement of the fundamental theorem: $M \cong M^{\mathrm{co}\mathbb{K}[G]} \otimes \mathbb{K}[G]$.

We discuss a first simple application to finite-dimensional Hopf algebras:

Corollary 3.1.7.
Let H be a finite-dimensional Hopf algebra. If $I \subset H$ is a right ideal and a right coideal, then $I = H$ or $I = (0)$.

Proof.
As a right ideal, I is a right submodule of H. Similarly, as a right coideal, it is a right H-subcomodule. The condition of a Hopf module is inherited, so I is a Hopf submodule. The fundamental theorem 3.1.5 for Hopf modules implies

$$I \cong I^{\mathrm{co}H} \otimes_{\mathbb{K}} H \ .$$

Taking dimensions, we find

$$\dim_{\mathbb{K}} H \cdot \dim_{\mathbb{K}} I^{\mathrm{co}H} = \dim_{\mathbb{K}} I \leq \dim_{\mathbb{K}} H$$

which only leaves the two possibilities $I = (0)$ or $I = H$. □

Definition 3.1.8

1. *Let H be a Hopf algebra. The $\mathbb{K}$-linear subspace*
$$\mathcal{I}_l(H) := \{x \in H \,|\, h \cdot x = \epsilon(h)x \quad \text{for all } h \in H\}$$
is called the space of <u>*left integrals*</u> *of the Hopf algebra H. Similarly,*
$$\mathcal{I}_r(H) := \{x \in H \,|\, x \cdot h = \epsilon(h)x \quad \text{for all } h \in H\}$$
is called the space of <u>*right integrals*</u> *of H.*

2. *Similarly, the subspace of the linear dual H^**
$$C\mathcal{I}_l(H) := \{\phi \in H^* \,|\, (\mathrm{id}_H \otimes \phi) \circ \Delta_H(h) = 1_H \phi(h) \quad \text{for all } h \in H\}$$
is called the space of <u>*left cointegrals*</u>*. Right cointegrals are defined analogously.*

3. *A Hopf algebra is called* <u>*unimodular*</u>*, if $\mathcal{I}_l(H) = \mathcal{I}_r(H)$.*

Remarks 3.1.9.

1. The space of left integrals is the space of left invariants for the left action of H on itself by multiplication. Alternatively, $h \in \mathcal{I}_l(H) \subset H \cong \mathrm{Hom}_{\mathbb{K}}(\mathbb{K}, H)$ is a morphism of left H-modules. A similar statement holds for right integrals.

2. Even if a Hopf algebra H is cocommutative, it might not be unimodular. For an example, see Montgomery, p. 17.

3. Let H be finite-dimensional. Then $\phi \in H^*$ is a left integral for the dual Hopf algebra H^*, if and only if
$$\mu^*(\beta, \phi) = \epsilon^*(\beta)\phi \quad \text{for all} \quad \beta \in H^* \ .$$
Using the definition of the bialgebra structure on H^*, this amounts to the equality
$$\beta(h_{(1)}) \cdot \phi(h_{(2)}) = \beta(1_H) \cdot \phi(h) \text{ for all } \quad \beta \in H^* \text{ and } h \in H \ .$$
Thus ϕ is a left integral of H^*, if and only if
$$h_{(1)}\langle \phi, h_{(2)} \rangle = \langle \phi, h \rangle 1_H \text{ for all } \quad h \in H \ ,$$
i.e. if ϕ is a left cointegral for H.

4. Let G be a *finite* group. Then the group algebra $\mathbb{K}[G]$ is a unimodular Hopf algebra, with integrals
$$\mathcal{I}_l = \mathcal{I}_r = \mathbb{K}\sum_{g\in G} g \ .$$
Indeed, for $I := \sum_{h\in G} h$ we have $g.I = \sum_{h\in G} gh = I = \epsilon(g)I$ for all $g \in G$.

5. The dual $\mathbb{K}G$ of the group algebra $\mathbb{K}[G]$ is a commutative Hopf algebra. Suppose that G is a finite group; then it can be identified with the commutative algebra of $\mathbb{K}$-valued functions on G. In this case, a right integral $\lambda \in \mathbb{K}[G]$ can be considered as an element in the bidual, $\lambda \in \mathbb{K}G^*$, i.e. a functional $\phi \mapsto \phi(\lambda)$ on functions of G. This is called a measure.

   On the space of functions on a group G, we have a left action of G by translations:
$$L_g : \quad \mathbb{K}G \to \mathbb{K}G$$
   with $L_g\phi(h) = \phi(hg)$. We compute, using that λ is a right integral:
$$\lambda(L_g\phi) = (L_g\phi)(\lambda) = \phi(\lambda \cdot g) = \phi(\lambda) = \lambda(\phi) \ .$$
   Thus the measure given by a right integral is left invariant.

6. The spaces of integrals for the Taft algebra are
$$\mathcal{I}_l = \mathbb{K}\sum_{j=0}^{N-1} g^j x^{N-1}$$
   and
$$\mathcal{I}_r = \mathbb{K}\sum_{j=0}^{N-1} \zeta^j g^j x^{N-1} \ .$$
   The Taft algebra is thus not unimodular.

We need some actions and coactions of the Hopf algebra H on the dual vector space H^*. Since we will use dualities, we assume H to be finite-dimensional.

Observation 3.1.10.

1. We consider H^* as a right H-comodule
$$\begin{array}{rcl} \rho : H^* & \to & H^* \otimes H \\ f & \mapsto & f_{(0)} \otimes f_{(1)} \end{array}$$
   with coaction derived from the coproduct in H:
$$\langle p, f_{(1)}\rangle \cdot \langle f_{(0)}, h\rangle = \langle p, h_{(1)}\rangle \cdot \langle f, h_{(2)}\rangle \quad \text{for all} \quad p \in H^*, h \in H \ .$$
   Graphically, this definition is simpler to understand and the proof that ρ is a coaction is then easy, see handwritten notes.

2. Consider for $x \in H$ the $\mathbb{K}$-linear endomorphism given by right multiplication with x
$$\begin{array}{rcl} m_x : H & \to & H \\ h & \mapsto & h \cdot x \end{array}$$

The transpose is a map $m_x^* : H^* \to H^*$, for each $x \in H$. One checks graphically that it defines the structure of a left H-module on H^*. We write

$$h \rightharpoonup h^* \quad \in H^*$$

for the image of $h^* \in H^*$ under the left action of $h \in H$. Thus

$$\langle h \rightharpoonup h^*, g\rangle = \langle h^*, gh\rangle \quad \text{for all} \quad g \in H \ .$$

One can perform this construction quite generally for an algebra in a rigid monoidal category. In this case, one has to take the *left* dual for this construction to work. This is again immediately obvious from the graphical proof.

3. In the same vein, the transpose of left multiplication defines a right action of H on H^*. We write

$$h^* \leftharpoonup h \quad \in H^*$$

for the image of $h^* \in H^*$ under the right action of $h \in H$. Thus

$$\langle h^* \leftharpoonup h, g\rangle = \langle h^*, hg\rangle \quad \text{for all} \quad g \in H \ .$$

One can perform this construction quite generally for an algebra in a rigid monoidal category. In this case, one has to take the *right* dual for this construction to work. This is again immediately obvious from the graphical proof.

4. Since the antipode is an antialgebra morphism, we can use it to turn left actions into right actions and vice versa.

In this way, we get a left action of H on H^* by

$$(h \rightharpoondown h^*) := (h^* \leftharpoonup S(h))$$

It obeys

$$\langle h \rightharpoondown h^*, g\rangle = \langle h^*, S(h)g\rangle \quad \text{for all} \quad g \in H \ .$$

Similarly, we get a right action of H on H^* by

$$(h^* \leftharpoondown h) := (S(h) \rightharpoonup h^*)$$

with

$$\langle h^* \leftharpoondown h, g\rangle = \langle h^*, gS(h)\rangle \quad \text{for all} \quad g \in H \ .$$

The following Lemma will be needed to show the existence of integrals:

Lemma 3.1.11.
Let H be a finite-dimensional Hopf algebra. Then H^* with right H action $\leftharpoondown$ and right coaction ρ from observation 3.1.10 is a Hopf module.

Proof.
The condition to have a Hopf module is

$$\rho(f \leftharpoondown h) = (f_{(0)} \leftharpoondown h_{(1)}) \otimes (f_{(1)} \cdot h_{(2)})$$

for all $f \in H^*$ and $h \in H$. By the definition of the coaction ρ, this amounts to showing for all $p \in H^*$ and $x \in H$:

$$\langle p, x_{(1)} \rangle \langle f \leftharpoondown h, x_{(2)} \rangle = \langle f_{(0)} \leftharpoondown h_{(1)}, x \rangle \langle p, f_{(1)} \cdot h_{(2)} \rangle \ .$$

We start with the right hand side:

$$\begin{aligned}
\langle f_{(0)} \leftharpoondown h_{(1)}, x \rangle \langle p, f_{(1)} h_{(2)} \rangle &= \langle f_{(0)}, xS(h_{(1)}) \rangle \langle h_{(2)} \rightharpoonup p, f_{(1)} \rangle \quad \text{[defn. of } \leftharpoondown \text{ and } \rightharpoonup\text{]} \\
&= \langle h_{(3)} \rightharpoonup p, x_{(1)} S(h_{(2)}) \rangle \cdot \langle f, x_{(2)} S(h_{(1)}) \rangle \quad \text{[defn. of } \rho\text{]} \\
&= \langle p, x_{(1)} \langle \epsilon, h_{(2)} \rangle \rangle \cdot \langle f, x_{(2)} S(h_{(1)}) \rangle \quad \text{[defn. of } \rightharpoonup \text{ and antipode]} \\
&= \langle p, x_{(1)} \rangle \langle f, x_{(2)} S(h) \rangle \quad \text{[counit]} \\
&= \langle p, x_{(1)} \rangle \langle f \leftharpoondown h, x_{(2)} \rangle \quad \text{[defn. of } \leftharpoondown\text{]}
\end{aligned}$$

□

Lemma 3.1.12.
Let H be a finite-dimensional Hopf algebra. Consider H^* as a right comodule with the H-coaction ρ. Then

$$(H^*)^{\mathrm{co}H} = \mathcal{I}_l(H^*) \ .$$

Proof.
We recall from remark 3.1.9 that elements $\beta \in \mathcal{I}_l(H^*)$ are left cointegrals for H: they are elements such that

$$\mu^*(h^*, \beta) = \epsilon^*(h^*)\beta \quad \text{for all} \quad h^* \in H^* \ .$$

This means that we have $\beta \in \mathcal{I}_l(H^*)$, if and only if for all $h \in H$ and $h^* \in H^*$, we have

$$\mu^*(h^*, \beta)(h) = h^*(h_{(1)}) \cdot \beta(h_{(2)}) = \epsilon^*(h^*)\beta(h) = h^*(1)\beta(h) \ .$$

On the other hand, we have for coinvariants under the coaction ρ

$$\rho(\beta) = \beta \otimes 1_H$$

and thus by definition of ρ

$$\langle h^*, h_{(1)} \rangle \cdot \langle \beta, h_{(2)} \rangle = \langle h^*, 1 \rangle \cdot \langle \beta, h \rangle$$

for all $h^* \in H$ and $h \in H$. □

Theorem 3.1.13.
Let H be a *finite-dimensional* Hopf algebra over a field $\mathbb{K}$.

1. Then $\dim \mathcal{I}_l(H) = \dim \mathcal{I}_r(H) = 1$.

2. The antipode S is bijective and $S(\mathcal{I}_l) = \mathcal{I}_r$.

3. For any non-zero left cointegral $\lambda \in \mathcal{I}_l(H^*) \setminus \{0\}$, the so-called Frobenius map

$$\begin{aligned} \Psi_\lambda : H &\to H^* \\ h &\mapsto (S(h) \rightharpoonup \lambda) = (\lambda \leftharpoondown h) \end{aligned}$$

is an isomorphism of right H-modules, where H is endowed with the regular right action, i.e. by multiplication, and H^* with the action $h^* \leftharpoondown h$.

Proof.

1. Consider H^* with the Hopf module structure described in lemma 3.1.11. By the fundamental theorem on Hopf modules,

$$H^* \cong (H^*)^{\mathrm{co}H} \otimes H \ .$$

Since H is finite-dimensional, we can take dimensions and find $\dim(H^*)^{\mathrm{co}H} = 1$. By lemma 3.1.12, we have $\dim \mathcal{I}_l(H^*) = \dim(H^*)^{\mathrm{co}H} = 1$. Thus the Hopf algebra H^* has left integrals. Since any finite-dimensional Hopf algebra can be written as the dual of a Hopf algebra, we get the first equality. The second equality follows analogously or from 2.

2. Again by the fundamental theorem 3.1.5 on Hopf modules, the map

$$\begin{array}{rcl} \mathcal{I}_l(H^*) \otimes H & \to & H^* \\ \lambda \otimes h & \mapsto & (\lambda \leftharpoonup h) \end{array} \tag{4}$$

is an isomorphism of Hopf-modules. In particular, it is a morphism of right H-modules. The compatibility with the right action is also shown graphically. Keeping $\lambda \in \mathcal{I}_l(H^*)\backslash\{0\}$ fixed, we deduce the third assertion.

3. Fix $\lambda \in \mathcal{I}_l(H^*) \setminus \{0\}$ and suppose that there is $h \in H$ such that $S(h) = 0$. Then

$$0 = (S(h) \rightharpoonup \lambda) \stackrel{\text{def}}{=} (\lambda \leftharpoonup h)$$

and thus by injectivity of the map (4), we have $\lambda \otimes h = 0$. This implies over a field $h = 0$. Thus the antipode S is injective and, as an endomorphism of a finite-dimensional vector space, bijective.

If $\Lambda \in \mathcal{I}_l(H)$, we have $h \cdot \Lambda = \epsilon(h)\Lambda$ for all $h \in H$. Applying the antipode, which is an antialgebra morphism and preserves the counit, we find

$$S(\Lambda) \cdot S(h) = \epsilon(h)S(\Lambda) = \epsilon(S(h))S(\Lambda) \quad \text{for all} \quad h \in H \ .$$

Since S is bijective, this implies that $S(\Lambda)$ is a right integral.

□

We find a different relation between the left and right integrals on a finite-dimensional Hopf algebra in the following

Observation 3.1.14.

1. Let $t \in \mathcal{I}_l(H)$ be a left integral. Then for any $h \in H$, also the element $th \in H$ is a left integral: we have for all $h' \in H$

$$h'(th) = (h't)h = \epsilon(h')th \ .$$

Since the subspace of left integrals is one-dimensional, $t \cdot h = t\alpha(h)$ with some linear form $\alpha \in H^*$.

2. One directly checks that $\alpha : H \to \mathbb{K}$ is a morphism of algebras and thus a group-like element of H^*.

3. Let now be $t' \in \mathcal{I}_r(H)$ a non-vanishing right integral. Then by theorem 3.1.13.2, the element St' is a left integral and thus for all $h \in H$

$$S(ht') = St' \cdot Sh = \alpha(Sh)\, St' \ .$$

The invertibility of the antipode implies $ht' = \alpha(Sh)t' = (S^*\alpha)(h)t$ for all $h \in H$. Here S^* is the antipode of the dual Hopf algebra H^*. Thus the inverse $\alpha^{-1} = S^*\alpha \in G(H^*)$ plays a similar role for right integrals.

Definition 3.1.15
Let H be a finite-dimensional Hopf algebra. The element $\alpha \in G(H^*)$ constructed in observation 3.1.14 is called the distinguished group-like element or the modular element of H^*.

Corollary 3.1.16.
A finite-dimensional Hopf algebra is unimodular, if and only if the distinguished group-like element α equals the counit, $\alpha = \epsilon$.

Proof.
Let $t \in \mathcal{I}_l(H) \setminus \{0\}$. If $\alpha = \epsilon$, then $t \cdot h = t\alpha(h) = t\epsilon(h)$ for all $h \in H$ so that t is a right integral as well. The converse is obvious. □

The third assertion of theorem 3.1.13 allows us to identify additional algebraic structure on any finite-dimensional Hopf algebra.

Definition 3.1.17
Let (A, μ, η) be a unital associative algebra in a monoidal category $\mathcal{C}$.

1. *A (Δ, ϵ)-Frobenius structure on A is the structure of a coassociative, counital coalgebra (Δ, ϵ) such that $\Delta : A \to A \otimes A$ is a morphism of A-bimodules.*

2. *Assume now that the monoidal category $\mathcal{C}$ is rigid. A κ-Frobenius structure on A is a pairing $\kappa \in \mathrm{Hom}_{\mathcal{C}}(A \otimes A, \mathbb{I})$ that is invariant (or associative) i.e. satisfies*

$$\kappa \circ (\mu \otimes \mathrm{id}_A) = \kappa \circ (\mathrm{id}_A \otimes \mu)\,,$$

and that is non-degenerate in the sense that

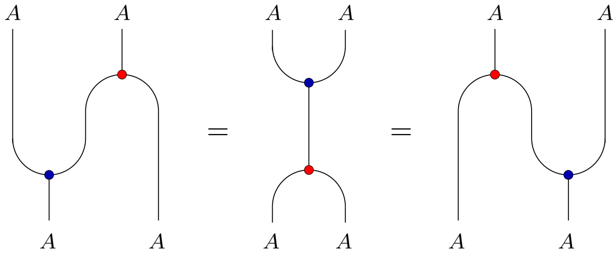
$$\Phi_\kappa := (\mathrm{id}_{^\vee A} \otimes \kappa) \circ (\tilde{b}_A \otimes \mathrm{id}_A) \ \in \mathrm{Hom}(A, {}^\vee A)$$

is an isomorphism.

3. *Assume again that the monoidal category $\mathcal{C}$ is rigid. A Φ_ρ-Frobenius structure on A is a left-module isomorphism $\Phi_\rho \in \mathrm{Hom}_{A-\mathrm{mod}}(A, {}^\vee A)$ between the left regular A-module (A, μ) and left A-module ${}^\vee A$ with the left dual action.*

Remarks 3.1.18.

1. Graphically, the bimodule condition in the (Δ, ϵ)-Frobenius structure reads



2. Note that, unlike in the case of bialgebras (which cannot be defined in any monoidal category), neither the coproduct Δ nor the counit ϵ is an algebra morphism.

3. Concerning the Φ_ρ-Frobenius structure, we remark that if $\Phi_\rho \in \mathrm{Hom}(A, {}^\vee A)$ is an isomorphism between the left regular A-module (A, μ) and left A-module ${}^\vee A$, then dual

$$\Phi_\rho^\vee \in \mathrm{Hom}(({}^\vee A)^\vee, A^\vee) = \mathrm{Hom}(A, A^\vee)$$

on A is a left-module isomorphism $\Phi_\rho \in \mathrm{Hom}(A, {}^\vee A)$ between the left regular A-module (A, μ) and right A-module ${}^\vee A$ with the right dual action. This will be shown graphically.

It turns out that the three concepts are equivalent:

Proposition 3.1.19.
In a rigid monoidal category $\mathcal{C}$ the notions of a (Δ, ϵ)-Frobenius structure and of a κ-Frobenius structure on an algebra (A, μ, η) are equivalent.

More concretely:

1. If $(A, \mu, \eta, \Delta, \epsilon)$ is an algebra with a (Δ, ϵ)-Frobenius structure, then $(A, \mu, \eta, \kappa_\epsilon)$ with

$$\kappa_\epsilon := \epsilon \circ \mu$$

is an algebra with κ-Frobenius structure.

2. If (A, μ, η, κ) is an algebra with κ-Frobenius structure, then $(A, \mu, \eta, \Delta_\kappa, \epsilon_\kappa)$ with

$$\Delta_\kappa := (\mathrm{id}_A \otimes \mu) \circ (\mathrm{id}_A \otimes \Phi_\kappa^{-1} \otimes \mathrm{id}_A) \circ (b_A \otimes \mathrm{id}_A) \qquad \text{and} \qquad \epsilon_\kappa := \kappa \circ (\mathrm{id}_A \otimes \eta)$$

with $\Phi_\kappa \in \mathrm{Hom}(A, A^\vee)$ the morphism that exists by the assumption that κ is non-degenerate is an algebra with (Δ, ϵ)-Frobenius structure.

Proof.
We present the proof that a (Δ, ϵ)-Frobenius structure gives a κ-Frobenius structure graphically. The converse statement is relegated to an exercise. □

Proposition 3.1.20.
In a rigid monoidal category $\mathcal{C}$ the notions of a κ-Frobenius structure and of a Φ_ρ-Frobenius structure on an algebra (A, μ, η) are equivalent.

More specifically, for any algebra A in $\mathcal{C}$ the following holds:

1. There exists a non-degenerate pairing on A, if and only if A is isomorphic to ${}^\vee A$ as an object of $\mathcal{C}$.

2. There exists an invariant pairing on A, if and only if there exists a morphism from A to ${}^\vee A$ that is a morphism of left A-modules.

Proof.
Given a morphism $\Phi \in \mathrm{Hom}_{\mathcal{C}}(A, {}^\vee A)$, we define a pairing on A by

$$\kappa_\Phi := \tilde{d}_A \circ (\mathrm{id}_A \otimes \Phi)\,.$$

Conversely, using the dualities, we find for any pairing a morphism $\psi \in \mathrm{Hom}(A, {}^\vee A)$ such that the operations are inverse.

A pairing is obviously non-degenerate, if and only if the morphism Φ is an isomorphism. Similarly, invariance of the pairing amounts to the fact that Φ is a morphism of left modules. This can be seen graphically and is relegated to an exercise. □

Definition 3.1.21
A Frobenius algebra in a rigid monoidal category $\mathcal{C}$ is an associative unital algebra A in $\mathcal{C}$ together with the choice of one of the following three equivalent structures:

1. *A (Δ, ϵ)-Frobenius structure on A.*

2. *A κ-Frobenius structure on A.*

3. *A Φ_ρ-Frobenius structure on A.*

Example 3.1.22.
It is instructive to write down explicitly a distinguished Frobenius algebra structure on the group algebra $\mathbb{K}[G]$ of a finite group.

1. The bilinear form is defined on the distinguished basis by

$$\kappa(g, h) = \delta_{gh,e} \qquad \text{for all } g, h \in G\,.$$

   This form is obviously non-degenerate and invariant, $\kappa(gh, l) = \delta_{ghl,e} = \kappa(g, hl)$ for all $g, h, l \in G$.

2. The corresponding Φ_ρ-Frobenius structure is the morphism

$$\begin{array}{rccl} \Phi_\rho: & \mathbb{K}[G] & \to & \mathbb{K}(G) = \mathbb{K}[G]^* \\ & g & \mapsto & \delta_{g^{-1}} \end{array}$$

   To show that this is indeed a morphism of left modules, we have to show $\Phi_\rho(hg) = h \rightharpoonup \Phi_\rho(g)$. Indeed, evaluating this on $x \in G$, we find

$$(h \rightharpoonup \delta_{g^{-1}})(x) = \delta_{g^{-1}}(xh) = \delta_{g^{-1}h^{-1}}(x) = \delta_{(hg)^{-1}}(x) \qquad \text{for all } x \in G\,.$$

3. We can finally deduce the (Δ_F, ϵ_F)-Frobenius structure, where we added an index F to the Frobenius coproduct and counit to distinguish them from the Hopf coproduct and counit. We find
$$\epsilon_F(g) = \delta_{g,e} \quad \text{and} \quad \Delta_F(g) = \sum_{h \in G} gh^{-1} \otimes h$$
which is indeed different from the coproduct and counit giving the Hopf algebra structure on $\mathbb{K}[G]$. Note that here, in contrast to the Hopf coproduct, the product in the group enters, cf. observation 2.1.13.

We can now state:

Theorem 3.1.23.
Let H be a finite-dimensional Hopf-algebra with left integral $\lambda \in H^*$. Then H is a Frobenius algebra with bilinear pairing
$$\kappa(h, h') := \lambda(h \cdot h') \quad \text{for } h, h' \in H \ .$$

Proof.
From the associativity and bilinearity of the product of the group, it is obvious that the form is bilinear and invariant. To show non-degeneracy, assume that there is $a \in H$ such that
$$0 = \kappa(a, h) = \lambda(ah) = \langle h \rightharpoonup \lambda, a \rangle \quad \text{for all} \ \ h \in H \ .$$
But $(H \rightharpoonup \lambda) = H^*$ by equation (4) in the proof of theorem 3.1.13, and the pairing between the vector space H and its dual H^* is non-degenerate. □

Example 3.1.24.
Consider the case of a group algebra $H = \mathbb{K}[G]$. Then the cointegral $\lambda \in H^*$ is the projection to the component of the neutral element: $\lambda(g) = \delta_{g,e}$ for all $g \in G$. Indeed,
$$(\mathrm{id}_H \otimes \lambda) \circ \Delta(g) = g\lambda(g) = e\delta_{g,e} = 1_H \lambda(g) \quad \text{for all } g \in G \ .$$
This yields the Frobenius structure on $\mathbb{K}[G]$ discussed in example 3.1.22.

Proposition 3.1.25.
Let H be a finite-dimensional Hopf algebra. Recall from theorem 3.1.13 that for a non-zero $\lambda \in \mathcal{I}_l(H^*)$
$$\begin{array}{rcl} \Psi_\lambda : H & \to & H^* \\ h & \mapsto & (S(h) \rightharpoonup \lambda) \end{array}$$
is an isomorphism of right H-modules. As a consequence, $h \mapsto (\lambda \leftharpoonup h)$ is a linear isomorphism $H \to H^*$.

1. Let λ be a left integral in H^*. We can find $\Lambda \in H$ such $\lambda \leftharpoonup \Lambda = \epsilon$ equals the counit ϵ. Then Λ is a right integral.

2. Conversely, if $I \in H$ is a right integral, then $\langle \lambda, I \rangle \neq 0$. If we normalize $I \in H$ such that $\langle \lambda, I \rangle = 1$, we have $\lambda \leftharpoonup I = \epsilon$.

Proof.

1. We first assume that I is a right integral. Then for all $h \in H$

$$\langle \lambda \leftharpoonup I, h \rangle = \langle \lambda, I \cdot h \rangle = \langle \lambda, I \rangle \epsilon(h)$$

   and thus $\lambda \leftharpoonup I = \langle \lambda, I \rangle \epsilon$. By injectivity, since $I \neq 0$, we conclude $\langle \lambda, I \rangle \neq 0$. Normalizing I, we find the identity $\lambda \leftharpoonup I = \epsilon$.

2. Conversely, suppose that we have $\Lambda \in H$ such that

$$\langle \lambda, \Lambda h \rangle = \epsilon(h) \quad \text{for all} \quad h \in H \ .$$

   Applying this to $h = 1_H \in H$, we find

$$\langle \lambda, \Lambda \rangle = \langle \lambda, \Lambda 1_H \rangle = \epsilon(1_H) = 1 \ .$$

   Thus

$$\langle \lambda, \Lambda h \rangle = \epsilon(h) \langle \lambda, \Lambda \rangle = \langle \lambda, \epsilon(h) \Lambda \rangle \ .$$

   By the injectivity of the map $h \mapsto (\lambda \leftharpoonup h)$, we conclude $\Lambda h = \epsilon(h) \Lambda$ for all $h \in H$. Thus Λ is a right integral.

□

3.2 Integrals and semisimplicity

We now need the important notion of semi-simplicity.

Definition 3.2.1

1. *A module M over a $\mathbb{K}$-algebra A is called simple, if it has no non-trivial submodules, i.e. the only submodules of M are (0) and M itself.*

2. *A module M over a $\mathbb{K}$-algebra A is called semisimple, if every submodule $U \subset M$ has a* complement *D, i.e. if we can find for any submodule U a submodule D such that $D \oplus U = M$.*

3. *An algebra is called semisimple , if it is semisimple as a left module over itself.*

Remarks 3.2.2.

1. A $\mathbb{K}$-vector space is a semisimple module over the $\mathbb{K}$-algebra $\mathbb{K}$.

2. One has a similar notion of semisimplicity for right modules. It turns out that an algebra is semisimple as a right module over itself, if and only if it is semisimple as a left module over itself.

Proposition 3.2.3.
Let A be a $\mathbb{K}$-algebra and M an A-module. Then the following assertions are equivalent:

(i) M is a direct sum of simple submodules.

(ii) M is a (not necessarily direct) sum of simple submodules.

(iii) M is semisimple, i.e. any submodule $U \subset M$ has a complement D.

For the proof, we refer to the lecture notes on advanced algebra.

Corollary 3.2.4.
Any quotient and any submodule of a semisimple module is semisimple.

Proof.
Suppose we are given a submodule $U \subset M$ of a semisimple module M. Consider the canonical surjection $M \to M/U$. The image of a simple submodule of M is then either zero or simple. Thus the quotient module is a sum of simple modules and thus semisimple by proposition 3.2.3.

Next, find a complement D of U. Then the submodule U is isomorphic to the quotient $U \cong M/D$ and by the result just obtained semisimple. $\square$

We next need the important notion of a projective module. We recall that a collection of morphisms of A-modules

$$0 \to N' \xrightarrow{f} N \xrightarrow{g} N'' \to 0$$

is called a short exact sequence, if f is injective, g is surjective and $\mathrm{Im}\, f = \ker g$.

Proposition 3.2.5.
Let A be a $\mathbb{K}$-algebra. Then the following assertions about an A-module M are equivalent:

1. For every diagram with A-modules N_1, N_2

$$\begin{tikzcd} & M \arrow[dl, dotted] \arrow[d] & \\ N_1 \arrow[r] & N_2 \arrow[r] & 0 \end{tikzcd}$$

   with exact line, there is a lift such that the diagram commutes. (The lift is indicated by the dotted arrow. The lift is, in general, not unique.)

2. There is an A-module N such that $M \oplus N$ is a free A-module.

3. Any short exact sequence of the form

$$0 \to N' \to N \xrightarrow{f} M \to 0$$

   splits, i.e. there is a morphism $s : M \to N$ such that $f \circ s = \mathrm{id}_M$. Then $N \cong N' \oplus s(M)$.

4. For any short exact sequence of modules

$$0 \to T' \to T \to T'' \to 0$$

   the sequence of $\mathbb{K}$-vector spaces

$$0 \to \mathrm{Hom}_{\mathbb{K}}(M, T') \to \mathrm{Hom}_{\mathbb{K}}(M, T) \to \mathrm{Hom}_{\mathbb{K}}(M, T'') \to 0$$

   is exact. (Note that the sequence

$$0 \to \mathrm{Hom}_{\mathbb{K}}(M, T') \to \mathrm{Hom}_{\mathbb{K}}(M, T) \to \mathrm{Hom}_{\mathbb{K}}(M, T'')$$

   is exact for any module M.)

Proof.

1⇒ 3 The split is given by the lift in the specific diagram

$$\begin{tikzcd} & M \arrow[dl, dotted] \arrow[d, "\mathrm{id}"] & \\ N \arrow[r] & M \arrow[r] & 0 \end{tikzcd}$$

3⇒ 2 Any A-module M is a quotient of a free module, e.g. by the surjection

$$\begin{array}{rcl} \oplus_{m\in M} A & \to & M \\ a_m & \mapsto & a_m.m \end{array}$$

Take a surjection $N \to M$ with kernel N' and N a free module. Since the short exact sequence $0 \to N' \to N \to M \to 0$ splits, we have $N \cong M \oplus N'$, where N is a free module.

2⇒ 4 We first note that assertion 4 holds in the case when M is a free module: then $\mathrm{Hom}_{\mathbb{K}}(M, N) \cong \mathrm{Hom}_{\mathbb{K}}(\oplus_{i\in I} R, N) \cong \prod_{i\in I} N$ for any module N, where the index set I labels a basis of M. The maps are simply in each component the given maps.

In particular, the sequence

$$0 \to \mathrm{Hom}_{\mathbb{K}}(M \oplus N, T') \to \mathrm{Hom}_{\mathbb{K}}(M \oplus N, T) \to \mathrm{Hom}_{\mathbb{K}}(M \oplus N, T'') \to 0$$

is exact. Using the universal property of the direct sum, this amounts to

$$\begin{aligned} &0 \to \mathrm{Hom}_{\mathbb{K}}(M, T') \times \mathrm{Hom}_{\mathbb{K}}(N, T') \to \mathrm{Hom}_{\mathbb{K}}(M, T) \times \mathrm{Hom}_{\mathbb{K}}(N, T) \\ &\to \mathrm{Hom}_{\mathbb{K}}(M, T'') \times \mathrm{Hom}_{\mathbb{K}}(N, T'') \to 0 \ . \end{aligned}$$

The kernel of a Cartesian product of maps is the product of kernels; the image of the Cartesian product of maps is the Cartesian product of the images. This implies the exactness of the sequence in 4.

4⇒ 1 From the surjectivity of the line, we get a short exact sequence

$$0 \to \ker((N_1 \to N_2)) \to N_1 \xrightarrow{f} N_2 \to 0$$

By 4, we get a short exact sequence

$$0 \to \mathrm{Hom}_R(M, \ker(N_1 \to N_2)) \to \mathrm{Hom}_R(M, N_1) \to \mathrm{Hom}_R(M, N_2) \to 0$$

where the last arrow is

$$\begin{array}{rrcl} f_* : & \mathrm{Hom}_R(M, N_1) & \to & \mathrm{Hom}_R(M, N_2) \\ & \varphi & \mapsto & f \circ \varphi =: f_*(\varphi) \end{array}$$

The surjectivity of this morphism amounts to property 1.

□

Definition 3.2.6

An A-module with one of the four equivalent properties from proposition 3.2.5 is called a projective module.

Proposition 3.2.7.
Let A be a $\mathbb{K}$-algebra. Then the following assertions are equivalent:

1. The algebra A is semisimple, i.e. seen as a left module over itself, it is a direct sum of simple submodules.
2. Any A-module is semisimple, i.e. direct sum of simple submodules.
3. The category $A-$mod is semisimple, i.e. all A-modules are projective.

As a consequence of this result, we need to understand only *simple* modules to understand the representation category of a semisimple algebra.

Proof.

3.⇒ 2. Suppose that the category $A-$mod is semisimple. Let M be an A-module. Any submodule $U \subset M$ yields a short exact sequence

$$0 \to U \to M \to M/U \to 0 \ ,$$

which splits, since the module M/U is projective. Then $M \cong U \oplus M/U$ and the submodule U has a complement in M.

2.⇒ 1. Trivial, since 1. is a special case of 2.

1.⇒ 3. We have to show that every module is projective, i.e. direct summand of a free module. Since every module is a homomorphic image of a free module F, we have a short exact sequence:

$$0 \to \ker \pi \to F \xrightarrow{\pi} M \to 0$$

A being semisimple by assumption, implies that also the direct sum F is semisimple. Thus the submodule $\ker \pi$ has a complement which is isomorphic to M, $F \cong M \oplus \ker \pi$. Thus M is projective by property 2 of a projective module.

□

Lemma 3.2.8.
Let $\mathcal{C}$ be an abelian category. Then a sequence $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ is exact in $\mathcal{C}$, if for any object $X \in \mathcal{C}$ the sequence

$$\mathrm{Hom}_{\mathcal{C}}(X, A) \xrightarrow{\alpha_*} \mathrm{Hom}_{\mathcal{C}}(X, B) \xrightarrow{\beta_*} \mathrm{Hom}_{\mathcal{C}}(X, C)$$

of abelian groups is exact.

Proof.
Let $X = A$ and find from the exact Hom-sequence $\beta \circ \alpha = \beta_* \circ \alpha_*(\mathrm{id}_A) = 0$. Thus $\mathrm{Im}\, \alpha \subset \ker \beta$.

Next put $X = \ker \beta$ with inclusion map $\iota : \ker \beta \to B$. Since ι is the embedding of the kernel of β, we have $\beta_*(\iota) = \beta \circ \iota = 0$. By exactness of the Hom sequence, there exists $\varphi \in \mathrm{Hom}_{\mathcal{C}}(\ker \beta, A)$ such that $\alpha \circ \varphi = \alpha_*(\varphi) = \iota$. Thus $\ker \beta \subset \mathrm{Im}\, \alpha$. □

Lemma 3.2.9.
Let $\mathcal{C}$ and $\mathcal{D}$ be abelian categories and $F : \mathcal{C} \to \mathcal{D}$ be an additive functor left adjoint to $G : \mathcal{D} \to \mathcal{C}$. Then F is a right exact functor and G is a left exact functor.

Proof.
Let $0 \to A \to B \to C \to 0$ be an exact sequence in $\mathcal{D}$ and let $X \in \mathcal{C}$. Then we have the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}(F(X), A) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}(F(X), B) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}(F(X), C) \arrow[d, "\cong"] \\
0 \arrow[r] & \mathrm{Hom}(X, G(A)) \arrow[r] & \mathrm{Hom}(X, G(B)) \arrow[r] & \mathrm{Hom}(X, G(C))
\end{tikzcd}$$

The vertical arrows are the adjunction isomorphisms and isomorphisms of abelian groups. The top row is exact since the Hom-functor is left exact, thus the bottom row is exact as well. By lemma 3.2.8, this implies that $0 \to G(A) \to G(B) \to G(C)$ is exact. Thus any right adjoint functor is left exact.

In particular, $F^{\mathrm{opp}} : \mathcal{C}^{\mathrm{opp}} \to \mathcal{D}^{\mathrm{opp}}$ is a right adjoint and thus left exact. But this amounts to F being right exact. □

Lemma 3.2.10.
Let $\mathcal{C}$ be an abelian monoidal category. Suppose that the object X is rigid. Then the functor $- \otimes X$ of tensoring with X is exact, i.e. if

$$0 \to U \to V \to W \to 0$$

is an exact sequence in $\mathcal{C}$, then

$$0 \to U \otimes X \to V \otimes X \to W \otimes X \to 0$$

is exact in $\mathcal{C}$ as well.

Proof.
This follows from lemma 3.2.9, since the functor of tensoring with a rigid object has a left and a right adjoint by example 2.5.22. □

For the following propositions, the reader might wish to keep the category $\mathcal{C} = H{-}\mathrm{mod}_{fd}$ of finite-dimensional modules over a finite-dimensional Hopf algebra in mind.

Lemma 3.2.11.
Let $\mathcal{C}$ be a rigid abelian tensor category. Let P be a projective object and let M be any object. Then the object $P \otimes M$ is projective.

Proof.
By rigidity, we have adjunction isomorphisms

$$\mathrm{Hom}(P \otimes M, N) \cong \mathrm{Hom}(P, N \otimes M^\vee) \ .$$

Thus the functor $\mathrm{Hom}(P \otimes M, -)$ is isomorphic to the concatenation of the functor $- \otimes M^\vee$ (which is exact by lemma 3.2.10) with the functor $\mathrm{Hom}(P, -)$ which is exact by property 4 of the projective object P. □

Corollary 3.2.12.
A $\mathbb{K}$-Hopf algebra is semi-simple, if and only if the trivial module $(\mathbb{K}, \epsilon)$ is projective.

Proof.
If the trivial module $\mathbb{I} = (\mathbb{K}, \epsilon)$ – which is the tensor unit in $H-$mod – is projective, then by lemma 3.2.11 any module $M \cong M \otimes \mathbb{I}$ is projective. The converse is trivial. □

Theorem 3.2.13 (Maschke)**.**
Let H be a finite-dimensional Hopf algebra. Then the following statements are equivalent:

1. H is semisimple.
2. The counit takes non-zero values on the space of left integrals, $\epsilon(\mathcal{I}_l(H)) \neq 0$.

Proof.

1. Suppose that H is semisimple. Then any module is projective, in particular the trivial module. Thus the exact sequence of left H-modules
$$(0) \to \ker \epsilon \to H \xrightarrow{\epsilon} \mathbb{K} \to (0) \tag{5}$$
splits. Thus $H = \ker \epsilon \oplus I$ with I a left ideal of H.
Take $z \in I$ and any $h \in H$. Then $h - \epsilon(h)1 \in \ker \epsilon$. Since I is a left ideal, we have $h \cdot z \in I$. The direct sum decomposition $H = \ker \epsilon \oplus I$ implies
$$I \ni h \cdot z = \underbrace{(h - \epsilon(h)1) \cdot z}_{\in \ker \epsilon} + \underbrace{\epsilon(h)z}_{\in I} = \epsilon(h)z \ .$$
Thus $z \in \mathcal{I}_l(H)$ is a left integral in H.
Since $\dim_K I = 1$, we may choose $z \neq 0$. Then $z \notin \ker \epsilon$ and thus $\epsilon(\mathcal{I}_l(H)) \neq 0$.
2. Conversely, let Λ be a left integral and assume that $\epsilon(\Lambda) \neq 0$. Replacing Λ by a scalar multiple, we can assume that $\epsilon(\Lambda) = 1$. Then
$$\begin{array}{rccc} s: & \mathbb{K} & \to & H \\ & \lambda & \mapsto & \lambda\Lambda \end{array}$$
obeys $\epsilon \circ s(\lambda) = \lambda\epsilon(\Lambda) = \lambda$ and is a morphism of left H modules, since Λ is a left integral, so that the exact sequence (5) splits. Thus the trivial module is projective and the claim follows from corollary 3.2.12.

□

Example 3.2.14.
Consider the group algebra $\mathbb{K}[G]$ of a finite group G with two-sided integral $\Lambda = \sum_{g \in G} g$. Then
$$\epsilon(\Lambda) = \sum_{g \in G} \epsilon(g) = |G| \ .$$
Thus the group algebra $\mathbb{K}[G]$ is semisimple, if and only if $\mathrm{char}(\mathbb{K}) \not| \, |G|$.

Corollary 3.2.15.
A finite-dimensional semisimple Hopf algebra is unimodular.

Proof.
Since H is semisimple, we can choose a left integral $t \in H$ such that $\epsilon(t) \neq 0$. Then for any $h \in H$, we have

$$\alpha(h)\epsilon(t)t = \alpha(h)t^2 = (th)t = t(ht) = \epsilon(h)t^2 = \epsilon(h)\epsilon(t)t \ ,$$

where we used the definition of a left integral and of the distinguished group-like element α of H^*. Since $\epsilon(t) \neq 0$, we have $\alpha(h) = \epsilon(h)$ for all $h \in H$ which implies unimodularity by corollary 3.1.16. □

We recall the notion of a separable algebra over a field $\mathbb{K}$. To this end, let A be an associative unital $\mathbb{K}$-algebra. The algebra $A^e := A \otimes A^{\mathrm{opp}}$ is called the enveloping algebra of A. If B is an A-bimodule, it is a left module over A^e by

$$(a_1 \otimes a_2).b := (a_1.b).a_2 \ .$$

Conversely, any A^e-left module M carries a canonical structure of an A-bimodule with left action $a.m := (a \otimes 1).m$ and right action $m.a := (1 \otimes a).m$. Thus the categories of A^e-left modules and A-bimodules are canonically isomorphic as $\mathbb{K}$-linear abelian categories.

Proposition 3.2.16.
Let $\mathbb{K}$ be a field and A be an associative unital $\mathbb{K}$-algebra. Then the following properties are equivalent:

1. A is projective as an A^e-module.
2. The short exact sequence of A^e-modules

   $$0 \to \ker \mu \to A^e \xrightarrow{\mu} A \to 0$$

   splits. Put differently, the multiplication epimorphism

   $$\mu : A \otimes_{\mathbb{K}} A \to A$$

   has a right inverse as a morphism of bimodules:

   $$\varphi : A \to A \otimes_{\mathbb{K}} A$$

   with $\mu \circ \varphi = \mathrm{id}_A$ and $\varphi(abc) = a \cdot \varphi(b) \cdot c$ for all $a, b, c \in A$.
3. Given any extension of fields $\mathbb{K} \subset E$, the E-algebra $A \otimes_{\mathbb{K}} E$ induced by extension of scalars is semisimple.

For the proof of this statement, we refer to Chapter 10 of R.S. Pierce, Associative Algebras, Springer Graduate Texts in Mathematics 88.

Definition 3.2.17
A $\mathbb{K}$-algebra A that has one of the properties of proposition 3.2.16 is called separable.

Remarks 3.2.18.

1. The choice of a right inverse φ of the multiplication $\mu : A \otimes_{\mathbb{K}} A \to A$ is called the choice of a separability structure of A.

2. We can describe φ by the element $e := \varphi(1) \in A^e$. Indeed, since φ is a morphism of A-bimodules, $\varphi(a) = (a \otimes 1)e$. Obviously, $\mu(e) = \mu(s(1)) = 1$ and $(a \otimes 1)e = e(1 \otimes a)$ for all $a \in A$.

   With the multiplication in A^e, we have $e^2 = e$. The element $e \in A^e$ is therefore called a separability idempotent.

3. Separable algebras over fields are finite-dimensional and semisimple.

   More precisely, a $\mathbb{K}$-algebra A is separable, if and only if
   $$A \cong A_1 \oplus \cdots \oplus A_r$$
   is a direct sum of finite-dimensional simple $\mathbb{K}$-algebras where all $Z(A_i)/\mathbb{K}$ are separable extensions of fields.

4. We present an example: for any field $\mathbb{K}$, the full matrix algebra $\mathrm{Mat}_n(\mathbb{K})$ is a separable $\mathbb{K}$-algebra.

   Introduce matrix units ϵ_{ij}, i.e. ϵ_{ij} is the matrix with zero entries, except for one in the i-th line and j-th column. Fix some index $1 \leq j \leq n$; then
   $$e^{(j)} := \sum_{i=1}^{n} \epsilon_{ij} \otimes \epsilon_{ji} \in \mathrm{Mat}_n(\mathbb{K})^{\mathrm{opp}} \otimes \mathrm{Mat}_n(\mathbb{K})$$
   obeys
   $$\mu(e) = \sum_{i=1}^{n} \epsilon_{ij}\epsilon_{ji} = \sum_{i=1}^{n} \epsilon_{ii} = 1 \in \mathrm{Mat}_n(\mathbb{K})$$
   and for all $k, l = 1, \ldots n$
   $$\sum_{i=1}^{n} \epsilon_{ij} \otimes \epsilon_{ji} \cdot \epsilon_{kl} = \epsilon_{kj} \otimes \epsilon_{jl} = \sum_{i=1}^{n} \epsilon_{kl}\epsilon_{ij} \otimes \epsilon_{ji}$$
   so that all elements $e^{(j)}$ are separability idempotents.

Proposition 3.2.19.
Let H be a finite-dimensional semisimple $\mathbb{K}$-Hopf algebra.

1. H is a separable $\mathbb{K}$-algebra.

2. Any Hopf subalgebra $K \subset H$ such that H is free over K is semisimple as well.

Proof.

1. We have to show that for any field extension $\mathbb{K} \subset E$, the algebra $H \otimes_{\mathbb{K}} E$ is semisimple as well. Note that $H \otimes E$ is an E-Hopf algebra with morphisms
   $$\begin{aligned} \Delta(h \otimes \alpha) &:= \Delta(h) \otimes \alpha \in H \otimes H \otimes E \cong (H \otimes E) \otimes_E (H \otimes E) \\ \epsilon(h \otimes \alpha) &:= \epsilon(h) \otimes \alpha \\ S(h \otimes \alpha) &:= S(h) \otimes \alpha \end{aligned}$$

for all $h \in H$ and all $\alpha \in E$. This implies that the ideal of left integrals is obtained by extension of scalars as well,

$$\mathcal{I}_l(H \otimes_{\mathbb{K}} E) = \mathcal{I}_l(H) \otimes_{\mathbb{K}} E$$

and thus that the counit ϵ is non-zero on $\mathcal{I}_l(H \otimes_{\mathbb{K}} E)$. Now Maschke's theorem 3.2.13 implies that the Hopf algebra $H \otimes_{\mathbb{K}} E$ is semisimple.

2. Since H is semisimple, find $t \in \mathcal{I}_i(H)$ with $\epsilon(t) \neq 0$. Since H is free as an K-module, find a K-basis $\{h_i\}$ of ${}_K H$ and write $t = \sum_{i \in I} k_i h_i$ with $k_i \in K$. Then for all $k \in K$, we have

$$\sum_{i \in I} (k k_i) h_i = kt = \epsilon(k) t = \sum_{i \in I} (\epsilon(k) k_i) h_i \ .$$

Comparison of coefficients shows $kk_i = \epsilon(k) k_i$ for all $i \in I$ and all $k \in K$. Thus $k_i \in \mathcal{I}_l(K)$ are integrals for all $i \in I$.

Now $0 \neq \epsilon(t) = \sum_{i \in I} \epsilon(k_i)\epsilon(h_i)$ implies that $\epsilon(k_i) \neq 0$ for some $i \in I$. Thus, by Maschke's theorem 3.2.13, the Hopf algebra K is semisimple.

□

Observation 3.2.20.

1. We comment on the results in a language using bases. Let A be a Frobenius algebra. It is finite-dimensional and let $(l_i)_{i=1,\dots N}$ be a $\mathbb{K}$-basis of A. Since the Frobenius form κ is non-degenerate, we can find another basis $(r_i)_{i=1,\dots,N}$ such that

$$\kappa(l_i, r_i) = \delta_{ij} \ .$$

Such a pair of bases is called a pair (r_i, l_i) of <u>dual bases</u> for the Frobenius form κ.

2. Since $(l_i)_{i=1,\dots N}$ is a basis, we can write any $x \in A$ as a linear combination, $x = \sum_{i=1}^N x_i l_i$. Now

$$\kappa(x, r_j) = \sum_{i=1}^N x_i \kappa(l_i, r_j) = x_j$$

and thus

$$x = \sum_{i=1}^N \kappa(x, r_i) l_i \qquad \text{for all } x \in A \ ; \tag{6}$$

similarly,

$$x = \sum_{i=1}^N \kappa(l_i, x) r_i \qquad \text{for all } x \in A \ . \tag{7}$$

3. Conversely, given a pair of bases (r_i, l_i) such that equation (6) holds for all $x \in A$, we find with $x = l_j$ that $\kappa(l_i, r_i) = \delta_{ij}$ and conclude that (7) holds for all $x \in A$.

4. Consider for any dual bases the element

$$C := \sum_{i=1}^N r_i \otimes l_i \in A \otimes A \ .$$

We claim that it is a Casimir element, i.e. $xC = Cx$ for all $x \in A$. Indeed,

$$l_i x = \sum_{i=1}^{N} \kappa(l_i x, r_i) l_i$$

implies

$$Cx = \sum_{i=1}^{N} r_i \otimes l_i x = \sum_{i,j=1}^{N} \kappa(l_i x, r_j) r_i \otimes l_i \ .$$

Similarly, we find

$$xC = \sum_{i=1}^{N} x r_i \otimes l_i = \sum_{i,j=1}^{N} \kappa(l_i, x r_j) r_i \otimes l_i \ .$$

The invariance of the Frobenius form κ now implies $xC = Cx$.

Remark 3.2.21.
We can give explicitly a separability idempotent of a finite-dimensional semisimple Hopf algebra.

1. Let $\lambda \in H^*$ be a non-zero left integral and let $\Lambda \in H$ be a right integral that $\lambda(\Lambda) = 1$, c.f. proposition 3.1.25. Then we have

$$\begin{array}{rcl} S(\Lambda_{(1)})\langle \lambda, \Lambda_{(2)} x\rangle & = & S(\Lambda_{(1)})\Lambda_{(2)} x_{(1)} \langle \lambda, \Lambda_{(3)} x_{(2)}\rangle \quad [\lambda \text{ cointegral}] \\ & = & x_{(1)} \ \langle \lambda, \Lambda x_{(2)}\rangle \quad [S \text{ antipode}] \\ & = & x\langle \lambda, \Lambda\rangle = x \quad [\Lambda \text{ right integral, normalization}] \end{array}$$

   It follows that the components $\Lambda^i_{(1)}$ of any representation of $\Delta(\Lambda)$

$$\Delta(\Lambda) = \sum_i \Lambda^i_{(1)} \otimes \Lambda^i_{(2)} \in H \otimes H$$

   form a generating system of H. We can thus find a form for $\Delta(\Lambda)$ such that the components $(\Lambda^i_{(1)})$ form a basis. Thus $(S(\Lambda_{(1)}), \Lambda_{(2)})$ form a pair of dual bases for the standard Frobenius structure on on the Hopf algebra H given by λ.

2. Assume now that H is semisimple. By Maschke's theorem $\kappa := \epsilon(\Lambda) \neq 0$. Then

$$e := \kappa^{-1} \cdot S(\Lambda_{(1)}) \otimes \Lambda_{(2)} \in H \otimes H$$

   is a separability idempotent. Indeed,

$$\mu(e) := \kappa^{-1} S(\Lambda_{(1)}) \cdot \Lambda_{(2)} = \kappa^{-1} 1_H \epsilon(\Lambda) = 1_H$$

   by the defining property of the antipode. The Casimir property of a separability idempotent follows directly from observation 3.2.20.4, since it is built from a pair of dual bases.

3.3 Powers of the antipode

Observation 3.3.1.
Let V be a finite-dimensional $\mathbb{K}$-vector space. Under the canonical identification

$$\begin{array}{rcl} V^* \otimes V & \to & \mathrm{End}_{\mathbb{K}}(V) \\ \beta \otimes v & \mapsto & (w \mapsto \beta(w) v) \end{array}$$

the trace becomes $\mathrm{Tr}(\beta \otimes v) = \beta(v)$. Indeed, with dual bases $(e_i)_{i\in I}$ of V and $(e^i)_{i\in I}$ of V^*, we find $\beta = \sum_i \beta_i e^i$ and $v = \sum_i v^i e_i$. The corresponding linear map is $\sum_{i,j} \beta_i v^j e_j \otimes e^i$ which has trace $\sum_i \beta_i v^i = \beta(v)$.

Lemma 3.3.2.
Let H be a finite-dimensional Hopf algebra with $\lambda \in \mathcal{I}_l(H^*)$ and a right integral $\Lambda \in H$ such that $\lambda(\Lambda) = 1$. Let F be a linear endomorphism of H. Then

$$\mathrm{Tr}(F) = \langle \lambda, F(\Lambda_{(2)})S(\Lambda_{(1)}) \rangle \ .$$

Proof.
We know by remark 3.2.21.1 that for all $x \in H$, we have

$$F(x) = \langle \lambda, F(x)S(\Lambda_{(1)}) \rangle \Lambda_{(2)} \ .$$

Thus under the identification $H^* \otimes H \cong \mathrm{End}(H)$, the endomorphism F corresponds to

$$\langle \lambda, F(-)S(\Lambda_{(1)}) \rangle \otimes \Lambda_{(2)} \ ;$$

thus

$$\mathrm{Tr}(F) = \langle \lambda, F(\Lambda_{(2)})S(\Lambda_{(1)}) \rangle \ .$$

$\square$

We need to understand the powers of the antipode. We first need another structure: for any element $h \in H$, left multiplication yields a $\mathbb{K}$-linear endomorphism

$$\begin{array}{rccc} L_h : & H & \to & H \\ & x & \mapsto & hx \end{array}$$

We thus define a linear form

$$\begin{array}{rccc} \mathrm{Tr}_H : & H & \to & \mathbb{K} \\ & h & \mapsto & \mathrm{Tr}(L_h) \ . \end{array}$$

Proposition 3.3.3.
Let H be a finite-dimensional Hopf algebra with $\lambda \in \mathcal{I}_l(H^*)$ and a right integral $\Lambda \in H$ such that $\lambda(\Lambda) = 1$.

1. We have
$$\mathrm{Tr} S^2 = \langle \epsilon, \Lambda \rangle \langle \lambda, 1 \rangle \ .$$
2. If $S^2 = \mathrm{id}_H$, then $\mathrm{Tr}_H = \langle \epsilon, \Lambda \rangle \lambda$.

Proof.

1. Taking S^2 in lemma 3.3.2, we find
$$\mathrm{Tr}(S^2) = \langle \lambda, S^2(\Lambda_{(2)})S(\Lambda_{(1)}) \rangle = \langle \lambda, S(\Lambda_{(1)} \cdot S(\Lambda_{(2)})) \rangle = \langle \epsilon, \Lambda \rangle \cdot \langle \lambda, 1 \rangle \ .$$
2. The identity $S^2 = \mathrm{id}_H$ implies by proposition 2.5.6
$$h_{(2)}S(h_{(1)}) = \langle \epsilon, h \rangle 1 \quad \text{for all } h \in H \ .$$
Taking $F = L_h$, we find
$$\mathrm{Tr}_H(h) = \mathrm{Tr}(L_h) = \langle \lambda, h\Lambda_{(2)}S(\Lambda_{(1)}) \rangle = \langle \epsilon, \Lambda \rangle \cdot \langle \lambda, h \rangle \ ,$$
where we used in the last step the previous equation for $h = \Lambda$.

□

Corollary 3.3.4.

1. H and H^* are semisimple, if and only if $\mathrm{Tr}S^2 \neq 0$.

2. If $S^2 = \mathrm{id}_H$ and $\mathrm{char}\mathbb{K}$ does not divide $\dim H$, then H and H^* are semisimple.

Proof.

1. By Maschke's theorem 3.2.13, H is semisimple, if and only if $\langle \epsilon, \Lambda \rangle \neq 0$. Similarly, again by Maschke's theorem, H^* is semisimple, if and only if $\langle \epsilon^*, \Lambda^* \rangle = \langle \lambda, 1 \rangle \neq 0$. Together with proposition 3.3.3.1, this implies the assertion.

2. If $S^2 = \mathrm{id}_H$, then by proposition 3.3.3
$$\dim H = \mathrm{Tr}S^2 = \langle \epsilon, \Lambda \rangle \langle \lambda, 1 \rangle$$
which is non-zero by the assumption on the characteristic of $\mathbb{K}$. Now Maschke's theorem implies the assertion.

□

Remark 3.3.5.
We have seen in corollary 2.5.9.1 that $S^2 = \mathrm{id}_H$ for a cocommutative Hopf algebra. Thus a cocommutative finite-dimensional Hopf algebra over a field of characteristic zero is always semisimple and cosemisimple.

In particular, for H cocommutative over an algebraically closed field of charactersitic zero, all simple comodules are one-dimensional and H as a right comodule decomposes into a direct sum of these simple comodules. We find a basis $(b_i)_{i\in I}$ of H and elements $\tilde{b}_i \in H$ such that
$$\Delta(b_j) = b_j \otimes \tilde{b}_j \ .$$
Using coassociativity of the coregular coaction, we find
$$b_j \otimes \Delta(\tilde{b}_j) = b_j \otimes \tilde{b}_j \otimes \tilde{b}_j$$
so that all elements $\tilde{b}_j$ are group-like. Moreover, $\tilde{b}_j = b_j$, as can be seen e.g. in the dual Hopf algebra. Thus H is a group algebra of a finite group.

Observation 3.3.6.

1. Consider a Frobenius algebra A with bilinear form κ. Since κ is non-degenerate, it provides a bijection
$$\begin{array}{ccc} A & \to & A^* \\ h & \mapsto & \kappa(-,h) \ . \end{array}$$
Consider for fixed $x \in A$ the linear form
$$y \mapsto \kappa(x,y) \ .$$
Using the bijection $A \to A^*$ above, we find $\rho(x) \in A$ such that
$$\kappa(x,y) = \kappa(y, \rho(x)) \quad \text{for all } y \in A \ .$$
The map $\rho : A \to A$ is obviously $\mathbb{K}$-linear and a bijection.

2. The map $\rho : A \to A$ is a morphism of algebras. Indeed, using the definition of ρ and the invariance (I) of κ, we find

$$\begin{aligned}\kappa(z, \rho(xy)) &= \kappa(xy, z) \overset{(I)}{=} \kappa(x, yz) = \kappa(yz, \rho(x)) \overset{(I)}{=} \kappa(y, z\rho(x)) \\ &= \kappa(z\rho(x), \rho(y)) \overset{(I)}{=} \kappa(z, \rho(x)\rho(y))\end{aligned}$$

Since the Frobenius form κ is non-degenerate, this implies $\rho(xy) = \rho(x)\rho(y)$ for all $x, y \in A$.

Definition 3.3.7
The automorphism ρ is called the Nakayama automorphism *of A with respect to the Frobenius structure κ.*

Remark 3.3.8.
If A is commutative, the Nakayama involution is the identity. A Frobenius algebra is called symmetric, if the Nakayama automorphism equals the identity, i.e. if $\kappa(x, y) = \kappa(y, x)$ for all $x, y \in A$.

Group algebras are examples of symmetric algebras.

Our strategy will now be to compute the Nakayama automorphism for the Frobenius algebra structure of a finite-dimensional Hopf algebra given by the right cointegral. (One can show that the Nakayama automorphism has, in this case, always finite order.) We need two lemmas. Denote by S^{-1} the composition inverse of the antipode, $S \circ S^{-1} = S^{-1} \circ S = \mathrm{id}_H$.

Lemma 3.3.9.
Let $\gamma \in H^*$ a non-zero right integral of a finite-dimensional Hopf algebra H and let $\Gamma \in H$ be the left integral such that $\langle \gamma, \Gamma \rangle = 1$. Denote $t := S(\Gamma)$ and $\alpha \in H^*$ the distinguished group like element which is an algebra morphism $H \to \mathbb{K}$.

1. Then $(S^{-1}(t_{(2)}), t_{(1)})$ is a pair of dual bases for γ.

2. We have for the Nakayama automorphism for the Frobenius structure given by the right integral:
$$\rho(h) = \langle \alpha, h_{(1)} \rangle \; S^{-2}(h_{(2)}) \; .$$

Proof.

- We already know that for the Frobenius form given by a left integral λ for H^* we have a dual basis $(S\Lambda_{(1)}, \Lambda_{(2)})$. Applying this to the dual of the Hopf algebra H^{copp} which has antipode S^{-1}, we find the assertion.

- As for any pair of dual bases, we have
$$\rho(h) = S^{-1}(t_{(2)}) \; \langle \gamma, t_{(1)}\rho(h) \rangle = S^{-1}(t_{(2)}) \; \langle \gamma, ht_{(1)} \rangle$$
where in the second step applied the definition of a Nakayama automorphism. Applying S^2, we find
$$\begin{aligned}S^2\rho(h) &= \langle \gamma, ht_{(1)} \rangle S(t_{(2)}) \\ &= \langle \gamma, h_{(1)}t_{(1)} \rangle h_{(2)}t_{(2)}S(t_{(3)}) \quad [\gamma \text{ right cointegral}] \\ &= \langle \gamma, h_{(1)}t \rangle h_{(2)} = \langle \gamma, \alpha(h_{(1)})t \rangle h_{(2)} \quad [\text{antipode, } \alpha \text{ distinguished element}] \\ &= \langle \alpha, h_{(1)} \rangle \, h_{(2)} \quad [\text{normalization}]\end{aligned}$$
Applying S^{-2} yields the claim.

□

Similarly, we have

Lemma 3.3.10.
Let $a \in G(H)$ be the distinguished group-like element of H and t, γ as before.

1. Then $(S(t_{(1)})a, t_{(2)})$ is a pair of dual bases for γ.
2. We have for the Nakayama automorphism for the Frobenius structure given by the right integral:
$$\rho(h) = a^{-1}S^2(h_{(1)})\langle\alpha, h_{(2)}\rangle a \ .$$

Proof.

- Using the definitions, we find for all $h \in H$:
$$\begin{aligned} S(t_{(1)})a\langle\gamma, t_{(2)}h\rangle &= S(t_{(1)})t_{(2)}h_{(1)}\langle\gamma, t_{(3)}h_{(2)}\rangle \quad [\gamma \text{ right cointegral}] \\ &= h_{(1)}\langle\gamma, th_{(2)}\rangle = h_{(1)}\epsilon(h_{(2)}) = h \end{aligned}$$
so that we have dual bases.
- By the fact that we have dual bases, we can write
$$\rho(h) = S(t_{(1)})a\langle\gamma, t_{(2)}\rho(h)\rangle = S(t_{(1)})a\langle\gamma, ht_{(2)}\rangle \ ,$$
where in the second identity, we applied the definition of the Nakayama automorphism. Applying S^{-2} and conjugating with a, we find
$$\begin{aligned} aS^{-2}(\rho(h))a^{-1} &= a\langle\gamma, ht_{(2)}\rangle S^{-1}(t_{(1)}) \\ &= h_{(1)}t_{(2)}\langle\gamma, h_{(2)}t_{(3)}\rangle S^{-1}(t_{(1)}) \quad [\gamma \text{ cointegral}] \\ &= h_{(1)}\langle\gamma, h_{(2)}t\rangle = h_{(1)}\langle\alpha, h_{(2)}\rangle \ . \end{aligned}$$
Conjugating with a^{-1} and applying S^2 yields the claim.

□

Observation 3.3.11.
If H is a finite-dimensional Hopf algebra, then H^* is a finite-dimensional Hopf algebra as well. We then have the structure of a left and right H^*-module on H by
$$h^* \rightharpoonup h := h_{(1)}\langle h^*, h_{(2)}\rangle \qquad \text{and} \qquad h \leftharpoonup h^* := \langle h^*, h_{(1)}\rangle h_{(2)} \ .$$
This can be checked again graphically.

Theorem 3.3.12 (Radford,1976)**.**
Let H be a finite-dimensional Hopf algebra over a field $\mathbb{K}$. Let $a \in G(H)$ and $\alpha \in G(H^*)$ be the distinguished grouplike elements. Then the following identity holds:
$$S^4(h) = a(\alpha^{-1} \rightharpoonup h \leftharpoonup \alpha)a^{-1} = \alpha^{-1} \rightharpoonup (aha^{-1}) \leftharpoonup \alpha$$

Proof.

- We first show the identity

$$a(\alpha^{-1} \rightharpoonup h \leftharpoonup \alpha)a^{-1} = \alpha^{-1} \rightharpoonup (aha^{-1}) \leftharpoonup \alpha$$

 We transform the left hand side by using the definition of the H^*-actions:

$$a(\alpha^{-1} \rightharpoonup h \leftharpoonup \alpha)a^{-1} = \langle \alpha, h_{(1)} \rangle a h_{(2)} a^{-1} \langle \alpha^{-1}, h_{(3)} \rangle \ .$$

 We transform the right hand side, using the definition of the H^* actions and the fact that a is group-like:

$$\begin{aligned} \alpha^{-1} \rightharpoonup (aha^{-1}) \leftharpoonup \alpha &= \langle \alpha, a h_{(1)} a^{-1} \rangle a h_{(2)} a^{-1} \langle \alpha^{-1}, a h_{(3)} a^{-1} \rangle \\ &= \langle \alpha, h_{(1)} \rangle a h_{(2)} a^{-1} \langle \alpha^{-1}, h_{(3)} \rangle \ , \end{aligned}$$

 where in the last identity we used that α as a group-like element of H^* is a morphism of algebras $H \to \mathbb{K}$, cf. remark 2.6.9.1.

- The two lemmata 3.3.9 and 3.3.10 on the Nakayama automorphism ρ for the right cointegral imply

$$\langle \alpha, h_{(1)} \rangle \ S^{-2}(h_{(2)}) = \rho(h) = a^{-1} S^2(h_{(1)}) \langle \alpha, h_{(2)} \rangle a \ .$$

 Applying S^2 to this equation and conjugating with $a \in G(H)$, we get

$$a \cdot \langle \alpha, h_{(1)} \rangle \ h_{(2)} \cdot a^{-1} = S^4(h_{(1)}) \langle \alpha, h_{(2)} \rangle \ .$$

 Multiplying this equation with the non-zero scalar $\langle \alpha^{-1}, h_{(3)} \rangle$, we find

$$\begin{aligned} a(\alpha^{-1} \rightharpoonup h \leftharpoonup \alpha)a^{-1} &= a \cdot \langle \alpha, h_{(1)} \rangle \ h_{(2)} \langle \alpha^{-1}, h_{(3)} \rangle \cdot a^{-1} \\ &= S^4(h_{(1)}) \langle \alpha, h_{(2)} \rangle \langle \alpha^{-1}, h_{(3)} \rangle \\ &= S^4(h_{(1)}) \langle \alpha \cdot \alpha^{-1}, h_{(2)} \rangle \\ &= S^4(h) \end{aligned}$$

□

Corollary 3.3.13.
Let H be a finite-dimensional Hopf algebra.

1. The order of the antipode S of H is finite.
2. If H is unimodular, then S^4 coincides with the inner automorphism of H induced by a grouplike element. In particular, the order of the antipode is at most $4 \cdot \dim H$.
3. If both H and H^* are unimodular, then $S^4 = \mathrm{id}_H$.

Proof.

1. Since H and H^* are finite-dimensional and since distinct powers of a group-like element are linearly independent, every group-like element in H or H^* has finite order. By Radford's formula 3.3.12, S^4 has finite order and thus S has finite order.

2. By corollary 3.1.16, the Hopf algebra H is unimodular, if and only if the distinguished group-like element α equals the counit. The action of the counit on H is trivial, thus for unimodular Hopf algebras, Radford's formula reads $S^4(h) = aha^{-1}$. The last assertion follows by applying the same reasoning to the Hopf algebra H^*.

□

We finally derive a result relating the order of the antipode S of H to semisimplicity of the Hopf algebra H.

Lemma 3.3.14.
Let A be a Frobenius algebra with bilinear form κ and dual bases (r_i, l_i) as in observation 3.2.20. Suppose that $e \in A$ has the property that $e^2 = \alpha e$ with $\alpha \in \mathbb{K}$. Consider an $\mathbb{K}$-linear endomorphism f of the subspace $eA := \{ea \,|\, a \in A\}$ of A. Then

1. $\alpha \mathrm{Tr}(f) = \sum \kappa(f(el_i), r_i)$.
2. $\alpha \mathrm{Tr}(f) = \sum \kappa(l_i, f(er_i))$.

Proof.
Using the defining property of dual bases, we find

$$\alpha e x = e^2 x = e\left(\sum_i \kappa(ex, r_i) l_i\right) = \sum_i \kappa(ex, r_i) e l_i \ .$$

Thus, since f is linear,

$$\alpha f(ex) = \sum_i \kappa(ex, r_i) f(el_i)$$

so that under the isomorphism

$$(eA)^* \otimes (eA) \to \mathrm{End}_{\mathbb{K}}(eA)$$

we have

$$\sum_i \kappa(-, r_i) \otimes f(el_i) \mapsto \alpha f \ .$$

Combined with lemma 3.3.2, this shows the first formula. The second formula is shown analogously. □

Definition 3.3.15
Let A be a unital associative $\mathbb{K}$-algebra. Let V be a finite-dimensional left A-module with algebra map $\rho : A \to \mathrm{End}_{\mathbb{K}}(V)$. Then the linear form

$$\begin{array}{rccc} \chi_V : & A & \to & \mathbb{K} \\ & a & \mapsto & \mathrm{Tr}\rho(a) \end{array}$$

is called the <u>character</u> of the module V.

Remarks 3.3.16.
The following properties are easy to check:

1. $\chi_V(1_A) = \dim_{\mathbb{K}} V$ for any A-module V.

2. Let V, W be A-modules. Any isomorphism of modules $V \cong W$ implies identity of characters, $\chi_V = \chi_W$. Note that the converse is, in general, wrong.

3. Let V, W be A-modules. Then we have $\chi_{V\oplus W} = \chi_V + \chi_W$, as a consequence of the behaviour of the trace on direct sums of vector spaces.

4. Suppose that we consider modules over a Hopf algebra H. Then $\chi_{V\otimes W} = \chi_V \cdot \chi_W$ with the convolution product in $\mathrm{Hom}_{\mathbb{K}}(H, \mathbb{K})$. Indeed,

$$\chi_{V\otimes W}(h) = \mathrm{Tr}_{V\otimes W}\rho_V(h_{(1)}) \otimes \rho_W(h_{(2)}) = \chi_V(h_{(1)}) \cdot \chi_W(h_{(2)}) = (\chi_V \cdot \chi_W)(h) \ .$$

5. Suppose again that we consider modules over a Hopf algebra H. Then for a trivial module $T = (V, \epsilon \otimes \mathrm{id}_V)$, we have $\chi_T(h) = \epsilon(h) \dim V$.

6. For the character of the right dual module $V^\vee$, we have

$$\chi_{V^\vee}(h) = \mathrm{Tr}_{V^*}\rho_V(S(h))^t = \mathrm{Tr}_V\rho_V(S(h)) = \chi_V(S(h)) \ .$$

   For the character of the left dual module, we have

$$\chi_{{}^\vee V}(h) = \mathrm{Tr}_{V^*}\rho_V(S^{-1}(h))^t = \mathrm{Tr}_V\rho_V(S^{-1}(h)) = \chi_V(S^{-1}(h)) \ .$$

Lemma 3.3.17.
Let H be a Hopf algebra. It is a left module over itself. Then

1. $\chi_H^2 = \dim H \cdot \chi_H$.

2. $S^2\chi_H = \chi_H$, where we use S for the antipode of H^* as well,

as identities of elements in the Hopf algebra H^*.

Proof.

1. Let V be any H-module and V_ϵ the trivial H-module structure on the vector space underlying V. Then the bialgebra property of H implies that the linear map

$$\begin{array}{rcl} H \otimes V_\epsilon & \to & H \otimes V \\ h \otimes v & \mapsto & h_{(1)} \otimes h_{(2)}.v \end{array}$$

   defines an isomorphism of H-modules. This implies

$$\chi_H \chi_V = \chi_H \chi_{V_\epsilon} = \chi_H \dim V \ ,$$

   where all products are convolution products and where we used that the counit ϵ is the unit of H^*. Then specialize to $V = H$.

2. Let $h \in H$. Then

$$\langle S^2(\chi_H), h\rangle = \langle \chi_H, S^2 h\rangle = \mathrm{Tr}_H(L_{S^2(h)}) \ .$$

   Since S^2 is an algebra automorphism, we have

$$\mathrm{Tr}_H(L_{S^2(h)}) = \mathrm{Tr}_H(L_h) = \langle \chi_H, h\rangle \ .$$

□

Since $S^2\chi_H = \chi_H$ and since S^2 is a an algebra morphism of H^*, we have by lemma 3.3.17.2 for any $\beta \in H^*$

$$S^2(\chi_H\beta) = S^2(\chi_H) \cdot S^2(\beta) = \chi_H \cdot S^2(\beta)$$

so that S^2 restricts to an endomorphism of the linear subspace $\chi_H H^* \subset H^*$.

Lemma 3.3.18.
Let H be a Hopf algebra. Let $\gamma \in H^*$ be a nonzero right integral and $\Gamma \in H$ be a left integral, normalized such that $\langle \gamma, \Gamma \rangle = 1$. Then

$$\mathrm{Tr}_{H^*}(S^2) = \langle \epsilon, \Gamma \rangle \langle \gamma, 1 \rangle = (\dim H) \mathrm{Tr} S^2|_{\chi_H H^*} \ .$$

Proof.
By applying proposition 3.3.3 to $H^{\mathrm{opp,copp}}$, we find

$$\mathrm{Tr}_H S^2 = \langle \gamma, 1 \rangle \cdot \langle \epsilon, \Gamma \rangle \ .$$

We denote by $\tilde{\Gamma} \in H^{**}$ the image of $\Gamma \in H$ in the bidual of H. Now γ is a Frobenius form with dual bases $(\Gamma_{(1)}, S(\Gamma_{(2)}))$ which implies that $\tilde{\Gamma}$ is a Frobenius form for H^* with dual bases $(S(\gamma_{(1)}), \gamma_{(2)})$.

Now lemma 3.3.14 applies to $e := \chi_H$ with $\alpha = \dim H$, thus yielding

$$\begin{aligned}
\dim H \cdot \mathrm{Tr}(S^2)|_{\chi_H H^*} &= \langle \tilde{\Gamma}, S^2(\chi_H \gamma_{(2)}) S(\gamma_{(1)}) \rangle \\
&= \langle \tilde{\Gamma}, S^2(\chi_H) S^2(\gamma_{(2)}) S(\gamma_{(1)}) \rangle \quad [S^2 \text{ algebra morphism}] \\
&= \langle \tilde{\Gamma}, \chi_H S(\gamma_{(1)} \cdot S(\gamma_{(2)})) \rangle \quad [S^2\chi_H = \chi_H] \\
&= \langle \gamma, 1 \rangle \cdot \langle \chi_H, \Gamma \rangle
\end{aligned}$$

By the same lemma 3.3.14, taking $f = L_\Gamma$ and $e = 1$ with $\alpha = 1$, we have for the second factor

$$\begin{aligned}
\chi_H(\Gamma) &= \langle \gamma, S(\Gamma_{(2)}) \Gamma \Gamma_{(1)} \rangle \\
&= \langle \gamma, \epsilon(\Gamma_{(2)}) \Gamma \Gamma_{(1)} \rangle \quad [\Gamma \text{ is a left integral of } H] \\
&= \langle \gamma, \Gamma\Gamma \rangle = \epsilon(\Gamma) \langle \gamma, \Gamma \rangle = \epsilon(\Gamma)
\end{aligned}$$

Combining the two results yields

$$\dim H \cdot \mathrm{Tr}(S^2)|_{\chi_H H^*} = \langle \gamma, 1 \rangle \cdot \langle \chi_H, \Gamma \rangle = \langle \gamma, 1 \rangle \cdot \langle \epsilon, \Gamma \rangle$$

which finishes the proof of the lemma. □

Theorem 3.3.19 (Larson-Radford, 1988)**.**
Let $\mathbb{K}$ be a field of characteristic zero. Let H be a finite-dimensional $\mathbb{K}$-Hopf algebra. Then the following statements are equivalent:

1. H is semisimple.

2. H^* is semisimple.

3. $S^2 = \mathrm{id}_H$.

Proof.
We have already seen in corollary 3.3.4.2 that 3. implies 1. and 2. One can show that 2 implies 1, see Larson and Radford. Here we only show that 1. and 2. together imply 3. Suppose that H and H^* are both semisimple and thus, by corollary 3.2.15, unimodular. By corollary 3.3.13.3, then $S^4 = (S^2)^2 = \mathrm{id}$.

Hence the eigenvalues of S^2 on H and of $S^2|_{\chi_H H^*}$ are all ± 1. Call them $(\mu_j)_{1\leq j\leq n}$ with $n := \dim H$ and $(\eta_i)_{1\leq i\leq m}$ with $m := \dim \chi_H H^*$. Thus

$$\mathrm{Tr}_{H^*}(S^2) = \sum_{j=1}^{n} \mu_j \quad \text{and} \quad \mathrm{Tr}(S^2|_{\chi_H H^*}) = \sum_{i=1}^{m} \eta_i \ ,$$

By lemma 3.3.18,

$$\sum_{j=1}^{n} \mu_j = n \sum_{i=1}^{m} \eta_i \ . \tag{8}$$

This implies

$$n \cdot |\sum_{i=1}^{m} \eta_i| \leq \sum_{j=1}^{n} |\mu_j| = n \ .$$

For a semisimple Hopf algebra, we have seen in corollary 3.3.4 that $0 \neq \mathrm{Tr}S^2 = \sum_{j=1}^{n} \mu_j$ and thus by the equality (8) we find $\sum_{i=1}^{m} \eta_i \neq 0$. This implies $\sum_{i=1}^{m} \eta_i = \pm 1$ and, as a further consequence of equation (8) we have $\sum_{j=1}^{n} \mu_j = \pm n$. Since $S^2(1_H) = 1_H$, we have at least one eigenvalue $+1$. Thus all eigenvalues of S^2 on H have to be $+1$ which amounts to $S^2 = \mathrm{id}_H$. □

There are some important results we do not cover in these lectures. The following theorem is proven in [Schneider]:

Theorem 3.3.20 (Nichols-Zoeller, 1989)**.**
Let H be a finite-dimensional Hopf algebra, and let $R \subset H$ be a Hopf subalgebra. Then H is a free R-module.

Corollary 3.3.21 ("Langrange's theorem for Hopf algebras")**.**
If $R \subset H$ are finite-dimensional Hopf algebras, then the order of R divides the order of H.

We finally refer to chapter 4 of Schneider's lecture notes [Schneider] for a character theory for finite-dimensional semisimple Hopf algebras that closely parallels the character theory for finite groups.

4 Quasi-triangular Hopf algebras and braided categories

4.1 Interlude: topological field theory

In this subsection, we introduce the notion of a topological field theory and investigate low-dimensional topological field theories. To this end, we need more structure on monoidal categories.

Let $(\mathcal{C}, \otimes, a, l, r)$ be a tensor category. From the tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, we can get the functor $\otimes^{\mathrm{opp}} = \otimes \circ \tau$ with

$$V \otimes^{\mathrm{opp}} W := W \otimes V \quad \text{and } f \otimes^{\mathrm{opp}} g := g \otimes f \ .$$

It defines a tensor product: given an associator a for $\otimes$, one verifies that $a^{\mathrm{opp}}_{U,V,W} := a^{-1}_{W,V,U}$ is an associator for the tensor product $\otimes^{\mathrm{opp}}$. Similarly, one obtains left and right unit constraints.

Definition 4.1.1

1. *A commutativity constraint for a tensor category $(\mathcal{C}, \otimes)$ is a natural isomorphism*

$$c : \otimes \to \otimes^{\mathrm{opp}}$$

of functors $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$. Explicitly, we have for any pair (V, W) of objects of $\mathcal{C}$ an isomorphism

$$c_{V,W} : V \otimes W \to W \otimes V$$

such that for all morphisms $V \xrightarrow{f} V'$ and $W \xrightarrow{g} W'$ the diagrams

$$\begin{tikzcd}
V \otimes W \arrow[r, "c_{V,W}"] \arrow[d, "f\otimes g"'] & W \otimes V \arrow[d, "g\otimes f"] \\
V' \otimes W' \arrow[r, "c_{V',W'}"] & W' \otimes V'
\end{tikzcd}$$

commute.

2. *Let $\mathcal{C}$ be, for simplicity, a strict tensor category. A braiding is a commutatitivity constraint such that for all objects U, V, W the compatibility relations with the tensor product*

$$\begin{aligned}
c_{U\otimes V,W} &= (c_{U,W} \otimes \mathrm{id}_V) \circ (\mathrm{id}_U \otimes c_{V,W}) \\
c_{U,V\otimes W} &= (\mathrm{id}_V \otimes c_{U,W}) \circ (c_{U,V} \otimes \mathrm{id}_W)
\end{aligned}$$

hold.

If the category is not strict, the following two hexagon axioms have to hold:

$$\begin{tikzcd}
 & U\otimes(V\otimes W) \arrow[r, "c_{U,V\otimes W}"] & (V\otimes W)\otimes U \arrow[dr, "a_{V,W,U}"] & \\
(U\otimes V)\otimes W \arrow[ur, "a_{U,V,W}"] \arrow[dr, "c_{U,V}\otimes\mathrm{id}_W"'] & & & V\otimes(W\otimes U) \\
 & (V\otimes U)\otimes W \arrow[r, "a_{V,U,W}"] & V\otimes(U\otimes W) \arrow[ur, "\mathrm{id}_V\otimes c_{U,W}"'] &
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & (U\otimes V)\otimes W \arrow[r, "c_{U\otimes V,W}"] & W\otimes(U\otimes V) \arrow[dr, "a^{-1}_{W,U,V}"] & \\
U\otimes(V\otimes W) \arrow[ur, "a^{-1}_{U,V,W}"] \arrow[dr, "\mathrm{id}_U\otimes c_{V,W}"'] & & & (W\otimes U)\otimes V \\
 & U\otimes(W\otimes V) \arrow[r, "a^{-1}_{U,W,V}"] & (U\otimes W)\otimes V \arrow[ur, "c_{U,W}\otimes\mathrm{id}_V"'] &
\end{tikzcd}$$

3. *A braided tensor category is a tensor category together with the structure of a braiding.*

4. *With c_{UV}, also c^{-1}_{VU} is a braiding. In case, the identity $c_{U,V} = c^{-1}_{V,U}$ holds, the braided tensor category is called symmetric.*

Remarks 4.1.2.

1. Graphically, we represent the braiding by overcrossings and its inverse by undercrossings. Overcrossings and undercrossings have to be distinguished.

2. The flip map
$$\begin{array}{rccc} \tau : & V \otimes W & \to & W \otimes V \\ & v \otimes w & \mapsto & w \otimes v \end{array}$$
defines a symmetric braiding on the monoidal category $\mathrm{vect}(\mathbb{K})$ of $\mathbb{K}$-vector spaces. It also induces a symmetric braiding on the category $\mathbb{K}[G]$-mod of $\mathbb{K}$-linear representations of a group. More generally, flip maps give a symmetric braiding on the category $H-$mod for any cocommutative Hopf algebra. Since the universal enveloping algebra $\mathrm{U}(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is cocommutative, the category of $\mathbb{K}$-linear representations of $\mathfrak{g}$ is symmetric, as well.

3. There are tensor categories that do not admit a braiding. For example, for G a non-abelian group, the category $\mathrm{vect}(G)$ of G-graded vector spaces does not admit a braiding since
$$\mathbb{K}_g \otimes \mathbb{K}_h \cong \mathbb{K}_{gh} \quad \text{and} \quad \mathbb{K}_h \otimes \mathbb{K}_g \cong \mathbb{K}_{hg}$$
are not isomorphic, if $gh \neq hg$.

4. The category $\mathrm{vect}(G)$ admits the flip as a braiding, if the group G is abelian. In the case of $G = \mathbb{Z}_2$, objects are $\mathbb{Z}_2$-graded vector spaces $V_0 \oplus V_1$. We can introduce another symmetric braiding c: on homogeneous components, it is the flip up to signs:
$$\begin{array}{rcc} c : V_i \otimes W_j & \to & W_j \otimes V_i \\ v_i \otimes w_j & \mapsto & (-1)^{ij} w_j \otimes v_i \end{array}$$
This category is called the category of super vector spaces. In particular, a tensor category can admit inequivalent braidings.

5. Recall from definition 2.5.15 that the monoidal category of cobordisms has disjoint union as the tensor product. It admits symmetric braiding given by the morphism represented by the bordism of two cylinders exchanging the order of the two tensorands $M_1 \coprod M_2 \to M_2 \coprod M_1$.

Remarks 4.1.3.

1. As in any tensor category, we can consider algebras and coalgebras in a braided tensor category $(\mathcal{C}, \otimes, c)$. Now, we have the notion of a commutative associative unital algebra (A, μ, η): here the product is required to obey $\mu \circ c_{A,A} = \mu$. We also have the opposed algebra with multiplication $\mu^{\mathrm{opp}} := \mu \circ c_{A,A}$. Similarly, we have the notion of the coopposed coalgebra with coproduct $\Delta^{\mathrm{copp}} := c_{C,C} \circ \Delta$, and the notion of a cocommative coassociative counital coalgebra with coproduct obeying $c_{C,C} \circ \Delta = \Delta$.

2. Another construction that uses the braiding is the following: Suppose that we have two associative unital algebras (A, μ, η) and (A', μ', η') in a braided tensor category $\mathcal{C}$. Then the tensor product $A \otimes A'$ can be endowed with the structure of an associative algebra with product
$$A \otimes A' \otimes A \otimes A' \xrightarrow{\mathrm{id}_A \otimes c_{A,A'} \otimes \mathrm{id}_{A'}} A \otimes A \otimes A' \otimes A' \xrightarrow{\mu \otimes \mu'} A \otimes A' \ .$$
A unit is then $\eta \otimes \eta'$.

   Dually, also the tensor product of two counital coassociative coalgebras can be endowed with the structure of a coalgebra.

3. Hence, in braided tensor categories, it makes sense to consider an object H which has both the structure of an associative unital algebra and of a coassociative counital coalgebra such that the coproduct $\Delta : H \to H \otimes H$ is a morphism of algebras. We are thus able to introduce the notion of a bialgebra and, moreover, of a Hopf algebra, in a braided category.

4. We will see in an exercise that the exterior algebra is a Hopf algebra in the symmetric tensor category of super vector spaces.

We again need functors and natural transformations with appropriate compatibilities:

Definition 4.1.4

1. *A tensor functor $(F, \varphi_0, \varphi_2)$ from a braided tensor category $\mathcal{C}$ to a braided tensor category $\mathcal{D}$ is called a braided tensor functor, if for any pair of objects (V, V') of $\mathcal{C}$, the square*

$$\begin{tikzcd} F(V) \otimes F(V') \arrow[r, "\varphi_2"] \arrow[d, "c_{F(V),F(V')}"'] & F(V \otimes V') \arrow[d, "F(c_{V,V'})"] \\ F(V') \otimes F(V) \arrow[r, "\varphi_2"] & F(V' \otimes V) \end{tikzcd}$$

commutes. If the braided tensor category has the property of being symmetric, a braided tensor functor is also called a symmetric tensor functor.

2. *As braided monoidal natural transformations, we take all monoidal natural transformations.*

Definition 4.1.5 *[Atiyah]*
Let $\mathbb{K}$ be a field. A topological field theory of dimension n is a symmetric monoidal functor

$$Z : \mathrm{Cob}(n) \to \mathrm{vect}(\mathbb{K}) \ .$$

Remarks 4.1.6.

1. The category $\mathrm{vect}(\mathbb{K})$ can be replaced by any symmetric monoidal category. (Interesting examples include e.g. categories of complexes of vector spaces.) Also variants of cobordism categories are in use: spin cobordisms, manifolds with principal bundle, unoriented manifolds, ...

2. Without loss of generality, one can suppose that the symmetric monoidal functor Z is strict.

3. As a motivation, consider a group G and the groupoid $*//G$. The functor category $[*//G, \mathrm{vect}(\mathbb{K})]$ is then the category of $\mathbb{K}$-linear representations of G. For this reason, topological field theories can be seen as representations of cobordism categories.

4. We deduce from the definition that a topological field theory Z of dimension n is given by the following data:

   (a) For every oriented closed manifold M of dimension $(n-1)$, a $\mathbb{K}$-vector space $Z(M)$.

(b) For every oriented bordism B from an $(n-1)$-manifold M to another $(n-1)$-manifold N, a $\mathbb{K}$-linear map $Z(B) : Z(M) \to Z(N)$.

(c) A collection of coherent isomorphisms

$$Z(\emptyset) \cong \mathbb{K} \qquad Z(M \coprod N) \cong Z(M) \otimes Z(N).$$

Functoriality implies that we can glue cobordisms and get the composition of linear maps. Moreover, these data are required to satisfy a number of natural coherence properties which we will not make explicit.

5. A closed oriented manifold M of dimension n can be regarded as a bordism from the empty $(n-1)$-manifold to itself, $M : \emptyset \to \emptyset$. Thus

$$Z(M) : \mathbb{K} \cong Z(\emptyset) \to Z(\emptyset) \cong \mathbb{K}$$

and thus $Z(M) \in \mathrm{Hom}_{\mathbb{K}}(\mathbb{K}, \mathbb{K}) \cong \mathbb{K}$ is a number: an invariant assigned to every closed oriented manifold of dimension n.

Observation 4.1.7.
Let Z be an n-dimensional topological field theory. For any closed oriented $(n-1)$-dimensional manifold M, $Z(M)$ is a vector space. The cylinder on M gives a bordism $d_M : \overline{M} \coprod M \to \emptyset$ which is a right evaluation. Similarly, we get a right coevaluation $b_M : \emptyset \to M \coprod \overline{M}$.

Applying the functor Z, we get a vector space $Z(M)$ together with another vector space $Z(\overline{M})$ that is a right dual.

We need two lemmas to understand the vector spaces appearing in topological field theory:

Lemma 4.1.8.
Let V be an object in a tensor category. Let $(V^\vee, d_V, b_V)$ and $(\tilde{V}^\vee, \tilde{d}_V, \tilde{b}_V)$ be two right duals. Then $V^\vee$ and $\tilde{V}^\vee$ are canonically isomorphic: there is a unique isomorphism $\varphi : V^\vee \to \tilde{V}^\vee$, such that the two diagrams

$$\begin{tikzcd} V^\vee \otimes V \arrow[rr, "\varphi\otimes \mathrm{id}_V"] \arrow[dr, "d_V"'] & & \tilde{V}^\vee \otimes V \arrow[dl, "\tilde{d}_V"] \\ & \mathbb{I} & \end{tikzcd} \qquad \begin{tikzcd} V \otimes V^\vee \arrow[rr, "\mathrm{id}_V\otimes\varphi"] & & V \otimes \tilde{V}^\vee \\ & \mathbb{I} \arrow[ul, "b_V"] \arrow[ur, "\tilde{b}_V"'] & \end{tikzcd}$$

commute.

Proof.
For simplicity, we assume that the tensor category is strict. The axioms of a duality imply that

$$\alpha : V^\vee \xrightarrow{\mathrm{id}_{V^\vee}\otimes \tilde{b}_V} V^\vee \otimes V \otimes \tilde{V}^\vee \xrightarrow{d_V \otimes \mathrm{id}_{\tilde{V}^\vee}} \tilde{V}^\vee$$

and

$$\beta : \tilde{V}^\vee \xrightarrow{\mathrm{id}_{\tilde{V}^\vee}\otimes b_V} \tilde{V}^\vee \otimes V \otimes V^\vee \xrightarrow{\tilde{d}_V \otimes \mathrm{id}_{V^\vee}} V^\vee$$

are inverse to each other. Uniqueness is easy to see. $\square$

Lemma 4.1.9.
A $\mathbb{K}$-vector space V has a right dual, if and only if it is finite-dimensional.

Proof.
Consider the element

$$b_V(1) = \sum_{i=1}^{N} b_i \otimes \beta_i \in V \otimes V^* \quad \text{with } \ b_i \in V \quad \text{ and } \quad \beta_i \in V^*$$

which is necessarily a finite linear combination. Then by the axioms of a duality

$$v = (\mathrm{id}_V \otimes d_V)(b_V(1) \otimes \mathrm{id}_V)(v) = \sum_{i=1}^{N} b_i \beta_i(v) \ .$$

This shows that the vectors $(b_i)_{i=1,\dots N}$ are a finite set of generators for V and thus that V is finite-dimensional. The converse is obvious. $\square$

Corollary 4.1.10.
Let Z be a topological field theory of dimension n. Then for every closed $(n-1)$-manifold M, the vector space $Z(M)$ is finite-dimensional, and the pairing $Z(\overline{M}) \otimes Z(M) \to \mathbb{K}$ is perfect: that is, it induces an isomorphism α from $Z(\overline{M})$ to the dual space of $Z(M)$.

In low dimensions, it is possible to describe topological field theories very explicitly.

Example 4.1.11 (Topological field theories in dimension 1)**.**

- Let Z be a 1-dimensional topological field theory. Then Z assigns a finite-dimensional vector space $Z(M)$ to every closed oriented 0-manifold M, i.e. to a finite set of oriented points. Since the functor Z is monoidal, it suffices to know its values $Z(\bullet, +)$ and $Z(\bullet, -)$ on the positively and negatively oriented point which are finite-dimensional vector spaces dual to each other. Thus

  $$Z(M) \cong (\bigotimes_{x \in M_+} V) \otimes (\bigotimes_{y \in M_-} V^\vee)$$

  with $V := Z(\bullet, +)$.

- To fully determine Z, we must also specify Z on 1-manifolds B with boundary. Since Z is a symmetric monoidal functor, it suffices to specify $Z(B)$ when B is *connected.* In this case, the 1-manifold B is diffeomorphic either to a closed interval $[0, 1]$ or to a circle S^1.

- There are five cases to consider, depending on how we interpret the one-dimensional manifold B with boundary, the interval, as cobordism:

  (a) Suppose that $B = [0, 1]$, regarded as a bordism from $(\bullet, +)$ to itself. Then $Z(B)$ is the identity map $\mathrm{id}_V : V \to V$.

  (b) Suppose that $B = [0, 1]$, regarded as a bordism from $(\bullet, -)$ to itself. Then $Z(B)$ is the identity map $\mathrm{id} : V^\vee \to V^\vee$.

  (c) Suppose that $B = [0, 1]$, regarded as a bordism from $(\bullet, +) \coprod (\bullet, -)$ to the empty set. Then $Z(B)$ is a linear map from $V \otimes V^\vee$ into the ground field $\mathbb{K}$: the evaluation map $(v, \lambda) \mapsto \lambda(v)$. Since the order matters, we also consider the related bordism from $(\bullet, -) \coprod (\bullet, +)$ to the empty set. Then $Z(B)$ is a linear map from $V^\vee \otimes V$ into the ground field $\mathbb{K}$: the evaluation map $(\lambda, v) \mapsto \lambda(v)$.

(d) Suppose that $B = [0,1]$, regarded as a bordism from the empty set to $(\bullet,+)\coprod(\bullet,-)$. Then $Z(B)$ is a linear map from $\mathbb{K}$ to $Z((\bullet,+)\coprod(\bullet,-)) \cong V \otimes V^\vee$. Under the canonical isomorphism $V\otimes V^\vee \cong \mathrm{End}(V)$, this linear map is given by the coevalution $x \mapsto x\mathrm{id}_V$. Again, we can exchange the order of the objects.

(e) Suppose that $B = S^1$, regarded as a bordism from the empty set to itself. Then $Z(B)$ is a linear map from $\mathbb{K}$ to itself, which we can identify with an element of $\mathbb{K}$. To compute this element, decompose the circle $S^1 \cong \{z \in \mathbb{C} : |z| = 1\}$ into two intervals

$$S^1_- = \{z \in \mathbb{C} : (|z| = 1) \wedge \mathrm{Im}\,(z) \le 0\} \qquad S^1_+ = \{z \in \mathbb{C} : (|z| = 1) \wedge \mathrm{Im}\,(z) \ge 0\},$$

with intersection

$$S^1_- \cap S^1_+ = \{\pm 1\} \subseteq S^1.$$

It follows that $Z(S^1)$ is given as the composition of the linear maps

$$\mathbb{K} \simeq Z(\emptyset) \xrightarrow{Z(S^1_-)} Z(\pm 1) \xrightarrow{Z(S^1_+)} Z(\emptyset) \simeq \mathbb{K} \ .$$

These maps were described by (c) and (d) above. We thus get a map

$$\begin{array}{rcccl} \mathbb{K} \cong Z(\emptyset) & \to & Z(\pm 1) \cong V \otimes V^\vee & \to & Z(\emptyset) \cong \mathbb{K} \\ \lambda & \mapsto & \lambda \sum v^i \otimes v_i & \mapsto & \lambda \sum_i v^i(v_i) = \lambda \cdot \dim V \end{array}$$

where we have chosen a basis $(v_i)_{i\in I}$ of V and a dual basis $(v^i)_{i\in I}$ of V^*. Consequently, $Z(S^1)$ is given by the dimension of V.

In physical language, we have a quantum mechanical system which has only ground states and thus trivial Hamiltonian. Then the only invariant of the system is the degeneracy $\dim V$ of the space of ground states.

Example 4.1.12 (Topological field theories in dimension 2)**.**

- A two-dimensional topological field theory assigns a vector space $Z(M)$ to every closed, oriented 1-manifold M. Such a manifold is diffeomorphic to a disjoint union of circles, $M \cong (S^1)^{\amalg n}$ for some $n \ge 0$. Since Z is monoidal, $Z(M) \cong A^{\otimes n}$ with $A := Z(S^1)$ by Lemma 4.1.9 a finite-dimensional vector space.

- One can show (see e.g. [Kock, Proposition 1.4.13]) that the monoidal category Cob(2) is generated under composition and disjoint union by six cobordisms: cap or disc, trinion, also called pair of pants, the cylinder, the trinion with two outoing circles, a disc with one ingoing circle and two exchanging cylinders. Applying the functor Z to these cobordisms, we get the following linear maps:

$$\begin{array}{rl} \text{cap} & \eta : \mathbb{K} \to A \\ \text{trinion} & \mu : A \otimes A \to A \\ \text{cylinder} & I_A : A \to A \\ \text{opposite trinion} & \Delta : A \to A \otimes A \\ \text{opposite cap} & \epsilon : A \to \mathbb{K} \\ \text{exchanging cylinder} & \tau : A \otimes A \to A \otimes A \end{array}$$

- One can also classify all relations between the generators, see [Kock, 1.4.24-1.4.28]. The relations can be summarized that category Cob(2) is the free symmetric monoidal category on a commutative Frobenius object (Chapter 3). The relations imply that A has the structure of a commutative (Δ,ϵ)-Frobenius algebra.

- In fact, the converse is true as well: given a commutative Frobenius algebra A, one can construct a 2-dimensional topological field theory Z such that $A = Z(S^1)$ and the multiplication and Frobenius form on A are given by evaluating Z on a pair of pants and a disk, respectively.

- In a categorical language, we thus arrive at the following classification result for two-dimensional topological field theories: the topological field theories are described by the category $[\mathrm{Cob}(2), \mathrm{vect}(\mathbb{K})]_{\text{symm. monoidal}}$ of symmetric monoidal functors. This category is equivalent to the category of commutative $\mathbb{K}$-Frobenius algebras.

- As a general reference for this example, we refer to the book [Kock].

The structure of three-dimensional topological field theories is much more involved. Here, the notion of a monoidal category, e.g. the representation category of a Hopf algebra, will enter, as we will see later.

Before turning to this, we consider a generalization and a specific construction:

Example 4.1.13 (Open/closed topological field Theories in dimension 2)**.**

- We define a larger category $\mathrm{Cob}(2)^{o/cl}$ of open-closed cobordisms:
  - Objects are compact oriented 1-manifolds which are allowed to have boundaries. These are disjoint unions of oriented intervals and oriented circles.
  - As a bordism $B : M \to N$, we consider a smooth oriented two-dimensional manifold B, together with an orientation preserving smooth map
  $$\phi_B : \quad \overline{M} \coprod N \to \partial B$$
  which is a diffeomorphism to its image. The map is not required to be surjective. In particular, we have parametrized and unparametrized intervals on the boundary circles of M. The unparametrized intervals are also called <u>free boundaries</u> and constitute physical boundaries of two-manifolds. The other boundaries are cut-and-paste boundaries and implement (aspects of) locality of the topological field theory.
  Two bordisms B, B' give the same morphism, if there is an orientation-preserving diffeomorphism $\phi : B \to B'$ such that the following diagram commutes:
  $$\begin{tikzcd} B \arrow[rr, "\phi"] & & B' \\ & \overline{M} \coprod N \arrow[ul, "\phi_B"] \arrow[ur, "\phi'_B"'] & \end{tikzcd}$$
  Thus the diffeomorphism respects parametrizations of intervals on boundary circles and parametrizations of whole boundary circles.
  - For any object M, the identity morphism id_M is represented by the cylinder over M.
  - Composition is again by gluing.

- Again, disjoint union endows $\mathrm{Cob}(2)^{o/cl}$ with the structure of a symmetric monodial category with the empty set as the tensor unit.

- An <u>open-closed TFT</u> is defined as a symmetric monoidal functor
$$Z : \mathrm{Cob}(2)^{o/cl} \to \mathrm{vect}(\mathbb{K}) \; .$$

Again $C := Z(S^1)$ is a commutative Frobenius algebra. One can again write generators and relations for the cobordism category and finds that the image $O := Z(I)$ of the interval I carries the structure of a Frobenius algebra. C is called the bulk Frobenius algebra, O the boundary Frobenius algebra.

- The Frobenius algebra O is not necessarily commutative: given three disjoint intervals on the boundary of a disk, two of them cannot be exchanged by a diffeomorphism of the disc. This situation is thus rather different from three boundary circles in a sphere, where two of them can be continuously commuted. For this reason, the bulk Frobenius algebra C is commutative.

  Still, the boundary Frobenius algebra is symmetric, i.e. the bilinear form $\kappa_0 : O \otimes O \to \mathbb{K}$ is symmetric: $\kappa_O(a, b) = \kappa_O(b, a)$, as is shown graphically by cyclically exchanging two parametrized intervals on the boundary of a disc.

- The zipper gives a linear map $i_* : C \to O$ and the cozipper $i^* : O \to C$. We show graphically that

$$\begin{array}{rrcl} (1) & \mu_O \circ (i_* \otimes i_*) & = & i_* \circ \mu_C \\ (2) & (i^* \otimes i^*) \circ \Delta_O & = & \Delta_C \circ i^* \\ (3) & i_*(1_C) & = & 1_O \\ (4) & \epsilon_C \circ i^* & = & \epsilon_O \end{array}$$

  We summarize the relations: $i_* : C \to O$ is a unital algebra morphism. $i_* : O \to C$ is a counital morphism of coalgebras.

- One next shows that the image $i_*(C)$ is in the center $Z(O)$. Moreover, i_* and i^* are adjoints with respect to the Frobenius forms:

$$(6) \qquad \kappa_C(i^*\psi, \phi) = \kappa_O(\psi, i_*\phi) \quad \text{for all} \quad \psi \in O, \phi \in C\ .$$

- Finally we define graphically, using only structure in O, a map

$$\pi : O \to Z(O)\ .$$

  We then get a last relation, called the <u>Cardy relation</u>:

$$\pi = i_* \circ i^*\ .$$

- We are thus lead to the definition of a <u>knowledgeable Frobenius algebra</u>: A knowledgeable Frobenius algebra in a symmetric tensor category $\mathcal{D}$ consists of a commutative Frobenius algebra C in $\mathcal{D}$, a not necessarily commutative Frobenius algebra O in $\mathcal{D}$, a unital morphism of algebras

$$i_* : \quad C \to Z(O)$$

  such that $\pi = i_* \circ i^*$ with i^* the adjoint of i_* with respect to the Frobenius forms and π defined as before.

  Morphisms of knowledgeable Frobenius algebras are pairs of morphisms of Frobenius algebras, compatible with i_* and i^*. We thus get a category $Frob^{\mathrm{o/cl}}(\mathcal{D})$ of knowledgeable Frobenius algebras and an equivalence of categories

$$[\mathrm{Cob}(2)^{\mathrm{o/cl}}, \mathcal{D}]_{\text{symm. monodial}} = Frob^{\mathrm{o/cl}}(\mathcal{D})$$

  which classifies open/closed two-dimensional topological field theories.

- Given the bulk Frobenius algebra C, the boundary Frobenius algebra O is not uniquely determined. Rather, each choice of boundary Frobenius algebra determines a boundary condition for the two-dimensional closed topological field theory based on C. The category of all such boundary conditions carries a natural structure of an *algebroid*.

- As a general reference for this example, we refer to the paper [LP].

Example 4.1.14 (Topological field theories in dimension 2 from triangulations).

- We present a specific construction of a two-dimensional topological field theory. We construct it as a monoidal functor

$$Z^{tr}: \quad \mathrm{Cob}^{tr}(2) \to \mathrm{vect}(\mathbb{K})$$

on a category of cobordisms with polytope decomposition. We describe the domain category:

  - Objects are oriented closed one-dimensional manifolds with a polytope decomposition. Thus objects are disjoint unions of circles with a finite number n of marked points. Examples of objects are thus oriented standard circles $S^1 \subset \mathbb{C}$ with n marked points at roots of unity and inherited orientations for the intervals.
  - A morphism $M \stackrel{[B]}{\to} N$ is represented by a two-dimensional, oriented manifold B with a polytope decomposition. B is a manifold with boundary, together with orientation and triangulation preserving smooth maps

$$\phi_B: \quad \overline{M} \times [0, \epsilon) \coprod N \times [0, \epsilon) \to \partial B$$

    that parametrize a small neighborhood of the boundary respecting triangulations.

    Two-dimensional triangulated manifolds manifolds B and B' that differ by an equivalence which preserves the boundary triangulation, but not necessarily the triangulation in the interior are identified.
  - Since we can find triangulated cylinders relating the circles S^1_n and S^1_m for all $n, m \in \mathbb{Z}$, which are mutually inverses, we still have one isomorphism class of connected 1-manifolds.

There is an obvious monoidal functor

$$\mathrm{Cob}^{tr}(2) \to \mathrm{Cob}(2)$$

which forgets the triangulation. It turns out that this functor is full and essentially surjective: any one-dimensional and two-dimensional smooth manifold admits a polytope decomposition. The equivalence relation we impose on two-manifolds is strong enough to ensure that the functor is also faithful and thus an equivalence of categories.

- We now construct a two-dimensional topological field theory that is even more local than our previous construction: we start by associating vector spaces to the objects S^1_n. Our first input datum is thus

  - A $\mathbb{K}$-vector space V. We assume, for simplicity, from the very beginning that this vector space is finite-dimensional.

To an n-gone Δ_n with unoriented edges, we assign the vector space $H(\Delta_n) := V^{\otimes n}$. This does not depend on a starting point up to canonical isomorphisms of vector spaces. In this way, we get a finite-dimensional vector space $H(\Delta)$ for any n-gone. In a physical language, we associate degrees of freedom to edges, which could be e.g. achieved in a gauge theory by quantizing gauge field giving parallel transport.

Note, however, that $\tilde{Z}^{tr}(S^1_n) := H(\Delta_n)$ cannot be the value of a topological field theory on the isomorphism class of S^1, represented by the object S^1_n, since its value for isomorphic objects differs.

- We next have to construct linear maps for cobordisms with polytope decomposition. The cobordism is an oriented two-manifold, and all polygones in the triangulation inherit an orientation from this. The edges are unoriented.

  To associate a linear map $\tilde{Z}^{tr}(B)$ to a morphism represented by a triangulated surface $B : M_1 \to M_2$, we need to construct a vector in $(\tilde{Z}(M_1)^{tr})^* \otimes \tilde{Z}^{tr}(M_2)$.

  We keep the idea that attach to an n-gone Δ in the triangulation the vector space $H(\Delta)$. Suppose that the triangulation is such that we have n_i inner edges, n_1 ingoing edges in M_1 and n_2 outgoing edges in M_2 (which we assume all for the moment to be all positively oriented). Since inner edges appear for two polygones, with opposite relative orientation, we get from the triangulation a vector space isomorphic to

  $$H(B) := (V^{n_1})^* \otimes V^{n_2} \otimes V^{n_i} \otimes V^{n_i} \ .$$

  This vector space is huge and depends on the polytope decomposition.

- Let us first consider the case with empty boundary, $M = N = \emptyset$. Then we need to produce a number. To simplify the situation, assume that all polygones are triangles. As an additional datum, we fix

  - A tensor $t \in V \otimes V \otimes V$ that is invariant under the natural action of the group $\mathbb{Z}_3$ of cyclic permutations on $V^{\otimes 3}$.

  This naturally gives a vector in $H(B)$. To get a number from this vector, we need to get a linear form on this vector space. To construct it, we fix an additional datum:

  - A non-degenerate symmetric bilinear pairing $\mu : V \otimes V \to \mathbb{K}$.

  Each edge gives rise to two factors of V in $H(B)$; we apply μ to these two copies, and tensor over all edges. This gives the desired number.

- The prescription directly generalizes to the case with boundaries. In this case, the copies of V associated to the external edges are not paired and not affected by the contration. Therefore, we get a vector in $(H(M_1)^{tr})^* \otimes H(M_2)$.

- Since $Z^{tr}(B)$ has to be independent on inner triangulations, we have to impose conditions on t and μ. We present the conditions graphically. Any choice of (V, μ, t) satisfying this condition gives the algebraic structure of a (non-unital, non-counital) Frobenius algebra on V.

- We still do not have a topological field theory Z^{tr} since in such a theory, the cylinder has to act as the identity. One checks that one gets an idempotent on $H(S^1_1)$. One then gets a unit and a new multiplication. This multiplication is commutative, and one gets a commutative

Frobenius algebra, the center of the non-unital Frobenius algebra on V, in agreement with the general result 4.1.12. Notice that one cannot expect to get all two-dimensional topological field theories in this way, but only those having particularly strong locality properties: for example, the Frobenius form has to be the trace of the multiplication morphism. One has an extended topological field theory. It can be formalized in terms of higher categories. This is an active field of research, compare [L].

- References for this example include:
  - M. Fukuma, S. Hosono and H. Kawai: *Lattice topological field theory in two-dimensions.* Commun. Math. Phys. 161 (1994) 157-176
    `http://arxiv.org/abs/hep-th/9212154`
  - R. Lawrence: *An Introduction to Topological Field Theory.* Proc. Symp. Appl. Math. 51 (1996) 89-128.
    `http://www.ma.huji.ac.il/~ruthel/papers/amspub.pdf`

4.2 Braidings and quasi-triangular bialgebras

We note that in a braided category, a version of the Yang-Baxter equation holds:

Proposition 4.2.1.
Let U, V, W be objects in a strict braided tensor category. Then the following identity of morphisms $U \otimes V \otimes W \to W \otimes V \otimes U$ holds:

$$\begin{aligned}(c_{V,W} \otimes \mathrm{id}_U) \circ (\mathrm{id}_V \otimes c_{U,W}) \circ (c_{U,V} \otimes \mathrm{id}_W) \\ = (\mathrm{id}_W \otimes c_{U,V}) \circ (c_{U,W} \otimes \mathrm{id}_V) \circ (\mathrm{id}_U \otimes c_{V,W})\,.\end{aligned}$$

If the braided category is not strict, this amounts to a commuting diagram with 12 corners, a dodecagon. The reader should draw the graphical representation of this identity.

In particular $c_{V,V} \in \mathrm{Aut}\,(V \otimes V)$ is a solution of the Yang-Baxter equation. Thus any object V of a braided tensor category provides a group homomorphism $B_n \to \mathrm{Aut}(V^{\otimes n})$.

Proof.
The equality is a direct consequence of the hexagon axiom from definition 4.1.1 and the functoriality of the braiding:

$$\begin{array}{cll} & (c_{V,W} \otimes \mathrm{id}_U) \circ (\mathrm{id}_V \otimes c_{U,W}) \circ (c_{U,V} \otimes \mathrm{id}_W) & \\ = & (c_{V,W} \otimes \mathrm{id}_U) \circ c_{U,V\otimes W} & \text{[Hexagon axiom]} \\ = & c_{U,W\otimes V} \circ (\mathrm{id}_U \otimes c_{V,W}) & \text{[Functoriality, applied to } c_{U,V\otimes W}\text{]} \\ = & (\mathrm{id}_W \otimes c_{U,V}) \circ (c_{U,W} \otimes \mathrm{id}_V) \circ (\mathrm{id}_U \otimes c_{V,W}) & \text{[Hexagon axiom]} \end{array}$$

□

It is an obvious question to ask what kind of structure on a Hopf algebra induces the structure of a braiding on its representation category.

Definition 4.2.2

1. *Let H be a bialgebra. The structure of a quasi-cocommutative bialgebra is the choice of an invertible element R in the algebra $H \otimes H$ such that for all $x \in H$*

$$\Delta^{\mathrm{copp}}(x) R = R \Delta(x)\,. \tag{9}$$

R is called a universal R-matrix. A quasi-cocommutative Hopf algebra is a Hopf algebra together with the choice of a universal R-matrix.

2. *A quasi-cocommutative bialgebra H is called* quasi-triangular*, if its universal R-matrix obeys the relations in $H^{\otimes 3}$*

$$\begin{aligned}(\Delta \otimes \mathrm{id}_H)(R) &= R_{13} \cdot R_{23} \qquad (10)\\ (\mathrm{id}_H \otimes \Delta)(R) &= R_{13} \cdot R_{12} \qquad (11)\end{aligned}$$

with

$$R_{12} := R \otimes 1, \quad R_{23} := 1 \otimes R \quad \text{and} \quad R_{13} := (\tau_{H,H} \otimes \mathrm{id}_H)(1 \otimes R) \ .$$

It is convenient to extend this notation, e.g. by $R_{21} := \tau_{H,H}(R) \in H \otimes H$, and, by some abuse of notation $R_{21} := \tau_{H,H}(R) \otimes 1 \in H \otimes H \otimes H$.

3. *A morphism $f : (H, R) \to (H', R')$ of quasi-triangular Hopf algebras is a morphism $f : H \to H'$ of Hopf algebras such that $R' = (f \otimes f)(R)$.*

Remarks 4.2.3.

1. In Sweedler notation $R = R_{(1)} \otimes R_{(2)}$, the relations read

$$\begin{aligned} x_{(2)}R_{(1)} \otimes x_{(1)}R_{(2)} &= R_{(1)}x_{(1)} \otimes R_{(2)}x_{(2)} \\ (R_{(1)})_{(1)} \otimes (R_{(1)})_{(2)} \otimes R_{(2)} &= R_{(1)} \otimes R_{(1)'} \otimes R_{(2)}R_{(2)'} \\ R_{(1)} \otimes (R_{(2)})_{(1)} \otimes (R_{(2)})_{(2)} &= R_{(1)}R_{(1)'} \otimes R_{(2)'} \otimes R_{(2)} \end{aligned}$$

2. Cocommutative bialgebras have a distinguished structure of a quasi-triangular Hopf algebra with R-matrix $R = 1 \otimes 1$. A quasi-triangular structure on a Hopf algebra can thus be seen as a weakening of cocommutativity.

3. To see a non-trivial quasi-triangular structure, consider the cocommmutative Hopf algebra $\mathbb{K}[\mathbb{Z}_2]$ with $\mathbb{K}$ a field of characteristic different from 2. Write $\mathbb{Z}_2$ multiplicative as $\{1, g\}$. Then
$$R := \frac{1}{2}(1 \otimes 1 + 1 \otimes g + g \otimes 1 - g \otimes g)$$
is a universal R-matrix. A one-parameter family of R-matrices for the four-dimensional Taft Hopf algebra from observation 2.6.1 can be found in [Kassel, p. 174].

4. There is no universally accepted definition for the term quantum group. I would prefer to use the term for quasi-triangular Hopf algebras. Some authors use it as a synonym for Hopf algebras, some for certain subclasses of quasi-triangular Hopf algebras.

Theorem 4.2.4.
Let A be a bialgebra. Then the tensor category $A-$mod is braided, if and only if A is quasi-triangular. Both structures are in one-to-one correspondence.

Proof.

- Let A be quasi-triangular with R-matrix R. For any pair U, V of left A-modules, we define a linear map

$$\begin{aligned} c^R_{U,V} : \quad U \otimes V &\to V \otimes U \\ u \otimes v &\mapsto \tau_{U,V}(R.(u \otimes v)) = R_{(2)}.v \otimes R_{(1)}.u \, . \end{aligned}$$

This is a morphism of A-modules: we have, for all $u \in U, v \in V$ and $h \in A$:

$$\begin{array}{rcll} c^R_{U,V}(h.u \otimes v) & = & R_{(2)}h_{(2)}.v \otimes R_{(1)}h_{(1)}.u & \\ & = & h_{(1)}R_{(2)}.v \otimes h_{(2)}R_{(1)}.u & \text{[equation (9)]} \\ & = & h.c^R_{U,V}(u \otimes v)\,. & \end{array}$$

with inverse

$$c^{-1}(w \otimes v) = \overline{R}_{(1)}v \otimes \overline{R}_{(2)}w\,.$$

where $R^{-1} = \overline{R}_{(1)} \otimes \overline{R}_{(2)}$ is the multiplicative inverse of R in the algebra $A \otimes A$. To check the first hexagon axiom, we compute for $u \in U$, $v \in V$ and $w \in W$:

$$\begin{array}{cll} & (\mathrm{id}_V \otimes c^R_{U,W}) \circ (c^R_{U,V} \otimes \mathrm{id}_W)(u \otimes v \otimes w) & \\ = & (\mathrm{id}_V \otimes c^R_{UW})(R_{(2)}v \otimes R_{(1)}u \otimes w) & \text{[Defn. of } c^R\text{]} \\ = & R_{(2)}v \otimes R_{(2)'}w \otimes R_{(1)'}R_{(1)}u & \text{[Defn. of } c^R\text{]} \\ = & (R_{(2)})_{(1)}v \otimes (R_{(2)})_{(2)}w \otimes R_{(1)}u & \text{[equation (11) for } R\text{-Matrix]} \\ = & c^R_{U,V\otimes W}(u \otimes (v \otimes w)) & \text{[Defn. of } c^R\text{]} \end{array}$$

The other hexagon follows in complete analogy from equation (10).

- Conversely, suppose that the category A-mod is endowed with a braiding. Consider the element

$$R := \tau_{A,A}(c_{A,A}(1_A \otimes 1_A)) \in A \otimes A\,.$$

We have to show that R contains all information on the braiding on the category. To this end, let V be an A-module; for any vector $v \in V$, consider the A-linear map which realizes the isomorphism $V \cong \mathrm{Hom}_A(A, V)$:

$$\begin{array}{rccc} \overline{v}: & A & \rightarrow & V \\ & a & \mapsto & av \end{array}$$

Now consider two A-modules V, W and two vectors $v \in V$ and $w \in W$. The naturality of the braiding c applied to the morphism $\overline{v} \otimes \overline{w}$ implies

$$c_{V,W} \circ (\overline{v} \otimes \overline{w}) = (\overline{w} \otimes \overline{v}) \circ c_{A,A}$$

and thus

$$\begin{array}{rcll} c_{V,W}(v \otimes w) & = & c_{V,W}(\overline{v} \otimes \overline{w}(1_A \otimes 1_A)) & \\ & = & (\overline{w} \otimes \overline{v})c_{A,A}(1_A \otimes 1_A) & \text{[naturality, see above]} \\ & = & \tau_{V,W}(\overline{v} \otimes \overline{w}(R)) & \\ & = & \tau_{V,W}R.(v \otimes w) & \end{array}$$

This shows that all the information on a braiding is contained in the element $R \in A \otimes A$.

We have to derive the three relations on an R-matrix from the properties of a braiding. We have for the action of any $x \in A$ on $c_{A,A}(1 \otimes 1) \in A \otimes A$:

$$x.c_{A,A}(1 \otimes 1) = x.\tau_{A,A}(R) = \Delta(x) \cdot \tau_{A,A}(R)$$

On the other hand, the braiding $c_{A,A}$ is A-linear. Thus this expression equals

$$c_{A,A}(x.1 \otimes 1) = \tau_{A,A}R \cdot (\Delta(x) \cdot 1 \otimes 1) = \tau_{A,A}R \cdot \Delta(x)\,.$$

Thus $\Delta(x) \cdot \tau_{A,A}(R) = \tau_{A,A}R \cdot \Delta(x)$ which amounts to

$$R\Delta(x) = \tau_{A,A}\Delta(x)\tau_{A,A}(R) = \Delta^{\mathrm{opp}}(x)R\,.$$

One can finally derive the two hexagon properties (10) and (11) of an R-matrix from the hexagon axioms for the braiding.

$\square$

Let A be a quasi-triangular Hopf algebra. We conclude from proposition 4.2.1 that for any A-module V, the automorphism

$$c^R_{V,V} : V \otimes V \to V \otimes V$$

is a solution of the Yang-Baxter equation. This explains the name *universal* R-matrix. We note some properties of this R-matrix.

Proposition 4.2.5.
Let (H, R) be a quasi-triangular bialgebra.

1. Then the universal R-matrix obeys the following equation in $H^{\otimes 3}$:

$$R_{12}R_{13}R_{23} = R_{23}R_{13}R_{12}$$

   and we have

$$(\epsilon \otimes \mathrm{id}_H)(R) = 1 = (\mathrm{id}_H \otimes \epsilon)(R) \ .$$

2. If, moreover, H has an invertible antipode, then

$$(S \otimes \mathrm{id}_H)(R) = R^{-1} = (\mathrm{id}_H \otimes S^{-1})(R)$$

   and

$$(S \otimes S)(R) = R \ .$$

Proof.

1. We calculate, using the defining properties of the R-matrix:

$$\begin{array}{rcll} R_{12}R_{13}R_{23} & = & R_{12}(\Delta \otimes \mathrm{id})(R) & \text{[equation (10)]} \\ & = & (\Delta^{\mathrm{opp}} \otimes \mathrm{id})(R) \cdot R_{12} & \text{[equation (9)]} \\ & = & (\tau_{H,H} \otimes \mathrm{id})(\Delta \otimes \mathrm{id})(R) \cdot R_{12} & \text{[Defn. of } \Delta^{\mathrm{opp}}] \\ & = & (\tau_{H,H} \otimes \mathrm{id})(R_{13}R_{23}) \cdot R_{12} & \text{[equation (10)]} \\ & = & R_{23}R_{13}R_{12} \ . & \end{array}$$

   We now calculate in $H^{\otimes 3}$:

$$\begin{array}{rcll} 1 \otimes R & = & (((1\epsilon \otimes \mathrm{id}) \circ \Delta) \otimes \mathrm{id})\,(R) & [(\epsilon \otimes \mathrm{id}) \circ \Delta = \mathrm{id}] \\ & = & (1\epsilon \otimes \mathrm{id} \otimes \mathrm{id})(R_{13} \cdot R_{23}) & \text{[equation (10)]} \\ & = & (1\epsilon \otimes \mathrm{id} \otimes \mathrm{id})(R_{13}) \cdot R_{23} & \\ & = & (\mathrm{id} \otimes 1\epsilon \otimes \mathrm{id})(R_{23}) \cdot R_{23} & \\ & = & 1 \otimes ((1\epsilon \otimes \mathrm{id})(R) \cdot R) & \end{array}$$

   Since R is invertible, we get $(\epsilon \otimes \mathrm{id})(R) = 1$. The other equality is derived in complete analogy from equation (11).

2. Using the definition of the antipode, we have for all $x \in H$

$$\mu \circ (S \otimes \mathrm{id})\Delta(x) = \epsilon(x)1 \,.$$

Tensoring with the identity on H and using the relation just derived, we find

$$(\mu \otimes \mathrm{id}) \circ (S \otimes \mathrm{id} \otimes \mathrm{id})(\Delta \otimes \mathrm{id})(R) = (1\epsilon \otimes \mathrm{id})R = 1 \otimes 1 \ .$$

Thus, using equation (10)

$$1 \otimes 1 = (\mu \otimes \mathrm{id})(S \otimes \mathrm{id} \otimes \mathrm{id})(R_{13}R_{23}) = (S \otimes \mathrm{id})(R) \cdot S(1)R \ .$$

Using $S(1) = 1$, we get

$$(S \otimes \mathrm{id})(R) = R^{-1} \ .$$

Recall the notation $R_{21} = \tau_{H,H}(R)$. We observe that equation (9) implies $\Delta(x)R_{21} = R_{21} \cdot \Delta^{\mathrm{opp}}(x)$. Thus, there is a quasi-triangular Hopf algebra $(H, \mu, \Delta^{\mathrm{opp}}, S^{-1}, R_{21})$. The corresponding relation for this quasi-triangular Hopf algebra reads

$$(S^{-1} \otimes \mathrm{id})(R_{21}) = R_{21}^{-1}$$

which amounts to

$$(\mathrm{id}_H \otimes S^{-1})(R) = R^{-1} \ .$$

Finally, we use the two equations just derived to find

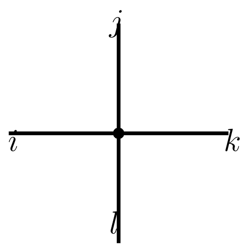
$$\begin{aligned}(S \otimes S)(R) &= (\mathrm{id} \otimes S)(S \otimes \mathrm{id})(R) \\ &= (\mathrm{id} \otimes S)(R^{-1}) \\ &= (\mathrm{id} \otimes S)(\mathrm{id}_H \otimes S^{-1})(R) = R\end{aligned}$$

□

4.3 Interlude: Yang-Baxter equations and integrable lattice models

Consider the following model: on the lattice points of the lattice $\mathbb{Z}^2 \subset \mathbb{R}^2$, we have "atoms". We are not interested in these atoms, but in their bonds to their nearest neighbour. We describe the state of a bond by a variable taking its values in the finite set $\{1, \ldots, n\}$.

Consider a vertex associated to an atom:



To such a vertex, we associate an energy $\epsilon_{ij}^{kl} \in \mathbb{R}$ which is allowed to depend on the type of bond i, j, k, l, but not on the vertex. We include the case that the energy depends on some external parameter $\epsilon_{ij}^{kl}(\lambda)$ which can be thought of as values of external magnetic or electric fields in some applications.

A lattice state is now a map that assigns to each bond a state:

$$\varphi : \quad \text{bonds} \quad \longrightarrow \{1, \ldots n\}$$

The energy of a state φ is obtained as the sum over atoms:

$$\epsilon_\lambda(\varphi) = \sum_{\text{atoms}} \epsilon_{\varphi(i)\varphi(l)}^{\varphi(j)\varphi(k)}(\lambda)\,.$$

To get a finite sum and thus well-defined expressions, we replace the lattice by a finite part with period boundary conditions, i.e. we consider vertices on $\mathbb{Z}_M \times \mathbb{Z}_N$. The partition function depends on an additional variable $\beta \in \mathbb{R}_+$, with the interpretation of an inverse temperature, $\beta = 1/kT$:

$$Z(\beta,\lambda) := \sum_{\text{states}} \mathrm{e}^{-\beta\epsilon_\lambda(\varphi)}\,.$$

The set of states is now the finite set $\{1,\dots,n\}^{N\cdot M}$. We endow it with the structure of a σ-algebra by the power set. We then get a family of probability measures with value

$$p_{\beta,\lambda}(\varphi) = \frac{1}{Z(\beta,\lambda)} e^{-\beta\epsilon_\lambda(\varphi)}$$

on the state φ. The random variables are then all measurable functions, i.e. all functions

$$Q : \{\text{set of states}\} \to \mathbb{R}\,.$$

The energy ϵ is one example. The expectation value of a random variable Q then is defined, as usual:

$$\mathbb{E}_{\beta,\lambda}[Q] = \frac{\sum\limits_{\text{states}} Q(\varphi) e^{-\beta\epsilon_\lambda(\varphi)}}{Z(\beta,\lambda)}\,.$$

For the special case of the energy, we have

$$\mathbb{E}_{\beta,\lambda}[\epsilon] = \frac{\sum\limits_{\text{states}} \epsilon_\lambda(\varphi) e^{-\beta\epsilon_\lambda(\varphi)}}{Z(\beta,\lambda)} = -\frac{\partial}{\partial\beta} \ln Z(\beta,\lambda)$$

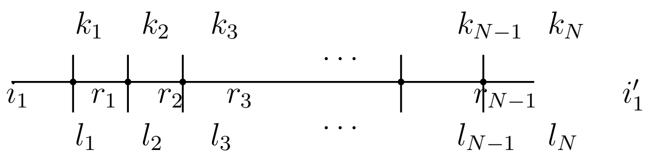
It is thus an important goal to compute the partition function. To this end, we introduce Boltzmann weights

$$R_{ij}^{kl}(\beta,\lambda) = \mathrm{e}^{-\beta\epsilon_{ij}^{kl}(\lambda)}\,.$$

We then get

$$\mathrm{e}^{-\beta\epsilon_\lambda(\varphi)} = \prod_{\text{atoms}} R_{ij}^{kl}\,.$$

Consider the contribution of the first line to the partition function where we temporarily allow different values for the leftmost and rightmost bond:



It equals

$$T_{i_1k_1\dots k_N}^{i_1'l_1\dots l_N} = \sum_{r_1\dots r_{N-1}} R_{i_1k_1}^{r_1l_1} R_{r_1k_2}^{r_2l_2} \dots R_{r_{N-1}k_N}^{i_1'l_N} \tag{12}$$

To eliminate indices, we introduce a complex vector space V freely generated on the set $\{1, \ldots, n\}$ with basis $\{v_1, \ldots v_n\}$ and a family of endomorphisms

$$R = R(\beta, \lambda): \quad \begin{array}{rcl} V \otimes V & \rightarrow & V \otimes V \\ v_i \otimes v_j & \mapsto & \sum_{k,l} R_{ij}^{kl} v_k \otimes v_l \,. \end{array}$$

Definition (12) leads to the definition of the endomorphism $T \in$ End $(V \otimes V^N)$ with

$$T = R_{01} R_{02} R_{03} \ldots R_{0n} \,.$$

Here we understand that the endomorphism R_{ij} acts on the i-th and j-th copy of V in the tensor product $V \otimes V^N$.

Periodic boundary conditions imply for the sum over the first line

$$\mathrm{Tr}_V(T)_{k_1 \ldots k_N}^{l_1 \ldots l_N} \,.$$

This endomorphism is called the row-to-row transfer matrix. To sum over all M lines, we take the matrix product and then the trace so that we find:

$$Z = tr_{V^{\otimes N}}(tr_V(T))^M \,.$$

This raises the problem of understanding the eigenvalues of the endomorphism $\mathrm{Tr}_V(T) \in \mathrm{End}(V^{\otimes N})$: in the thermodynamic limit, we take $M \rightarrow \infty$ so that $Z \sim \kappa_N^M$ with κ_N the eigenvalue with the largest modulus.

As usual in eigenvalue problems, we try to find as many endomorphisms of $V^{\otimes N}$ as possible commuting with $\mathrm{Tr}_V(T)$ which allows us to solve the eigenproblem separately on eigenspaces of these operators.

Definition 4.3.1
A vertex model with parameters λ is called integrable, if for any pair μ, ν of values for the parameters there is a value λ such that the equation

$$R_{12}(\lambda) R_{13}(\mu) R_{23}(\nu) = R_{23}(\nu) R_{13}(\mu) R_{12}(\lambda) \qquad \textit{(QYBE)}$$

holds in End $(V \otimes V \otimes V)$. *A specific case is the quantum Yang-Baxter equation with spectral parameters:*

$$R_{12}(\lambda - \mu) R_{13}(\lambda - \nu) R_{23}(\mu - \nu) = R_{23}(\mu - \nu) R_{13}(\lambda - \nu) R_{12}(\lambda - \mu)$$

Bialgebras are not enough to describe such a structure; Etingof and Varchenko have instead proposed algebroids.

Lemma 4.3.2.
Consider the tensor product $V \otimes V \otimes V^N$ and denote the index for the first copy of V by 0 and the index for the second copy of V by $\overline{0}$. Then the following equation holds in End $(V \otimes V \otimes V^N)$

$$R_{0\overline{0}}(\lambda) T_0(\mu) T_{\overline{0}}(\nu) = T_{\overline{0}}(\nu) T_0(\mu) R_{0\overline{0}}(\lambda) \,.$$

Proof.
We suppress spectral parameters and calculate:

$$\begin{array}{rcl} R_{0\overline{0}} T_0 T_{\overline{0}} & = & R_{0\overline{0}} R_{01} R_{02} \ldots R_{0N} R_{\overline{0}1} \ldots R_{\overline{0}N} \\ & = & R_{0\overline{0}} R_{01} R_{\overline{0}1} R_{02} \ldots R_{0N} R_{\overline{0}2} \ldots R_{\overline{0}N} \\ & \underset{\text{(QYBE)}}{=} & R_{\overline{0}1} R_{01} R_{0\overline{0}} R_{02} \ldots R_{0N} R_{\overline{0}2} \ldots R_{\overline{0}N} \end{array}$$

Here, we first used that the endomorphisms R_{0j} and $R_{\bar{0}1}$ for $j \geq 2$ act on different factors of the tensor product $V \otimes V \otimes V^N$ and thus commute. Then we applied the integrability equation (QYBE) on the indices $0, \bar{0}, 1$. Repeating this N-times, we get

$$\begin{aligned} &= R_{\bar{0}1} R_{\bar{0}2} \ldots R_{\bar{0}N} R_{01} \ldots R_{0N} R_{0\bar{0}} \\ &= T_{\bar{0}} T_0 R_{0\bar{0}} \end{aligned}$$

□

Proposition 4.3.3.
Suppose that for the integrable lattice model, the endomorphism $R(\lambda)$ is invertible for all values λ of the parameters. Then the endomorphism

$$C(\lambda) := \mathrm{Tr}_V T(\lambda) \in \text{ End } (V^{\otimes N})$$

commutes with $C(\mu)$ for all values λ, μ.

We thus have a set of commuting endomorphisms which make the eigenproblem for any operator $C(\lambda)$ more tractable, hence the name integrable.

Proof.
We take the trace $\mathrm{Tr}_{V_0 \otimes V_{\bar{0}}}$ over the relation in lemma 4.3.2 and use the cyclicity of the trace to get the following equation in End $(V^{\otimes N})$.

$$\begin{aligned} C(\mu) \cdot C(\nu) &= \mathrm{Tr}_{V_0 \otimes V_{\bar{0}}} T_0(\mu) T_{\bar{0}}(\nu) \\ &= \mathrm{Tr}_{V_0 \otimes V_{\bar{0}}} R_{0\bar{0}}(\lambda)^{-1} T_{\bar{0}}(\nu) T_0(\mu) R_{0\bar{0}}(\lambda) \\ &= C(\nu) C(\mu) \end{aligned}$$

□

Example 4.3.4.
We consider the case of two possible states for each bond. One represents the state of a bond by assigning to it a direction, denoted by an arrow. A famous model is then the XXX model or six vertex model. In this case, one assigns Boltzmann weight zero to all vertices, except for those with two ingoi[...] six configurations, hence the name of th[...]

XXX model

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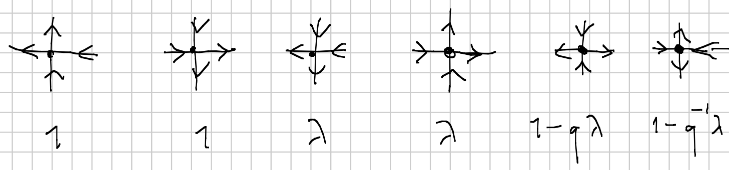


Since now V is two-dimensional, the R-matrix is a 4×4-matrix

$$R(q, \lambda) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \lambda & 1 - q\lambda & 0 \\ 0 & 1 - q^{-1}\lambda & \lambda & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

This model is integrable with the relation

$$\lambda - \mu + \nu + \lambda\mu\nu = (q + q^{-1})\lambda\nu \; .$$

4.4 The square of the antipode of a quasi-triangular Hopf algebra

We have seen in corollary 2.5.9 that for a cocommutative Hopf algebra, the square of the antipode obeys $S^2 = \mathrm{id}_H$. For quasi-triangular Hopf algebras, a weaker statement still holds. We assume throughout this chapter that the antipode S is invertible. This condition is automatically fulfilled by theorem 3.1.13.3 for finite-dimensional Hopf algebras. The reader might wish to check for which propositions the existence of a skew-antipode as in remark 2.5.8 is sufficient.

Proposition 4.4.1.
Let (H,R) be a quasi-triangular Hopf algebra. Then the element

$$u := S(R_{(2)}) \cdot R_{(1)} \in H$$

is invertible with inverse

$$u^{-1} := S^{-1}(\overline{R}_{(2)}) \cdot \overline{R}_{(1)} \ ,$$

where $R^{-1} = \overline{R}_{(1)} \otimes \overline{R}_{(2)}$. For all $h \in H$, we have

$$S^2(h) = uhu^{-1} = (Su)^{-1} h S(u)$$

so that the square of the antipode is an inner automorphism.

Proof.

- We first show that

$$(*) \qquad\qquad u \cdot h = S^2 h \cdot u \quad \text{ for all } h \in H \ .$$

By equation (9), we have

$$(R \otimes 1) \cdot (h_{(1)} \otimes h_{(2)} \otimes h_{(3)}) = (h_{(2)} \otimes h_{(1)} \otimes h_{(3)}) \cdot (R \otimes 1) \ ;$$

writing $R = a_i \otimes b_i$, this amounts to

$$a_i h_{(1)} \otimes b_i h_{(2)} \otimes h_{(3)} = h_{(2)} a_i \otimes h_{(1)} b_i \otimes h_{(3)} \ .$$

To this relation, we apply the linear map $\mathrm{id}_H \otimes S \otimes S^2$ and then multiply from right to left to get the equation

$$S^2(h_{(3)}) \cdot S(b_i h_{(2)}) \cdot a_i h_{(1)} = S^2(h_{(3)}) \cdot S(h_{(1)} b_i) h_{(2)} a_i \ .$$

The right hand side of this equation becomes

$$S^2(h_{(3)}) \cdot S(h_{(1)} b_i) h_{(2)} a_i = S^2(h_{(3)}) S(b_i) S(h_{(1)}) h_{(2)} a_i = S^2(h) S(b_i) a_i = S^2(h) \cdot u \ .$$

The left hand side becomes

$$S^2(h_{(3)}) \cdot S(b_i h_{(2)}) \cdot a_i h_{(1)} = S(h_{(2)} \cdot S h_{(3)}) S(b_i) a_i h_{(1)} = S(b_i) \cdot a_i \cdot h = u \cdot h \ .$$

This shows equation $(*)$.

- Next, we show that u is invertible. We write $R^{-1} := c_j \otimes d_j$ and set

$$v := \sum_j S^{-1}(d_j) \cdot c_j \equiv S^{-1}(d_j) \cdot c_j \ .$$

We calculate

$$\begin{aligned} uv &= uS^{-1}(d_j) c_j \stackrel{(*)}{=} S(d_j) u c_j \\ &= S(d_j) S(b_i) a_i c_j = S(b_i d_j) a_i c_j \end{aligned}$$

Now $a_i c_j \otimes b_i d_j = R \cdot R^{-1} = 1 \otimes 1$ so that $uv = 1$. Applying (*) to $h = v$, we find $S^2 v \cdot u = u \cdot v = 1$. Thus u is invertible and $S^2 h = uhu^{-1}$.

- Applying the anti-algebra morphism S to this relation yields

$$S^3 h = S(u^{-1})S(h)S(u) \ .$$

Replacing $S(h)$ by h, we find the second expression for S^2.

□

Definition 4.4.2
Let (H, R) be a quasi-triangular Hopf algebra. The element $u := S(R_{(2)}) \cdot R_{(1)} \in H$ is called the Drinfeld element of H.

Corollary 4.4.3.
We have $uS(u) = S(u)u$. This element is central in H.

Proof.
One directly sees that the element $Su \cdot u$ is central: $h \cdot S(u) \cdot u = S(u) \cdot u \cdot h$. For $h = u$, we get $u \cdot Su \cdot u = Su \cdot u \cdot u$ and, since u is invertible, the claim. □

We recall from proposition 2.5.13 that a finite-dimensional module V over a Hopf algebra with invertible antipode has a right dual $V^\vee$ defined on $V^* = \mathrm{Hom}_{\mathbb{K}}(V, \mathbb{K})$ with action $S(h)^t$ and a left dual ${}^\vee V$ defined on the same $\mathbb{K}$-vector space V^* with action by $S^{-1}(h)^t$.

Proposition 4.4.4.
Let (H, R) be a quasi-triangular Hopf algebra and u the Drinfeld element. Then we have, for any H-module V, an isomorphism of H-modules

$$\begin{array}{cccc} j_V : & V^\vee & \to & {}^\vee V \\ & \alpha & \mapsto & \alpha(u.-) \end{array}$$

where by $\alpha(u.-)$ we mean the linear form $v \mapsto \alpha(u.v)$ on V.

Proof.
The map j_V is bijective, because the Drinfeld element u is invertible. We have to show that it is a morphism of H-modules: we have for all $a \in H$ and $\alpha \in H^*, v \in V$

$$\begin{array}{rcl} \langle j_V(a.\alpha), v\rangle & = & \langle a.\alpha, u.v\rangle = \langle \alpha, S(a)u.v\rangle = \langle \alpha, S^2(S^{-1}(a))u.v\rangle \\ & \overset{(*)}{=} & \langle \alpha, uS^{-1}(a)v\rangle = \langle j_V(\alpha), S^{-1}(a).v\rangle = \langle a.j_V(\alpha), v\rangle \ . \end{array}$$

□

We comment on the relation to Radford's formula:

Remarks 4.4.5.

1. We have for the coproduct of the Drinfeld element $u = S(R_{(2)})R_{(1)}$

$$\Delta(u) = \overline{R}\,\overline{R}_{21}(u \otimes u) \ .$$

2. One can show that $g := u \cdot (Su)^{-1}$ is a group-like element of H and that

$$S^4(h) = ghg^{-1} \ .$$

For a proof, we refer to [Montgomery, p. 181].

Now denote by $a \in H$ and $\alpha \in H^*$ the distinguished group-like elements. Set

$$\tilde{\alpha} := (\alpha \otimes \mathrm{id}_H)(R) \in H \ .$$

Then $g = a^{-1}\tilde{\alpha} = \tilde{\alpha}a^{-1}$.

We now turn to a subclass of quasi-triangular Hopf algebras for which the braiding obeys additional constraints.

Definition 4.4.6
Let (H, R) be a quasi-triangular Hopf algebra.

1. *The invertible element*
$$Q := R_{21} \cdot R_{12} \in H \otimes H$$
is called the monodromy element. We write $Q = Q_{(1)} \otimes Q_{(2)}$ and note that
$$\Delta(h) \cdot Q = Q \cdot \Delta(h)$$
for all $h \in H$.

2. *The linear map*
$$\begin{array}{cccl} F_R : & H^* & \to & H \\ & \phi & \mapsto & (\mathrm{id}_H \otimes \phi)(R_{21} \cdot R_{12}) = (\mathrm{id}_H \otimes \phi)Q \end{array}$$
is called the Drinfeld map.

3. *A quasi-triangular Hopf algebra is called factorizable, if the Drinfeld map is an isomorphism of vector spaces.*

Remark 4.4.7.
The word factorizable is justified as follows: let $(b_i)_{i\in I}$ be a basis of H and $(b^i)_{i\in I}$ the dual basis of H^*. If H is factorizable, then the vectors $c_i := F_R(b^i)$ form another basis of H. We write

$$Q = \sum_{i,j} \lambda_{i,j} b_i \otimes c_j$$

with $\lambda_{i,j} \in \mathbb{K}$. We then have

$$c_k = F_R(b^k) = \sum_{i,j} \lambda_{i,j} b^k(b_i) \otimes c_j = \sum_{j\in I} \lambda_{k,j} c_j$$

and thus for the monodromy matrix

$$Q = \sum_{i\in I} b_i \otimes c_i \ ,$$

which explains the word factorizable.

We consider the following subspace of H^*:

$$C(H) := \{f \in H^* \mid f(xy) = f(yS^2(x)) \quad \text{for all } x, y \in H\}$$

We call this subspace the space of central forms or the space of class functions or the character algebra. We relate it to the center $Z(H)$ of H.

Lemma 4.4.8.
Let H be a finite-dimensional unimodular Hopf algebra with non-zero left cointegral $\lambda \in H^*$. Then by theorem 3.1.13 the map

$$\begin{array}{rcl} H & \to & H^* \\ a & \mapsto & \lambda(a \cdot -) = (\lambda \leftharpoonup a) \end{array}$$

is a bijection. It restricts to a bijection $Z(H) \cong C(H)$. In particular $\dim_{\mathbb{K}} Z(H) = \dim_{\mathbb{K}} C(H)$.

Proof.
If H is unimodular, we have $\alpha = \epsilon$ for the distinguished group-like element α. Then the Nakayama involution for the Frobenius structure given by the right cointegral reads by lemma 3.3.9

$$\rho(h) = \langle \alpha, h_{(1)} \rangle S^{-2}(h_{(2)}) = \langle \epsilon, h_{(1)} \rangle S^{-2}(h_{(2)}) = S^{-2}(h) \ .$$

For the left integral, one finds $\rho(h) = S^2(h)$ and thus

$$\lambda(a \cdot b) = \lambda(b \cdot S^2(a)) \ .$$

Thus

$$(\lambda \leftharpoonup a)(yS^2x) = \lambda(ayS^2x) = \lambda(xay) \ .$$

Thus $(\lambda \leftharpoonup a) \in C(H)$, if and only if for all $x \in H$, we have $\lambda(xa) = \lambda(ax)$. But this amounts to $a \in Z(H)$. □

Theorem 4.4.9 (Drinfeld)**.**
Let (H, R) be a quasi-triangular Hopf algebra with Drinfeld map $F_R : H^* \to H$. Then

1. For all $\beta \in C(H)$, we have $F_R(\beta) \in Z(H)$.
2. For all $\beta \in C(H)$ and $\alpha \in H^*$, we have

$$F_R(\alpha \cdot \beta) = F_R(\alpha) \cdot F_R(\beta) \ .$$

Proof.

- We calculate for $\beta \in C(h)$ and $h \in H$:

$$\begin{array}{rcll} h \cdot F_R(\beta) & = & h \cdot Q_{(1)} \beta(Q_{(2)}) & \\ & = & h_{(1)} Q_{(1)} \beta(S^{-1}(h_{(3)}) h_{(2)} Q_{(2)}) & \text{[Lemma 2.5.7]} \\ & = & Q_{(1)} h_{(1)} \beta(Q_{(2)} h_{(2)} S(h_{(3)})) & [Q\Delta = \Delta Q \text{ and } \beta \in C(H)] \\ & = & Q_{(1)} \beta(Q_{(2)}) \cdot h & \\ & = & F_R(\beta) \cdot h & \end{array}$$

- For the second statement, consider $\alpha \in H^*$ and $\beta \in C(H)$ and calculate

$$\begin{aligned}
F_R(\alpha\cdot\beta) &= R_2R_1'(\alpha\cdot\beta)(R_1R_2') \quad \text{[Defn. Drinfeld map]}\\
&= R_2R_1'(\alpha\otimes\beta)\Delta(R_1R_2') \quad \text{[Defn. product]}\\
&= R_2R_1'(\alpha\otimes\beta)\Delta(R_1)\cdot\Delta(R_2')\\
&= R_2r_2s_1t_1\alpha(R_1t_2)\beta(r_1s_2) \quad \text{[tensoriality of } R\text{, with } R=r=s=t]\\
&= R_2r_2s_1\beta(r_1s_2)t_1\alpha(R_1t_2)\\
&= R_2F_R(\beta)t_1\alpha(R_1t_2) \quad \text{[Defn. Drinfeld map]}\\
&= F_R(\alpha)\cdot F_R(\beta) \quad [F_R(\beta)\in Z(H)]
\end{aligned}$$

□

We will see below that any factorizable Hopf algebra is unimodular. From this fact we conclude

Corollary 4.4.10.
Let (H, R) be a factorizable Hopf algebra. Then the restriction of the Drinfeld map gives an algebra isomorphism

$$C(H) \xrightarrow{\cong} Z(H)\ .$$

Proof.
By theorem 4.4.9, the restriction of the Drinfeld map F to $C(H)$ is a morphism of algebras. It is injective, since the Drinfeld map is injective, due to the assumption that H is factorizable. Using the fact that H is unimodular, lemma 4.4.8 implies that this map is surjective and thus an isomorphism of algebras. □

We want to derive a few statements about semisimple factorizable Hopf algebras over an algebraically closed field $\mathbb{K}$ of characteristic zero. To this end, we refer to the following theorem that can be derived using character theory:

Theorem 4.4.11 (Kac-Zhu)**.**
Let H be a semisimple Hopf algebra and $e \in C(H)$ a primitive idempotent, i.e. there are no two non-zero elements e_1, e_2 such that $e_i^2 = e_i$, $e_1e_2 = e_2e_1 = 0$ and $e = e_1 + e_2$. Then $\dim_{\mathbb{K}} H^*e$ divides $\dim_{\mathbb{K}} H$.

We use it to show:

Proposition 4.4.12.
Let (H, R) be a semisimple factorizable Hopf algebra. If V is a simple left H-module, then $(\dim_K V)^2$ divides $\dim_{\mathbb{K}} H$.

Proof.
Since H is a semisimple algebra over an algebraically closed field, it is a direct sum of full matrix rings

$$H \cong M_{d_1}(\mathbb{K}) \oplus \ldots \oplus M_{d_t}(\mathbb{K})$$

and the center is

$$Z(H) \cong \mathbb{K} \oplus \ldots \oplus \mathbb{K}\ .$$

Denote by E_i the primitive idempotent in $Z(H)$ corresponding to the simple module V. Then the simple ideal $HE_i \cong M_{d_i}(\mathbb{K})$ has dimension $d_i^2 = (\dim_{\mathbb{K}} V)^2$.

Since semisimple Hopf algebras are unimodular by corollary 3.2.15, corollary 4.4.10 implies that the Drinfeld map F_R is an isomorphism of algebras $C(H) \xrightarrow{\cong} Z(H)$. Thus the element $e_i \in C(H)$ with $F_R(e_i) = E_i$ is a primitive idempotent of $C(H)$. By theorem 4.4.9

$$F_R(H^* e_i) = F_R(H^*)F_R(e_i) = HE_i \ .$$

Since F is bijective, we have

$$\dim_{\mathbb{K}}(H^* e_i) = \dim_{\mathbb{K}}(HE_i) = \dim_{\mathbb{K}}(V)^2 \ .$$

By the theorem of Kac-Zhu 4.4.11, therefore $\dim_{\mathbb{K}} V^2$ divides $\dim_{\mathbb{K}} H$. □

4.5 Yetter-Drinfeld modules

We now present an important class of examples of braided categories. It will lead us to a class of factorizable Hopf algebras of particular importance. Using this class, we will also be able to show that any factorizable Hopf algebra is unimodular. This class of Hopf algebras will also be naturally realized in the Turaev-Viro construction of three-dimensional topological field theories.

Some of the theory can be formulated for bialgebras. Whenever we deal with Hopf algebras, we will continue to assume that the antipode S is invertible (or that a skew antipode exists).

Definition 4.5.1

1. *Let H be a bialgebra. A* *Yetter-Drinfeld module* *is a triple (V, ρ_V, Δ_V) such that*

   - *(V, ρ_V) is a unital left H-module.*
   - *(V, Δ_V) is a counital left H-comodule:*

   $$\begin{array}{rccc} \Delta_V : & V & \to & H \otimes V \\ & v & \mapsto & v_{(-1)} \otimes v_{(0)} \end{array}$$

   - *The Yetter-Drinfeld condition*

   $$h_{(1)} \cdot v_{(-1)} \otimes h_{(2)}.v_{(0)} = (h_{(1)}.v)_{(-1)} \cdot h_{(2)} \otimes (h_{(1)}.v)_{(0)}$$

   holds for all $h \in H$.

2. *Morphisms of Yetter-Drinfeld modules are morphisms of left modules and left comodules. We write ${}^H_H\mathcal{Y}D$ for the category of Yetter-Drinfeld modules over the bialgebra H.*

Example 4.5.2.
Let us consider quite explicitly Yetter Drinfeld modules over the group algebra $\mathbb{K}[G]$ of a finite group G.

Since we have the structure of a comodule, any Yetter-Drinfeld module has a natural structure of a G-graded vector space,

$$V = \oplus_{g \in G} V_g \ ,$$

which is moreover endowed with a G-action. We evaluate the Yetter-Drinfeld condition for $g \in G$ and $v_h \in V_h$. We find for the left hand side using $\Delta(g) = g \otimes g$ and $\Delta_V(v_h) = h \otimes v_h$ that $gh \otimes g.v_h$. For the right hand side, we find

$$\sum_{x \in G} xg \otimes (g.v)_x \ .$$

The equality of the two terms implies that only the term with x such that $xg = gh$ contributes. Thus the Yetter-Drinfeld condition amounts to $g.v_h \in V_{ghg^{-1}}$. Thus the G-action has to cover for the grading the action of G on itself by conjugation.

We know that modules over a bialgebra form a tensor category, and so do comodules over a bialgebra. We can thus define as in proposition 2.4.7 and remark 2.4.8 on the tensor product of the vector spaces underlying two Yetter-Drinfeld modules V, W the structure of a module and of a comodule. We also note that the ground field with trivial action

$$\begin{array}{rcl} H \otimes \mathbb{K} & \to & \mathbb{K} \\ h \otimes \lambda & \mapsto & \epsilon(h)\lambda \end{array}$$

and trivial coaction

$$\begin{array}{rcl} \mathbb{K} & \to & H \otimes \mathbb{K} \\ \lambda & \mapsto & 1_H \otimes \lambda \end{array}$$

is trivially a Yetter-Drinfeld module and is a tensor unit for the tensor product.

Proposition 4.5.3.
Let H be a bialgebra. Then the category of Yetter-Drinfeld modules has a natural structure of a tensor category.

Proof.
Let V, W be Yetter-Drinfeld modules. We only have to show that the vector space $V \otimes W$ with action and coaction defined above obeys the Yetter-Drinfeld condition. This is done graphically. $\square$

Proposition 4.5.4.
Let H be a Hopf algebra. Then the category of Yetter-Drinfeld modules has a natural structure of a braided tensor category with the braiding of two Yetter-Drinfeld modules $V, W \in {}^H_H\mathcal{Y}D$ given by

$$\begin{array}{rcl} c_{V,W} : V \otimes W & \to & W \otimes V \\ v \otimes w & \mapsto & v_{(-1)}.w \otimes v_{(0)} \end{array}$$

Proof.
The following statements are shown graphically:

- The linear map $c_{V,W}$ is a morphism of modules and comodules and thus a morphism of Yetter-Drinfeld modules.
- The morphisms $c_{V,W}$ obey the hexagon axioms.
- Based on lemma 2.5.7, we show that the morphisms $c_{V,W}$ have inverses.

□

We define as in proposition 2.5.13 the right dual action of H on $V^* = \mathrm{Hom}_{\mathbb{K}}(V, \mathbb{K})$ as the pullback along S of the transpose of the action on V and the left dual action as the pullback of the transpose along S^{-1}. The right dual coaction maps $\beta \in V^*$ to the linear map $\Delta_V^\vee(\beta) \in H \otimes V^* \cong \mathrm{Hom}_{\mathbb{K}}(V, H)$

$$\begin{array}{rrcl} \Delta_V^\vee(\beta): & V & \mapsto & H \\ & v & \mapsto & S^{-1}(v_{(-1)})\beta(v_{(0)}) \end{array}$$

while the left dual coaction maps to

$$\begin{array}{rrcl} {}^\vee\Delta_V(\beta): & V & \mapsto & H \\ & v & \mapsto & S(v_{(-1)})\beta(v_{(0)}) \end{array}$$

Proposition 4.5.5.
Let H be a Hopf algebra. Then the category of *finite-dimensional* Yetter-Drinfeld modules ${}^H_H\mathcal{Y}D$ is rigid.

Proof.
We show the following statements by graphical calculations:

- The above definition indeed defines H-coactions on V^*.
- The coaction defined with S^{-1} has the property that the right evaluation and right coevaluation are morphisms of H-comodules. The statement that the coaction defined with S on the left dual is compatible with left evaluation and coevaluation follows in complete analogy.
- The left and right dual actions and coactions obey the Yetter-Drinfeld axiom.

□

This raises the question whether for any given Hopf algebra H the category ${}^H_H\mathcal{Y}D$ can be seen as the category of left modules over a quasi-triangular Hopf algebra $D(H)$.

Observation 4.5.6.

1. To investigate this in more detail, assume that the Hopf algebra H is finite-dimensional and recall from example 2.2.8.1 that a coaction of H then amounts to an action of H^*. Thus the quasi-triangular Hopf algebra $D(H)$ should account for an action of H and H^*. If the two actions would commute, $H^* \otimes H$ with the product structure for algebra and coalgebra would be an obvious candidate. This is not the case, and we have to encode the Yetter-Drinfeld condition in the product on $D(H)$.

2. It will be convenient to consider a slightly different category ${}_H\mathcal{Y}D^H$ of Yetter-Drinfeld modules: these are triples (V, ρ_V, Δ_V) such that

   - (V, ρ_V) is a unital left A-module.
   - (V, Δ_V) is a counital *right* A-comodule:

   $$\begin{array}{rrcl} \Delta_V: & V & \to & V \otimes H \\ & v & \mapsto & v_{(V)} \otimes v_{(H)} \end{array}$$

- The Yetter-Drinfeld condition holds in the form

$$h_{(1)}.v_{(V)} \otimes h_{(2)} \cdot v_{(H)} = (h_{(2)}.v)_{(V)} \otimes (h_{(2)}.v)_{(H)} \cdot h_{(1)} \; .$$

  Morphisms of Yetter-Drinfeld modules in ${}_H\mathcal{Y}D^H$ are morphisms of left modules and right comodules.

This observation leads to the following definition:

Definition 4.5.7
Let H be a finite-dimensional Hopf algebra. Endow the vector space $D(H) := H^* \otimes H$

- *with the structure of a counital coalgebra using the tensor product structure, i.e. for $f \in H^*$ and $a \in H$, we have*

$$\begin{array}{rcl} \epsilon(f \otimes a) & := & \epsilon(a) f(1) \\ \Delta(f \otimes a) & := & (f_{(1)} \otimes a_{(1)}) \otimes (f_{(2)} \otimes a_{(2)}) \end{array}$$

  This encodes the fact that the tensor product of Yetter-Drinfeld modules is the ordinary tensor product of modules and comodules.

- *Define an associative multiplication for $a, b \in H$ and $f, g \in H^*$ by*

$$(f \otimes a) \cdot (g \otimes b) := fg(S^{-1}(a_{(3)})?a_{(1)}) \otimes a_{(2)} b \; .$$

  The unit for this multiplication is $\epsilon \otimes 1 \in H^* \otimes H$.

A tedious, but direct calculation (see [Kassel, Chapter IX]) shows:

Proposition 4.5.8.
This defines a finite-dimensional Hopf algebra. Moreover, if (e_i) is any basis of H with dual basis (e^i) of H^*, then the element

$$R := \sum_i (1 \otimes e_i) \otimes (e^i \otimes 1) \in D(H) \otimes D(H) \; ,$$

which is independent of the choice of basis, is a universal R-matrix for $D(H)$.

Definition 4.5.9
We call the quasi-triangular Hopf algebra $(D(H), R)$ the Drinfeld double of the Hopf algebra H.

Remarks 4.5.10.

1. The Drinfeld double $D(H)$ contains H and H^* as Hopf subalgebras with embeddings

$$\begin{array}{rrcl} i_H : & H & \to & D(H) \\ & a & \mapsto & 1 \otimes a \end{array}$$

   and

$$\begin{array}{rrcl} i_{H^*} : & H^* & \to & D(H) \\ & f & \mapsto & f \otimes 1 \; . \end{array}$$

2. One checks that

$$\iota_{H^*}(f) \cdot \iota_H(a) = (f \otimes 1) \cdot (1 \otimes a) = f\epsilon(S^{-1}1_{(3)}?1_{(1)}) \otimes 1_{(2)} \cdot a = f \otimes a$$

and therefore writes $f \cdot a$ instead of $f \otimes a$. The multiplication on $D(H)$ is then determined by the straightening formula

$$a \cdot f = f(S^{-1}(a_{(3)}?a_{(1)}) \cdot a_{(2)} \ .$$

3. The Hopf algebra $D(H)$ is quasi-triangular, even if the Hopf algebra H does not admit an R-matrix. If (H,R) is already quasi-triangular, then one can show that the linear map

$$\begin{array}{rccl} \pi_R : & D(H) & \to & H \\ & fa & \mapsto & f(R_1)R_2 \cdot a \end{array}$$

is a morphism of Hopf algebras. The multiplicative inverse $\overline{R}$ of R gives a second projection $\pi_{\overline{R}} : D(H) \to H$. For more details, we refer to the article [S].

Proposition 4.5.11.
The Drinfeld double $D(H)$ of a finite-dimensional Hopf algebra H is factorizable.

Proof.
Writing in a short form

$$R = \sum_i (1 \otimes e_i) \otimes (e^i \otimes 1) = \sum_i e_i \otimes e^i$$

we find for the monodromy matrix of $D(H)$

$$Q = R_{21} \cdot R_{12} = \sum_{i,j} (e^i e_j) \otimes (e_i e^j) \ .$$

The family $e^i e_j = e^i \otimes e_j$ is a basis of $D(H) = H^* \otimes H$. Moreover,

$$S(e_i e^j) = S(e^j) \cdot S(e_i) = S(e^j) \otimes S(e_i)$$

and since S is invertible, also $e_i e^j$ is a basis of $D(H)$. Thus by remark 4.4.7 $D(H)$ is factorizable.
□

We can treat a left action of H^* for a right coaction of H by

$$\Delta_V : \quad V \xrightarrow{\mathrm{id}_V \otimes b_H} V \otimes H^* \otimes H \xrightarrow{\tau_{V,H^*} \otimes \mathrm{id}_H} H^* \otimes V \otimes H \xrightarrow{\rho \otimes \mathrm{id}_H} V \otimes H$$

and conversely,

$$\rho_V : \quad H^* \otimes V \xrightarrow{\mathrm{id}_{H^*} \otimes \Delta_V} H^* \otimes V \otimes H \xrightarrow{\mathrm{id}_{H^*} \otimes \tau_{V,H}} H^* \otimes H \otimes V \xrightarrow{d_H \otimes \mathrm{id}_V} V$$

Put differently, we have

$$f.v = \langle f, v_{(H)} \rangle v_{(V)} \ . \qquad (*)$$

The definition of the Drinfeld double $D(H)$ has been made in such a way that the following assertion holds:

Proposition 4.5.12.
Let H be a finite-dimensional Hopf algebra.

1. By treating the right H-coaction for a left H^*-action as above, any left $D(H)$-module becomes a Yetter-Drinfeld module in ${}_H\mathcal{Y}D^H$.

2. Conversely, any Yetter-Drinfeld module in ${}_H\mathcal{Y}D^H$ has a natural structure of a left module over the Drinfeld double $D(H)$.

Proof.
We note that the structure of a left $D(H)$-module on a vector space V consists of the structure of an H-module and of an H^*-module such that for all $f \in H^*$, $h \in H$ and $v \in V$ the following consequence of the straightening formula holds:

$$a.(f.v) = f(S^{-1}(a_{(3)}?a_{(1)}).\left(a_{(2)}.v\right) \ .$$

To show the second claim, we have to derive this relation from the Yetter-Drinfeld condition:

$$\begin{aligned}
f(S^{-1}(a_{(3)}?a_{(1)}).\left(a_{(2)}.v\right) &= \langle f, S^{-1}(a_{(3)})(a_{(2)}v)_{(H)}a_{(1)}\rangle(a_{(2)}v)_{(V)} \quad \text{[equation } (*)\text{]} \\
&= \langle f, S^{-1}(a_{(3)})a_{(2)}v_{(H)}\rangle a_{(1)}v_{(V)} \quad \text{[YD condition]} \\
&= \epsilon(a_{(2)})\langle f, v_{(H)}\rangle a_{(1)}v_{(V)} \quad \text{[lemma 2.5.7]} \\
&= \langle f, v_{(H)}\rangle a v_{(V)} = a.(f.v) \quad \text{[equation } (*)\text{]}
\end{aligned}$$

We leave the proof of the converse to the reader and refer for a more detailed account to [Kassel, Theorem IX.5.2], where Yetter-Drinfeld modules in ${}_H\mathcal{Y}D^H$ are called "crossed H-bimodules". □

Proposition 4.5.13.
Let H be a finite-dimensional Hopf algebra. Let $t \in H$ be a non-zero right integral and $T \in H^*$ a non-zero left integral. Then $T \otimes t$ is a left *and* right integral for $D(H)$. In particular, the Drinfeld double of a finite-dimensional Hopf algebra is unimodular.

Proof.
We use the following identity for the right integral $t \in H$

$$S^{-1}t_{(3)}a^{-1}t_{(1)} \otimes t_{(2)} = 1 \otimes t$$

where $a \in H$ is the distinguished group-like element. For the proof, we refer to [Montgomery, p. 192].

We then calculate for $f \in H^*$ and $h \in H$:

$$\begin{aligned}
(T \otimes t) \cdot (f \otimes h) &= Ttfh \\
&= Tf(S^{-1}t_{(3)}?t_{(1)}) \otimes t_{(2)} \cdot h \quad \text{[straightening formula]} \\
&= Tf(S^{-1}t_{(3)}a^{-1}t_{(1)}) \otimes t_{(2)} \cdot h \quad \text{[since } Tf = \langle f, a^{-1}\rangle T\text{]} \\
&= T\langle f, 1\rangle \otimes th \quad \text{[preceding identity]} \\
&= \langle f, 1\rangle \epsilon(h) T \otimes t
\end{aligned}$$

Thus $T \otimes t$ is a right integral for $D(H)$. The proof that it is a left integral for $D(H)$ is similar. □

Corollary 4.5.14.
Let H be a finite-dimensional Hopf algebra. Then the following assertions are equivalent:

1. $D(H)$ is semisimple.
2. H is semisimple.
3. H^* is semisimple.

Proof.
We have already used in the proof of theorem 3.3.19 that H is semisimple, if and only if H^* is semisimple. If both H and H^* are semisimple, then, by Maschke's theorem 3.2.13 $\epsilon(t) \neq 0$ and $\epsilon^*(T) \neq 0$. By proposition 4.5.13, this implies for $D(H)$ that $\epsilon(T \otimes t) \neq 0$ and thus by Maschke's theorem that $D(H)$ is semisimple. The converse follows by the same type of reasoning. □

We now use the Drinfeld double to derive a fundamental fact about the representation theory of finite-dimensional semisimple Hopf algebras.

Proposition 4.5.15.
Let $\mathbb{K}$ be an algebraically closed field of characteristic zero.

1. Let H be a semisimple Hopf algebra and V a simple $\mathcal{D}(H)$-module. Then $\dim_{\mathbb{K}} V$ divides $\dim_{\mathbb{K}} H$.
2. If (H, R) is a finite-dimensional semisimple quasi-triangular Hopf algebra, then for any simple left H-module V, $\dim_{\mathbb{K}} V$ divides $\dim_{\mathbb{K}} H$.

Proof.

1. By the theorem of Larson-Radford 3.3.19, semisimplicity of H amounts to $S^2 = \mathrm{id}_H$. This, in turn, implies that the antipode of the Drinfeld double $D(H)$ squares to the identity and thus, again by Larson-Radford 3.3.19 that $D(H)$ is semisimple. Now apply proposition 4.4.12 to the Hopf algebra $D(H)$ which is factorizable by proposition 4.5.11 to conclude that $(\dim_{\mathbb{K}} V)^2$ divides $\dim_{\mathbb{K}} D(H) = (\dim_{\mathbb{K}} H)^2$. Thus $\dim_{\mathbb{K}} V$ divides $\dim_{\mathbb{K}} H$.
2. Let now (H, R) be quasi-triangular. Since by remark 4.5.10.3 the Hopf algebra H is the epimorphic image of $D(H)$, any H-module can be pulled back to an $D(H)$-module. The pullback of a simple H-module is again simple.

□

We present an alternative point of view on the Drinfeld double of a *quasi-triangular* Hopf algebra.

Let H be a Hopf algebra with invertible antipode. For an invertible element $F \in H \otimes H$ consider the linear map

$$\begin{array}{rl} \Delta_F : & H \to H \otimes H \\ & \Delta_F(a) = F\Delta(a)F^{-1} \end{array}$$

This is obviously a morphism of algebras.

Lemma 4.5.16.

1. A sufficient condition for Δ_F to be coassociative is the identity

$$F_{12}(\Delta \otimes \mathrm{id}_H)(F) = F_{23}(\mathrm{id}_H \otimes \Delta)(F)$$

in $H^{\otimes 3}$.

2. A sufficient condition for ϵ to be a counit for Δ_F is the identity

$$(\mathrm{id} \otimes \epsilon)(F) = (\epsilon \otimes \mathrm{id})(F) = 1 \,.$$

3. Define

$$v := F_1 \cdot S(F_2) \quad \text{and} \quad v^{-1} := S(G_1)G_2$$

with $G = F^{-1}$ the multiplicative inverse in the algebra $H \otimes H$. Then S_F with $S_F(h) := v \cdot S(h) \cdot v^{-1}$ is an antipode for the coproduct Δ_F.

Proof.

1. To show coassociativity

$$(\Delta_F \otimes \mathrm{id}) \circ \Delta_F(a) = (\mathrm{id} \otimes \Delta_F) \circ \Delta_F(a) \ ,$$

we compute the two sides of this equation separately:

$$\begin{aligned} (\Delta_F \otimes \mathrm{id})\Delta_F(a) &= (\Delta_F \otimes \mathrm{id})(F\Delta(a)F^{-1}) \\ &= (\Delta_F \otimes \mathrm{id})(F) \cdot F_{12} \cdot (\Delta \otimes \mathrm{id})\Delta(a) \cdot F_{12}^{-1} \cdot (\Delta_F \otimes \mathrm{id})(F^{-1}) \end{aligned}$$

while for the right hand side, we find by an analogous computation

$$\begin{aligned} (\mathrm{id} \otimes \Delta_F)\Delta_F(a) &= (\mathrm{id} \otimes \Delta_F)(F\Delta(a)F^{-1}) \\ &= (\mathrm{id} \otimes \Delta_F)(F) \cdot F_{23} \cdot (\mathrm{id} \otimes \Delta)\Delta(a) \cdot F_{23}^{-1} \cdot (\mathrm{id} \otimes \Delta_F)(F^{-1}) \end{aligned}$$

A sufficient condition for coassociativity to hold is the identity

$$(\Delta_F \otimes \mathrm{id})(F) \cdot F_{12} = (\mathrm{id} \otimes \Delta_F)(F) \cdot F_{23}$$

which, by taking inverses in the algebra $H^{\otimes 3}$, implies

$$F_{12}^{-1}(\Delta \otimes \mathrm{id})(F^{-1}) = F_{23}^{-1}(\mathrm{id} \otimes \Delta_F)(F^{-1})$$

and which is equivalent to

$$F_{12}(\Delta \otimes \mathrm{id}_H)(F) = F_{23}(\mathrm{id}_H \otimes \Delta)(F) \ .$$

2. We leave the rest of the proofs to the reader.

□

Definition 4.5.17

Let H be a Hopf algebra with invertible antipode. An invertible element $F \in H \otimes H$ satisfying

$$\begin{aligned} F_{12}(\Delta \otimes \mathrm{id})(F) &= F_{23}(\mathrm{id} \otimes \Delta)(F) \\ (\mathrm{id} \otimes \epsilon)(F) &= (\epsilon \otimes \mathrm{id})(F) = 1 \end{aligned}$$

is called a 2-cocycle for H or a gauge transformation. We denote the twisted Hopf algebra with coproduct Δ_F, counit ϵ and antipode S_F by H^F.

Examples 4.5.18.

1. Let H be a finite-dimensional Hopf algebra with basis $\{e_i\}$ and dual basis $\{e^i\}$. Consider
$$\widetilde{H} := H^* \otimes H^{\mathrm{opp}}$$
and in $\widetilde{H} \otimes \widetilde{H}$ the basis-independent element
$$\widetilde{F} = \sum_{i=1}^{\dim H} (1_{H^*} \otimes e_i) \otimes (e^i \otimes 1_H)$$
A direct calculation shows that this element is a 2-cocycle for $\widetilde{H}$.

2. Let (H, R) be a finite-dimensional quasi-triangular Hopf algebra. Then
$$F_R := 1 \otimes R_2 \otimes R_1 \otimes 1$$
is a 2-cocycle for $H \otimes H$ with the tensor product Hopf algebra structure. For a proof, we refer to [S, Theorem 4.3].

The proof of the following theorem can be found in [S, Theorem 4.3]:

Theorem 4.5.19.
Let (H, R) be a finite-dimensional quasi-triangular Hopf algebra. The map
$$\begin{array}{rcl} \delta_R : D(H) & \to & (H \otimes H)^{F_R} \\ x & \mapsto & \pi_R(x_{(1)}) \otimes \pi_{\overline{R}}(x_{(2)}) \end{array}$$
with π_R and $\pi_{\overline{R}}$ as in remark 4.5.10.3 is a Hopf algebra morphism. It is bijective, if and only if the quasi-triangular Hopf algebra (H, R) is factorizable.

Corollary 4.5.20.
Let (H, R) be a factorizable Hopf algebra. Then H is unimodular.

Proof.
Let Λ be a left integral in H. Then $\Lambda \otimes \Lambda$ is a left integral in $(H \otimes H)^{F_R}$, since the algebra structure and the counit ϵ are not changed by the twist. Since H is factorizable, the Hopf algebra $(H \otimes H)^{F_R}$ is by theorem 4.5.19 isomorphic to $D(H)$. We have seen in proposition 4.5.13 that the Drinfeld double of any Hopf algebra is unimodular. Thus $\Lambda \otimes \Lambda$ is also a right integral of $(H \otimes H)^{F_R}$. Hence Λ is also a right integral of H. □

We finally present a more categorical point of view on the Drinfeld double. If we regard a monoidal category as a categorification of a ring, then this construction is the categorification of the construction of the center of the ring.

Observation 4.5.21.
Let $\mathcal{C}$ be a strict tensor category.

- We consider a category whose objects are pairs $(V, c_{-,V})$ consisting of an object V of $\mathcal{C}$ and a family of natural isomorphisms, called a <u>half-braiding</u> for V,
$$c_{X,V} : \quad X \otimes V \to V \otimes X$$
such that for all objects X, Y of $\mathcal{C}$ we have
$$c_{X \otimes Y,V} = (c_{X,V} \otimes \mathrm{id}_Y) \circ (\mathrm{id}_X \otimes c_{Y,V}) \; .$$

- A morphism $(V, c_{-,V}) \to (W, c_{-,W})$ is a morphism $f : V \to W$ such that for all objects X of $\mathcal{C}$ we have
$$(f \otimes \mathrm{id}_X) \circ c_{X,V} = c_{X,W} \circ (\mathrm{id}_X \otimes f) \ .$$
- Given two objects $V, W \in \mathcal{C}$, we define for any object X the morphism
$$c_{X,V\otimes W} : \quad X \otimes V \otimes W \to V \otimes W \otimes X$$
by
$$c_{X,V\otimes W} = (\mathrm{id}_V \otimes c_{X,W}) \circ (c_{X,V} \otimes \mathrm{id}_W) \ .$$
One shows that this forms a half-braiding for the object $V \otimes W$. We then set
$$(V, c_{-,V}) \otimes (W, c_{-,W}) := (V \otimes W, c_{-,V\otimes W}) \ .$$

Proposition 4.5.22.
This defines a strict monoidal category $Z(\mathcal{C})$, called the Drinfeld center of the category $\mathcal{C}$. It is braided with braiding given by the half-braiding

$$c_{V,W} : \quad (V, c_{-,V}) \otimes (W, c_{-,W}) \to (W, c_{-,W}) \otimes (V, c_{-,V}) \ .$$

The forgetful functor
$$\begin{array}{rrcl} F : & Z(\mathcal{C}) & \to & \mathcal{C} \\ & (V, c_{-,V}) & \mapsto & V \end{array}$$
is strict monoidal. (It is not, in general essentially surjective nor full, but faithful by the definition of morphisms of $Z(\mathcal{C})$.)

For the proof of this statement, we refer to the book [Kassel]. We now make contact to the double construction:

Theorem 4.5.23.
For any finite-dimensional Hopf algebra H, the braided tensor categories $Z(H-\mathrm{mod})$ and $D(H)-\mathrm{mod}$ are equivalent.

Proof.

- We indicate how to construct a functor
$$Z(H-\mathrm{mod}) \to \ {}_H\mathcal{Y}D^H \ .$$
To this end, we define on any object (V, c_{-V}) of the Drinfeld center $Z(H-\mathrm{mod})$ a right H-coaction. Consider
$$\begin{array}{rcl} \Delta_V : V & \to & V \otimes H \\ v & \mapsto & c_{H,V}(1_H \otimes v) \ . \end{array}$$
One checks that this defines a coassociative coaction.

Again the naturality of the braiding allows us to express the braiding in terms of the coaction Δ_V: consider for $x \in X$ the morphism $\overline{x} : H \to X$ with $\overline{x}(1) = x$, i.e. $\overline{x}(h) = h.x$. Then
$$\begin{aligned} c_{X,V}(x \otimes v) &= c_{X,V} \circ (\overline{x} \otimes \mathrm{id}_V)(1_H \otimes v) = (\mathrm{id}_V \otimes \overline{x}) \circ c_{H,V}(1_H \otimes v) \\ &= v_{(V)} \otimes v_{(H)} \cdot x = \Delta_V(v)(1_H \otimes x) \end{aligned}$$
which is exactly the braiding on the category ${}_H\mathcal{Y}D^H$.

- Next, we use the fact that the braiding is H-linear:

$$a.c_{X,V}(x \otimes v) = c_{X,V}(a.(x \otimes v))$$

for all $a \in H$ and $v \in V, x \in X$. Replacing the braiding $c_{X,V}$ by the expression just derived yields the equation

$$\Delta(a)\Delta_V(v)(1_H \otimes x) = \Delta_V(a_{(2)}v)(1_H \otimes a_{(1)})(1_H \otimes x) \ .$$

Setting $X = H$ and $x = 1_H$ yields

$$a_{(1)}.v_{(V)} \otimes a_{(2)} \cdot v_{(H)} = (a_{(2)}v)_V \otimes (a_{(2)}v)_{(H)} \cdot a_{(1)}$$

which is just the Yetter-Drinfeld condition in ${}_H\mathcal{Y}D^H$.

- For the the proof that this functor is essentially surjective and fully faithful, we refer to [Kassel].

□

We finally mention a categorical analogue of theorem 4.5.19.

Observation 4.5.24.

1. Suppose that A and B are two algebras over the same field $\mathbb{K}$. Then $A \otimes B$ is a $\mathbb{K}$-algebra as well. Deligne has shown how to construct the $\mathbb{K}$-linear category $A \otimes B$-mod from the two $\mathbb{K}$-linear categories A-mod and B-mod as the Deligne tensor product of categories such that we have
$$A \otimes B\text{-mod} \cong A\text{-mod} \boxtimes B\text{-mod} \ .$$
The Deligne product can be formed for any two $\mathbb{K}$-linear categories. For details, we refer to [D, section 5].

2. Let $\mathcal{C}$ be a braided tensor category. Using the braiding as a half-braiding gives a functor
$$\begin{array}{rcl} \mathcal{C} & \to & Z(\mathcal{C}) \\ V & \mapsto & (V, c_{-V}) \end{array}$$
which is obviously a braided monoidal functor.

3. Taking the inverse braiding
$$c^{\text{revd}}_{U,V} := c^{-1}_{V,U}$$
on the same monoidal category, gives another structure of braided tensor category $\mathcal{C}^{\text{revd}}$. We get another functor
$$\begin{array}{rcl} \mathcal{C}^{\text{revd}} & \to & Z(\mathcal{C}) \\ V & \mapsto & (V, c^{\text{revd}}_{-V}) \end{array}$$
which is again a braided monoidal functor.

4. Altogether, we obtain a braided monoidal functor
$$\mathcal{C}^{\text{revd}} \boxtimes \mathcal{C} \to Z(\mathcal{C}) \ .$$

5. Suppose that $\mathcal{C}$ is the category of representations of a quasi-triangular Hopf algebra (H, R). Then (H, R) is factorizable, if and only if the functor $\mathcal{C}^{\text{revd}} \boxtimes \mathcal{C} \to Z(\mathcal{C})$ is an equivalence of braided monoidal categories.

5 Topological field theories and quantum codes

5.1 Spherical Hopf algebras and spherical categories

We need to implement more structure on a Hopf algebra. Again, we assume that all Hopf algebras are finite-dimensional.

Definition 5.1.1

1. *A pivotal Hopf algebra is a pair (H, ω), where H is a Hopf algebra and $\omega \in G(H)$ is a group-like element, called the pivot such that*
$$S^2(x) = \omega x \omega^{-1} \ .$$
2. *A pivotal Hopf algebra is called spherical, if for all finite-dimensional representations V of H and all $\theta \in \mathrm{End}_H(V)$, we have*
$$\mathrm{Tr}_V \theta\omega = \mathrm{Tr}_V \theta\omega^{-1} \ .$$
In this case, the pivot ω is called a spherical element.

Remarks 5.1.2.

1. The pivot is not unique, but it is determined up to multiplication by an element in the group $G(H) \cap Z(H)$. The choice of a pivot is thus an additional structure on H. Note that we do not require that H has the structure of a quasi-triangular Hopf algebra, i.e. is endowed with an R-matrix.
2. Because of the theorem 3.3.19 of Larson-Radford, any finite-dimensional semisimple Hopf algebra H admits the element 1_H as a pivot.
3. For an example of a pivotal Hopf algebra that is not spherical, we refer to example 2.2 of [AAITC].

There is a corresponding categorical notion:

Definition 5.1.3

Let $\mathcal{C}$ be a right rigid monoidal category. A pivotal structure is a monoidal isomorphism
$$\omega : \quad \mathrm{id}_{\mathcal{C}} \to ?^{\vee\vee} \ .$$
A right rigid monoidal category together with a choice of pivotal structure is called a pivotal category.

Proposition 5.1.4.

Let $\mathcal{C}$ be a monoidal category that we assume to be strict. A pivotal category $\mathcal{C}$ is also left rigid, and left and right dualities strictly coincide:
$$V^\vee = {}^\vee V \quad \text{and} \quad f^\vee = {}^\vee f$$
for all objects V and morphisms f in $\mathcal{C}$. For this reason, we write
$$V^* = V^\vee = {}^\vee V \quad \text{and} \quad f^* = f^\vee = {}^\vee f$$
for dual objects and morphisms in a pivotal category.

Proof.
On objects, we put ${}^\vee V := V^\vee$. To define a left evaluation and a left coevaluation, we put

$$\tilde{b}_V : \quad \mathbb{I} \xrightarrow{b_{V^\vee}} V^\vee \otimes V^{\vee\vee} \xrightarrow{\mathrm{id}_{V^\vee} \otimes \omega_V^{-1}} V^\vee \otimes V$$

and

$$\tilde{d}_V : \quad V \otimes V^\vee \xrightarrow{\omega_V \otimes \mathrm{id}_{V^\vee}} V^{\vee\vee} \otimes V^\vee \xrightarrow{d_{V^\vee}} \mathbb{I} \; .$$

It is straightforward to show that these morphisms obey the axioms of a left duality. Since ω is natural, one has on morphisms ${}^\vee f = f^\vee$. □

Proposition 5.1.5.
Let H be a finite-dimensional Hopf algebra.

1. If $\omega \in G(H)$ is a pivot for H, then the action with ω endows the category $H{-}\mathrm{mod}_{fd}$ of finite-dimensional H-modules with a pivotal structure.

2. Conversely, if ω is a pivotal structure on the category $H{-}\mathrm{mod}_{fd}$, then $\omega_H(1_H)$ is a pivot for the Hopf algebra H.

Proof.

1. Assume that ω is a pivot. We know from proposition 2.5.13 that the category $H{-}\mathrm{mod}_{fd}$ is rigid. The right bidual of the H-module (V, ρ_V) is the H-module $(V, \rho_V \circ S^2)$. Use the pivot ω to define the linear isomorphism

$$\begin{array}{rccc} \omega_V : & V & \to & V \\ & v & \mapsto & \omega.v \; . \end{array}$$

   This is actually a morphism $V \to V^{\vee\vee}$ of H-modules:

$$a.\omega_V(v) = S^2(a) \cdot \omega.v = \omega \cdot a \cdot \omega^{-1} \omega.v = \omega \cdot a.v = \omega_V(a.v) \; .$$

   Implicitly assuming that the category of vector spaces has been replaced by an equivalent strict monoidal category, the natural transformation is monoidal by definition 2.4.6.3, if $\omega_V \otimes \omega_W = \omega_{V \otimes W}$. This holds, since $\omega.v \otimes \omega.w = \omega.(v \otimes w)$ for all $v \in V$ and $w \in W$, since ω is a grouplike element. For this reason, the natural transformation is invertible as well.

2. Suppose that $H{-}\mathrm{mod}_{fd}$ is a pivotal category. We canonically identify $H \cong H^{**}$ as a $\mathbb{K}$-vector space, on which $h \in H$ acts by $S^2(h)$. We consider the endomorphism

$$\omega_H : \quad H \to H^{**} \cong H \; .$$

   All right translations by $a \in A$

$$\begin{array}{rccc} R_a : & H & \to & H \\ & h & \mapsto & h \cdot a \end{array}$$

   are H-linear. The naturality of the pivotal structure ω thus implies $\omega(h \cdot a) = \omega_H(h) \cdot a$ for all $h, a \in H$. Since ω_H is a morphism of H-modules, we have

$$\omega_H(a \cdot b) = S^2(a) \omega_H(b) \; .$$

Altogether, we find

$$S^2(a)\omega_H(1) = \omega_H(a \cdot 1) = \omega_H(1 \cdot a) = \omega_H(1) \cdot a \ .$$

To show that $\omega := \omega_H(1)$ is a pivot for the Hopf algebra H, it remains to show that ω is grouplike. As in the proof of theorem 4.2.4, one shows that for $v \in V$, $\omega_V(v) = \omega_H(1).v = \omega.v$. Monoidality of the natural transformation now implies that ω is grouplike by reversing the arguments in the first part of the proof.

□

To understand the meaning of the notion of a spherical Hopf algebra, we need the notion of a trace which is also central for applications to topological field theory.

Lemma 5.1.6.
In any monoidal category, the monoid End $(\mathbb{I})$ is commutative.

Proof.
Identifying $\mathbb{I} \cong \mathbb{I} \otimes \mathbb{I}$, we see

$$\varphi \circ \varphi' = (\varphi \otimes \mathrm{id}_\mathbb{I}) \circ (\mathrm{id}_\mathbb{I} \otimes \varphi') = \varphi \otimes \varphi'$$

and

$$\varphi' \circ \varphi = (\mathrm{id}_\mathbb{I} \otimes \varphi') \circ (\varphi \otimes \mathrm{id}_\mathbb{I}) = \varphi \otimes \varphi' \ .$$

□

Definition 5.1.7
Let $\mathcal{C}$ be a pivotal category.

1. *Let X be an object of $\mathcal{C}$ and $f \in \mathrm{End}_\mathcal{C}(X)$. We define left and right* <u>*traces*</u>*:*

$$\begin{array}{rrcl} \mathrm{Tr}_l : & \mathrm{End}_\mathcal{C}(X) & \to & \mathrm{End}_\mathcal{C}(\mathbb{I}) \\ & f & \mapsto & d_V \circ (\mathrm{id}_{V^*} \otimes f) \circ \tilde{b}_V \\ \mathrm{Tr}_r : & \mathrm{End}_\mathcal{C}(X) & \to & \mathrm{End}_\mathcal{C}(\mathbb{I}) \\ & f & \mapsto & \tilde{d}_V \circ (f \otimes \mathrm{id}_{V^*}) \circ b_V \end{array}$$

2. *One also defines left and right* <u>*dimensions*</u>*:*

$$\dim_l X := \mathrm{Tr}_l \mathrm{id}_X \quad \text{and} \ \dim_r X := \mathrm{Tr}_r \mathrm{id}_X \ .$$

   Note that $\dim_l X \in \mathrm{End}_\mathcal{C}(\mathbb{I})$ and $\dim_r X \in \mathrm{End}_\mathcal{C}(\mathbb{I})$.

Some authors call this trace the <u>quantum trace</u> or the <u>categorical trace</u>.)

Lemma 5.1.8.
The two traces have the following properties:

1. The traces are symmetric: for any pair of morphisms $g : X \to Y$ and $f : Y \to X$ in $\mathcal{C}$, we have

$$\mathrm{Tr}_l(gf) = \mathrm{Tr}_l(fg) \quad \text{and} \quad \mathrm{Tr}_r(gf) = \mathrm{Tr}_r(fg) \ .$$

2. We have
$$\mathrm{Tr}_l(f) = \mathrm{Tr}_r(f^*) = \mathrm{Tr}_l(f^{**})$$
for any endomorphism f, and similar relations with left and right trace interchanged.

3. Suppose that
$$\alpha \otimes \mathrm{id}_X = \mathrm{id}_X \otimes \alpha \quad \text{for all} \quad \alpha \in \mathrm{End}_{\mathcal{C}}(\mathbb{I}) \text{ and all objects } X \in \mathcal{C} \ . \tag{13}$$
Then the traces are multiplicative for the tensor product:
$$\mathrm{Tr}_l(f \otimes g) = \mathrm{Tr}_l(f) \cdot \mathrm{Tr}_l(g) \quad \text{and} \quad \mathrm{Tr}_r(f \otimes g) = \mathrm{Tr}_r(f) \cdot \mathrm{Tr}_r(g)$$
for all endomorphisms f, g.

Remark 5.1.9.
Equation (13) always holds for $\mathbb{K}$-linear categories for which $\mathrm{End}_{\mathcal{C}}(\mathbb{I}) \cong \mathbb{K}\mathrm{id}_{\mathbb{I}}$ and thus in particular for categories of modules over Hopf algebras. It also holds for all braided pivotal categories, since $c_{X,\mathbb{I}} \cong c_{\mathbb{I},X} \cong \mathrm{id}_X$.

Proof.
We only show the assertions for one trace. The proof is best performed in graphical notation.
□

Corollary 5.1.10.
From the properties of the traces, we immediately deduce the following properties of the left and right dimensions:

1. Isomorphic objects have the same left and right dimension.
2. $\dim_l X = \dim_r X^* = \dim_l X^{**}$, and similarly with left and right dimension interchanged.
3. $\dim_l \mathbb{I} = \dim_r \mathbb{I} = \mathrm{id}_{\mathbb{I}}$.
4. Suppose, relation (13) holds. Then the dimensions are multiplicative:
$$\dim_l(X \otimes Y) = \dim_l X \cdot \dim_l Y \quad \text{and} \quad \dim_r(X \otimes Y) = \dim_r X \cdot \dim_r Y$$
for all objects X, Y of $\mathcal{C}$.
5. The dimension is additive for exact sequences: from
$$0 \to V' \to V \to V'' \to 0$$
we conclude $\dim V = \dim V' + \dim V''$.

Proof.
1. Choose $f : X \to Y$ and $g : Y \to X$ such that $\mathrm{id}_X = g \circ f$ and $\mathrm{id}_Y = f \circ g$. Then by the symmetry of the trace
$$\dim_l X = \mathrm{Tr}_l \mathrm{id}_X = \mathrm{Tr}_l g \circ f = \mathrm{Tr}_l f \circ g = \mathrm{Tr}_l \mathrm{id}_Y = \dim_l Y \ .$$

2. The axioms of a duality imply that $\mathrm{id}_X^* = \mathrm{id}_{X^*}$. Now the claim follows from the second identity of lemma 5.1.8.

3. Follows from the canonical identification $\mathbb{I} \cong \mathbb{I} \otimes \mathbb{I}^*$.

4. Follows from the identity $\mathrm{id}_{X\otimes Y} = \mathrm{id}_X \otimes \mathrm{id}_Y$ which is part of the definition of a tensor product.

5. We refer to [AAITC, 2.3.1].

□

Definition 5.1.11
A spherical category is a pivotal category whose left and right traces are equal,

$$\mathrm{Tr}_l(f) = \mathrm{Tr}_r(f)$$

for all endomorphisms f.

Remarks 5.1.12.

1. In a spherical category, we write $\mathrm{Tr}(f)$ and call it the trace of the endomorphism f.

2. In particular, left and right dimensions of all objects are equal, $\dim_l X = \dim_r X$. We call this element of $\mathrm{End}_{\mathcal{C}}(\mathbb{I})$ the dimension $\dim X$ of X.

3. For spherical categories, the graphical calculus has the following additional property: morphisms represented by diagrams are invariants under isotopies of the diagrams in the two-sphere $S^2 = \mathbb{R}^2 \cup \{\infty\}$. They are thus preserved under pushing arcs through the point ∞. Left and right traces are related by such an isotopy. This explains the name "spherical".

4. Suppose that $\mathcal{C} = H-$mod. Then the traces are given by

$$\mathrm{Tr}_l(f) = \mathrm{Tr}_V(f\rho_V(\omega^{-1})), \quad \mathrm{Tr}_r(f) = \mathrm{Tr}_V(f\rho_V(\omega)) \quad \text{for} \quad f \in \mathrm{End}_H(V).$$

   Thus, $H-$mod is a spherical category, whenever H is a spherical Hopf algebra. One can show [AAITC, Proposition 2.1] that it is sufficient to verify the trace condition on *simple* H-modules to show that a pivotal Hopf algebra is spherical.

We also consider analogous additional structure on *braided* tensor categories. Recall from remark 5.1.9 that for a braided category, the trace is always multiplicative.

Definition 5.1.13
Let $\mathcal{C}$ be a braided pivotal category.

1. *For any object X of $\mathcal{C}$, define the endomorphism*

$$\theta_X = (\mathrm{id}_X \otimes \tilde{d}_X) \circ (c_{X,X} \otimes \mathrm{id}_{X^*}) \circ (\mathrm{id}_X \otimes b_X) \ .$$

   This endomorphism is called the twist on the object X.

2. *A ribbon category is a braided pivotal category where all twists are selfdual, i.e.*

$$(\theta_X)^* = \theta_{X^*} \quad \textit{for all} \quad X \in \mathcal{C} \ .$$

Lemma 5.1.14.
Let $\mathcal{C}$ be a braided pivotal category.

1. The twist is invertible with inverse

$$\theta_X^{-1} = (d_X \otimes \mathrm{id}_X) \circ (\mathrm{id}_{X^*} \otimes c_{X,X}^{-1}) \circ (\tilde{b}_X \otimes \mathrm{id}_X) \ .$$

2. We have $\theta_{\mathbb{I}} = \mathrm{id}_{\mathbb{I}}$ and

$$\theta_{V \otimes W} = c_{W,V} \circ c_{V,W} \circ (\theta_V \otimes \theta_W) \, .$$

   The reader should draw the graphical representation of this relation.

3. The twist is natural: for all morphisms $f : X \to Y$, we have $f \circ \theta_X = \theta_Y \circ f$.

4. A braided pivotal category is a ribbon category, if and only if the identity

$$\theta_X = (d_X \otimes \mathrm{id}_X) \circ (\mathrm{id}_{X^*} \otimes c_{X,X}) \circ (\tilde{b}_X \otimes \mathrm{id}_X)$$

   holds.

Proof.

1.,2. This is best seen graphically.

3. Using properties of the duality, one shows for $f : U \to V$ that

$$\tilde{d}_V \circ (f \otimes \mathrm{id}_{V^*}) = \tilde{d}_U \circ (\mathrm{id}_U \otimes f^*)$$

   and similar relations for the other duality morphisms. The naturality of the twist now follows from these relations and the naturality of the braiding and its inverse.

4. Follows from a graphical calculation that is left to the reader.

□

Proposition 5.1.15.
A ribbon category $\mathcal{C}$ is spherical.

Proof.
To this end, one notes that

$$\begin{array}{rclcl} \overline{d}_V & = & d_V \circ c_{V,V^*} \circ (\theta_V^{-1} \otimes \mathrm{id}_{V^*}) & = & d_V \circ \circ c_{V,V^*} \circ (\mathrm{id}_V \otimes \theta_{V^*}^{-1}) \\ \overline{b}_V & = & (\theta_{V^*}^{-1} \otimes \mathrm{id}_V) \circ c_{V,V^*} \circ b_V & = & (\mathrm{id}_{V^*} \otimes \theta_V^{-1}) \circ c_{V,V^*} \circ b_V \end{array}$$

form another left duality. The proof of this assertion can be found in [Kassel, p. 351-353], with left and right duality interchanged as compared to our statement of the assertion. Since all (left) dualities are equivalent, the rest of the proof can now be easily performed graphically. □

We finally express these structures on the level of Hopf algebras.

Definition 5.1.16

A ribbon Hopf algebra is a quasi-triangular Hopf algebra (H,R) together with an invertible central element $\nu \in H$ such that

$$\Delta(\nu) = (R_{21} \cdot R)^{-1} \cdot (\nu \otimes \nu)\ ,\ \epsilon(\nu) = 1 \quad \text{and} \quad S(\nu) = \nu\,.$$

The element ν is called a ribbon element.

A ribbon element is not unique, but only determined up to multiplication by an element in $\{g \in G(H) \cap Z(H) \mid g^2 = 1\}$, see [AAITC, Definition 2.13].

Proposition 5.1.17.

1. Let (H,R,ν) be a ribbon Hopf algebra. For $V \in H{-}\mathrm{mod}_{fd}$, consider the endomorphism

$$\begin{array}{rcl} \theta_V : V & \to & V \\ v & \mapsto & \nu.v \end{array}$$

   This defines a twist that is compatible with the dualities.

2. Conversely, suppose that (H,R) is a quasi-triangular Hopf algebra and that there is an element $\nu \in H$ such that for any $V \in H{-}\mathrm{mod}_{fd}$ the endomorphism $\theta_V(v) := \nu.v$ is a twist on the braided category $H{-}\mathrm{mod}_{fd}$. Then ν is a ribbon element.

Proof.

- If ν is central and invertible, then all θ_V are H-linear isomorphisms. Conversely, for any algebra A any natural transformation $\theta : \mathrm{id}_{A-\mathrm{mod}} \to \mathrm{id}_{A-\mathrm{mod}}$ is given by the action of an element of the center $Z(A)$ of A, $\mathrm{End}(\mathrm{id}_{A-\mathrm{mod}}) = Z(A)$.

- We compute for $x \in V \otimes W$:

$$c_{W,V} c_{V,W} (\theta_V \otimes \theta_W)(x) = R_{21} R(\nu x_{(1)} \otimes \nu x_{(2)}) = \Delta(\nu) \cdot x = \theta_{V\otimes W}(x)\,.$$

  The compatibility of twist and braiding is thus equivalent to the property $R_{21}R(\nu \otimes \nu) = \Delta(\nu)$.

- It remains to show that

$$(\theta_{V^*} \otimes \mathrm{id}_V) b_V(1) = (\mathrm{id}_{V^*} \otimes \theta_V) b_V(1)\ .$$

  With $\{e_i\}$ a basis of V, this amounts to

$$\sum_i \nu e_i^* \otimes e_i = \sum_i e_i^* \otimes \nu e_i$$

  Evaluating this identity on any $v \in V$ yields

$$\begin{array}{rrcl} & \sum_i \nu e_i^*(v) \otimes e_i & = & \sum e_i^*(v) \otimes \nu e_i \\ \iff & S(\nu) \cdot v & = & \nu \cdot v \end{array}$$

  This shows that $S(\nu) = \nu$ is a sufficient condition. Applying this to $V = H$ and $v = 1$ shows that $S(\nu) = \nu$ is necessary as well.

□

Proposition 5.1.18.
Let V be any finite-dimensional module over a ribbon Hopf algebra H. Denote by u the Drinfeld element of H. Then $\nu^{-1}u$ is a spherical element and we have for the trace

$$\mathrm{Tr}(f) = \mathrm{Tr}_V \nu^{-1} u f \ .$$

In particular, the dimension is the trace over the action of the element $\nu^{-1}u$ on V.

Proof.
Using the definition of $\overline{d}_V$ from 5.1.15, we compute with $R = R_{(1)} \otimes R_{(2)}$

$$\overline{d}_V(v \otimes \alpha) = \langle R_{(2)}.\alpha, R_{(1)}\theta^{-1}v\rangle = \langle \alpha, S(R_{(2)} \cdot R_{(1)}\theta^{-1}v\rangle = \langle \alpha, u \cdot \nu^{-1}.v\rangle$$

Therefore

$$\mathrm{Tr} f = \overline{d}(f \otimes \mathrm{id}_{V^*})b_V = \sum_i \langle v^i, u \cdot \nu^{-1}.v_i\rangle$$

with v_i a basis of V. □

Remark 5.1.19.
One can show that the Drinfeld double of any finite-dimensional Hopf algebra is a ribbon algebra with the Drinfeld element as the ribbon element. Thus for doubles the categorical trace equals the trace in the usual sense of vector spaces. In particular, all categorical dimensions are non-negative integers.

5.2 Tanaka-Krein reconstruction

Let $\mathbb{K}$ be a field. In this subsection, we explain under what conditions a $\mathbb{K}$-linear ribbon category can be described as the category of modules over a $\mathbb{K}$-ribbon algebra. We assume that the field $\mathbb{K}$ is algebraically closed of characteristic zero and that all categories are essentially small, i.e. equivalent to a small category, a category in which the class of objects is a set.

Definition 5.2.1
Let $\mathcal{C}, \mathcal{D}$ be abelian tensor categories. A fibre functor is an exact faithful tensor functor $\Phi : \mathcal{C} \to \mathcal{D}$.

Examples 5.2.2.
1. Let H be a bialgebra over a field $\mathbb{K}$. The forgetful functor

$$\mathcal{F} : H{-}\mathrm{mod} \ \to \ \mathrm{vect}(\mathbb{K})\,,$$

   is a strict tensor functor. It is faithful, since by definition $\mathrm{Hom}_{H-\mathrm{mod}}(V, W) \subset \mathrm{Hom}_{\mathrm{vect}(\mathbb{K})}(V, W)$. It is exact, since the kernels and images in the categories $H{-}\mathrm{mod}$ and $\mathrm{vect}(\mathbb{K})$ are the same. Thus the forgetful functor $\mathcal{F}$ is a fibre functor.

2. There are tensor categories that do not admit a fibre functor to vector spaces.

We are looking for an inverse of the construction: give an a ribbon category, together with a fibre functor

$$\Phi : \quad \mathcal{C} \to \ \mathrm{vect}(\mathbb{K}) \ .$$

Can we find a ribbon Hopf algebra H such that $\mathcal{C} \cong H{-}\mathrm{mod}$ as a monoidal category?
We only sketch the proof of a slightly different result:

Theorem 5.2.3.
Let $\mathbb{K}$ be a field and $\mathcal{C}$ a $\mathbb{K}$-linear abelian tensor category and

$$\Phi : \mathcal{C} \to \mathrm{vect}_{fd}(\mathbb{K})$$

a fibre functor in the category of finite-dimensional $\mathbb{K}$-vector spaces. Then there is a $\mathbb{K}$-Hopf algebra H and an equivalence of tensor categories

$$\omega : \mathcal{C} \xrightarrow{\sim} \text{comod-}\ H\,,$$

such that the following diagram of monoidal functors commutes:

$$\begin{tikzcd} \mathcal{C} \arrow[dd, "\omega"'] \arrow[dr, "\Phi"] & \\ & \mathrm{vect}_{fd}(\mathbb{K}) \\ \text{comod}-H \arrow[ur, "\mathcal{F}"'] & \end{tikzcd}$$

where $\mathcal{F}$ is the forgetful functor.

Proof.
We only sketch the idea and refer for details to the book by Chari and Pressley.

- For any $\mathbb{K}$-vector space M, consider the functor

  $$\begin{array}{rrcl} \Phi \otimes M : & \mathcal{C} & \to & \mathrm{vect}_{fd}(\mathbb{K}) \\ & U & \mapsto & \Phi(U) \otimes_{\mathbb{K}} M \end{array} ,$$

  which is not monoidal, in general. We use these functors to construct a functor

  $$\begin{array}{rrcl} \mathrm{Nat}(\Phi, \Phi \otimes -) : & \mathrm{vect}_{fd}(\mathbb{K}) & \to & \mathrm{vect}_{fd}(\mathbb{K}) \\ & M & \mapsto & \mathrm{Nat}\ (\Phi, \Phi \otimes M) \end{array}$$

  which is representable: there is a vector space $H \in \mathrm{vect}_{fd}(\mathbb{K})$ and a natural isomorphism of functors $\tau : \mathrm{vect}_{fd}(\mathbb{K}) \to \mathrm{vect}_{fd}(\mathbb{K})$

  $$\mathrm{Hom}(H, -) \to \mathrm{Nat}\ (\Phi, \Phi \otimes -)$$

  i.e. natural isomorphisms of vector spaces

  $$\tau_M : \mathrm{Hom}_{\mathbb{K}}(H, M) \xrightarrow{\sim} \mathrm{Nat}\ (\Phi, \Phi \otimes M)\,.$$

- The natural identification

  $$e : \Phi \to \Phi \otimes \mathbb{K} \in \mathrm{Nat}(\Phi, \Phi \otimes \mathbb{K})$$

  gives a linear form $\epsilon := \tau_{\mathbb{K}}^{-1}(e) \in \ \mathrm{Hom}\ (H, \mathbb{K})$.

- Consider
$$\delta := \tau_H(\mathrm{id}_H) \in \ \mathrm{Nat}(\Phi, \Phi \otimes H)$$
which gives for any object U of $\mathcal{C}$ a natural $\mathbb{K}$-linear map
$$\delta_U : \Phi(U) \to \Phi(U) \otimes_{\mathbb{K}} H \ .$$
Consider the natural transformation
$$(\delta \otimes \mathrm{id}_H) \circ \delta : \quad \Phi \to \Phi \otimes H \to (\Phi \otimes H) \otimes H \cong \Phi \otimes (H \otimes H) \ .$$
Then define
$$\Delta := \tau^{-1}_{H \otimes H}\Big((\delta \otimes \mathrm{id}_H) \circ \delta\Big) \in \ \mathrm{Hom}(H, H \otimes H) \, .$$
One can show that ϵ and Δ endow the vector space H with the structure of a counital coassociative coalgebra.

- For the algebra structure on H, we use the monoidal structure on the functor Φ: consider
$$\begin{aligned} m_{U,V} \quad : \quad & \Phi(U) \otimes \Phi(V) \cong \Phi(U \otimes V) \xrightarrow{\delta_{U \otimes V}} \Phi(U \otimes V) \otimes H \\ & \cong \Phi(U) \otimes \Phi(V) \otimes H \end{aligned}$$
which is an element in
$$\mathrm{Nat}\ (\Phi^2, \Phi^2 \otimes H) \cong \ \mathrm{Hom}(H \otimes H, H) \ .$$
This gives an associative product with unit element
$$\eta : \mathbb{K} \cong \Phi(\mathbb{I}) \xrightarrow{\delta_{\mathbb{I}}} \Phi(\mathbb{I}) \otimes H \cong H \, .$$

- In a similar way, one uses the duality on $\mathcal{C}$ to obtain an antipode on H and shows that H becomes a Hopf algebra.

- One finally shows that $H-\mathrm{mod} \cong \mathcal{C}$ as monoidal categories.

□

Remark 5.2.4.
Deligne has characterized [D, Theorem 7.1] those $\mathbb{K}$-linear semisimple ribbon categories over a field $\mathbb{K}$ of characteristic zero that admit a fibre functor: these are those categories for which all objects have categorical dimensions that are non-negative integers.

5.3 Knots and links

Definition 5.3.1

1. *A link (German: Verschlingung) in $\mathbb{R}^3$ is a finite set of disjoint smoothly embedded circles (without parametrization and orientation).*

2. *A link with a single component is called a knot.*

3. *An isotopy of a link is a smooth deformation of $\mathbb{R}^3$ which does not induce intersections and self intersections.*

4. *A* <u>*framed link*</u> *is a link with a non-zero normal vector field.*

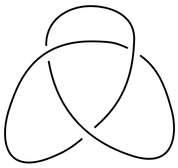
Links in the topological field theories of our interest are framed oriented links.

Examples 5.3.2.

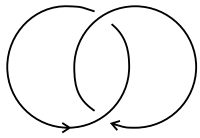
1. A special example is the so-called <u>unknot</u> which is given by the unit circle in the x-y-plane of $\mathbb{R}^3$.

2. Other important examples of well-known knots and links:

trefoil knot



Hopf link

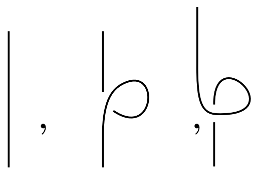


Remark 5.3.3.

1. If one projects a link $L \subset \mathbb{R}^3$ to the plane $\mathbb{R}^2$, we can represent the link by a link diagram. This is a set of circles in $\mathbb{R}^2$ with information about intersections which are, for a generic projection, only double transversal intersections.

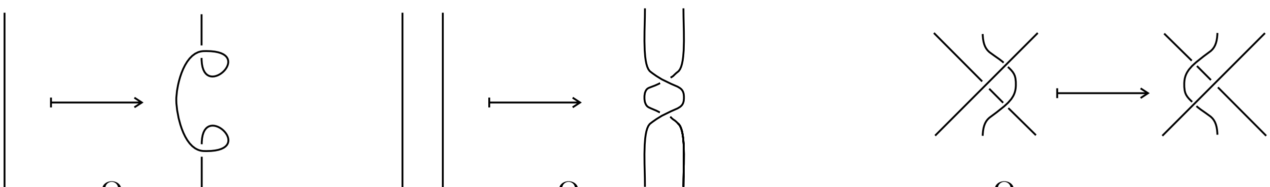
2. By taking the direction orthogonal to the plane containing the link diagram, we obtain a framing for the link represented by a link diagram. Thus any framed link can be represented by a link diagram.

3. Warning: if three knots differ locally by the following configurations,



then they are isotopic as knots, but not as framed knots.

4. Two link diagrams in $\mathbb{R}^2$ represent isotopic framed links in $\mathbb{R}^3$, if they are related by an isotopy of $\mathbb{R}^2$ or one of the <u>Reidemeister moves</u> $\Omega_0^{\pm 1}, \Omega_2^{\pm 1}, \Omega_3^{\pm 1}$



Ω_0 Ω_2 Ω_3

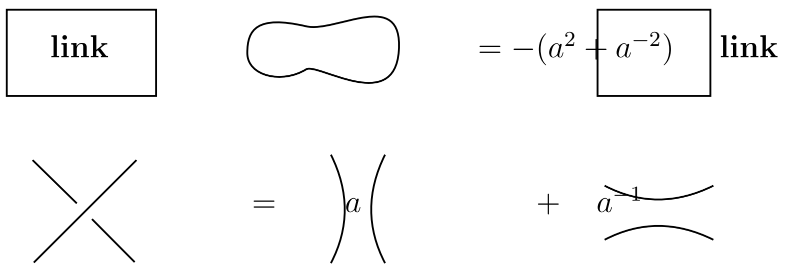
These moves are local, i.e. only affect a part of a link contained in a small disc.

5. We define the linking number $\mathrm{lk}(K, K')$ of two knots K, K' in a link as the sum of the signs ± 1 for each over and undercrossing. The matrix of linking numbers is a symmetric link invariant.

   For *framed* knots, one can define the self linking number: one deforms the knot along its normal vector field and defined the self linking number as the linking number of the original knot with its deformation.

We next present a famous link invariant.

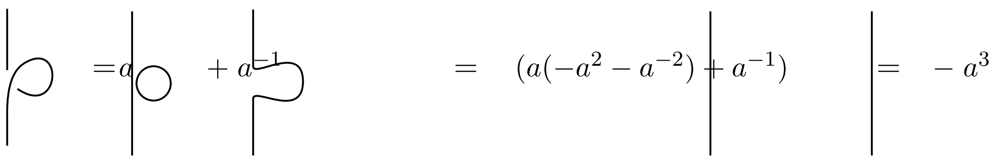
Definition 5.3.4
Fix $a \in \mathbb{C}^\times$. Let $E(a)$ be the complex vector space, freely generated by link diagrams up to isotopy of $\mathbb{R}^2$ modulo the two Kauffman relations:



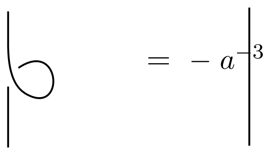
The vector space $E(a)$ is called the skein module. (skein is in German "Gebinde".) The class of a link diagram D determines a vector $\langle D \rangle(a) \in E(a)$.

Theorem 5.3.5.
1. The skein module is one-dimensional, $\dim_{\mathbb{C}} E(a) = 1$ and canonically identified with $\mathbb{C}$.
2. The skein class of a link is invariant under the Reidemester moves $\Omega_0^{\pm 1}, \Omega_2^{\pm 1}, \Omega_3^{\pm 1}$ and thus an isotopy invariant of links.

Proof.
1. The Kauffman relations are sufficient to unknot any knot. The unknot is the identified with the complex number $-a^2 - a^{-2}$.
2. To show invariance under Ω_0, we compute:

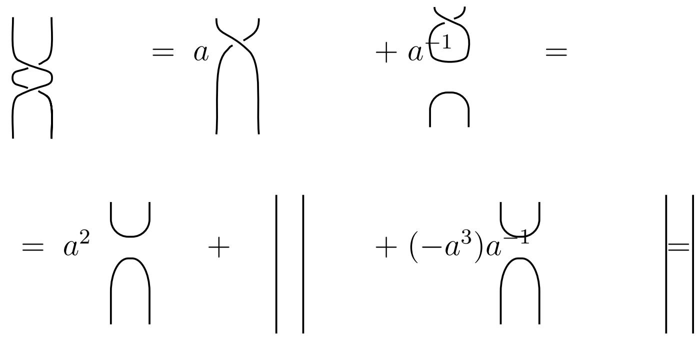


   In a similar way, we show for the opposite curl:



   We conclude invariant under the Reidemeister move $\Omega_0^{\pm 1}$.

3. Invariance under the Reidemeister move $\Omega_2^{\pm 1}$ is shown by a similar computation:



In the third identity, we used the result of 2. for the positive curl. We leave it to the reader to show invariance under the Reidemeister move $\Omega_3^{\pm 1}$.

□

Definition 5.3.6

Let L be a framed link. Choose any link diagram D representing L. Then the bracket polynomial of L is defined by

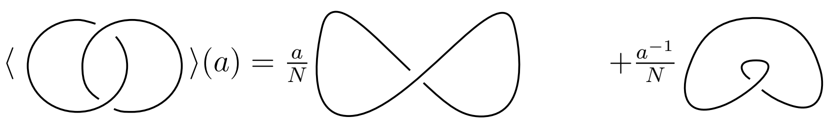
$$\langle L\rangle(a) = \frac{\langle D\rangle(a)}{-a^2 - a^{-2}} .$$

This is a Laurent polynomial in a. This function of a is an isotopy invariant of the link L.

Examples 5.3.7.

1. It is obvious that the unknot with trivial framing has bracket polynomial $\langle L\rangle(a) = 1$.

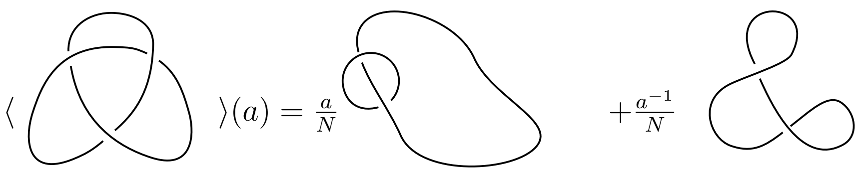
2. We obtain for the Hopf link:



$$= a(-a^{+3}) + a^{-1}(-a^{-3}) = -a^4 + a^{-4}$$

with $N := \frac{1}{-(a^2+a^{-2})}$. Here we used the results for the positive and negative curl obtained in the proof of theorem 5.3.5.

3. We obtain for the trefoil knot:



$$\begin{aligned} &= a(-a^{+4} - a^{-4}) + a^{-1}(a^{-3})^2 \\ &= -a^5 - a^{-3} + a^{-7} \end{aligned}$$

Here we used in the second equality the results for the Hopf link and the curls. We remark that the invariant of the trefoil knot and of its mirror are different. We conclude that the trefoil knot is not isotopic to its mirror (and not isotopic to the unknot).

In passing, we mention:

Definition 5.3.8
Let L be an oriented link in $\mathbb{R}^3$ without framing. Choose a framing for each component L_i such that the self-linking number of L_i is

$$-\sum_{j\neq i} lk(L_i, L_j)$$

to obtain a framed link L^f. The Jones polynomial for L is the Laurent polynomial

$$V_L(q) = \langle L^f \rangle (q^{-1/2}) \, .$$

To get a more categorical structure, we enlarge the geometric objects we consider and allow open ends.

Definition 5.3.9

1. *A (k,l)-tangle is a finite set of disjoint circles and intervals that are smoothly embedded in $\mathbb{R}^2 \times [0,1]$ such that*
   - *the end points of the intervals are precisely the points $(1,0,0), \ldots (k,0,0)$ and $(1,0,1), \ldots, (l,0,1)$.*
   - *The circles are contained in the open subset $(\mathbb{R}^2 \times (0,1))$.*
2. *Isotopies of tangles and framed tangles are defined in complete analogy to the case of links in definition 5.3.1.3.*

Remarks 5.3.10.

1. Since any link in $\mathbb{R}^3$ can be smoothly deformed to a link in $\mathbb{R}^2 \times (0,1)$, we identify links and $(0,0)$ tangles.
2. Tangle diagrams are projections of tangles to $\mathbb{R} \times [0,1]$ with only double transversal intersections. We only consider oriented tangle diagrams.
3. Tangle diagrams represent isotopic tangles, if they are related by an isotopy of $\mathbb{R} \times [0,1]$ or the Reidemeister moves $\Omega_0^{\pm 1}, \Omega_2^{\pm 1}, \Omega_3^{\pm 1}$.
4. One also defines for $a \in \mathbb{C}^*$ skein modules $E_{k,l}(a)$ for $k, l \in \mathbb{Z}_{\geq 0}$. Their dimension is known:
$$\dim E_{k,l}(a) = \begin{cases} 0 & k+l \text{ odd} \\ \begin{pmatrix} k+l \\ (k+l)/2 \end{pmatrix} \Big/ \frac{k+l}{2}+1 & k+l \text{ even} \end{cases}$$
   Again the skein class of a tangle is invariant under the Reidemeister moves. We thus contain invariants of tangles with values in finite-dimensional complex vector spaces.

Definition 5.3.11

1. *We define a category $\mathcal{T}$ of framed tangles:*

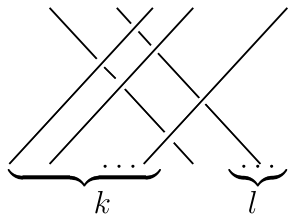
- *Its objects are the non-negative integers.*
- *A morphism $k \to l$ is an isotopy class of framed (k,l)-tangles.*

The composition is concatenation of tangles, followed by a rescaling to the interval $[0,1]$. The identity tangles are given by parallel lines.

2. *We endow $\mathcal{T}$ with a monoidal structure. On objects, we define $k \otimes l := k+l$; on morphisms, we take juxtaposition of tangles. The tensor unit is $0 \in \mathbb{Z}_{\geq 0}$.*

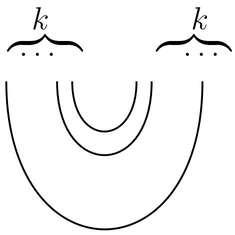
3. *The category $\mathcal{T}$ is endowed with the structure of a braided monoidal category by the following isomorphisms:*

$$c_{k,l} : k \otimes l \to l \otimes k$$

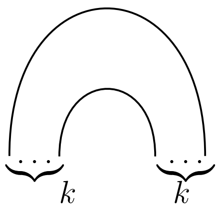


The axioms of a braiding follow from obvious isotopies.

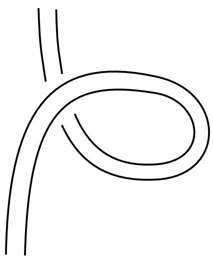
4. *The braided category $\mathcal{T}$ has the dualities*



bzw.



and the twist $\theta_k : k \to k$



which turn it into a ribbon category.

We note that braids form a monoidal subcategory of the category $\mathcal{T}$ of tangles. We also have the following generalization of the bracket polynomials from definition 5.3.6:

Observation 5.3.12.
For $a \in \mathbb{C}^\times$ define the skein category $\mathcal{S}(a)$. Its objects are the non-negative integers, its morphisms are the skein modules

$$\mathrm{Hom}_{\mathcal{S}(a)}(k,l) = E_{k,l}(a)\,.$$

With similar definitions as for the tangle category $\mathcal{T}$, one obtains a $\mathbb{C}$-linear ribbon category. Because morphisms are invariant under Reidemeister moves, we get a family of ribbon functors

$$P(a) : \mathcal{T} \to \mathcal{S}(a)$$

called <u>skein functors</u> which generalize the bracket polynomial.

Let now $\mathcal{C}$ be a ribbon category. We describe the category $\mathcal{T}_\mathcal{C}$ of $\mathcal{C}$-coloured framed oriented tangles.

Observation 5.3.13.

1. Tangles are framed and oriented. Each component of a tangle is labelled with an object of $\mathcal{C}$. Isotopies preserve the orientation, framing and labelling.

2. The objects of $\mathcal{T}_\mathcal{C}$ are finite sequences of pairs

$$(V_1, \epsilon_1) \dots (V_n, \epsilon_n) \qquad V_i \in \mathcal{C} \quad \epsilon_i \in \{\pm 1\},$$

   including the empty sequence.

3. Morphisms are isotopy classes of framed oriented tangles. If the source object has label $\epsilon = +1$, the tangle is upward directed and labelled with V. It has to end on either an object $(V, +1)$ at $t = 1$ or at $(V, -1)$ at $t = 0$, where $t \in [0, 1]$ parametrizes the tangle.

4. The category $\mathcal{T}_\mathcal{C}$ is endowed with a ribbon structure in complete analogy to the ribbon structure on the category $\mathcal{T}$ of framed oriented tangles.

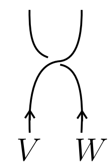
Proposition 5.3.14.
Let $\mathcal{C}$ be a ribbon category. Then there is a *unique* braided tensor functor

$$F = F_\mathcal{C} : \quad \mathcal{T}_\mathcal{C} \to \mathcal{C},$$

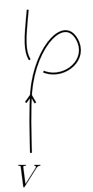
such that

1. F acts on objects as $F(V, +) = V$ and $F(V, -) = V^*$.

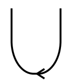
2. For all objects V, W of $\mathcal{C}$, we have



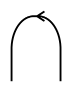
$$\longmapsto \qquad c_{V,W}$$



$$\longmapsto \qquad \theta_V$$



$$\longmapsto \qquad b_V$$



$$\longmapsto \qquad d_V$$

Proof.
One can show that any tangle can be decomposed into the building blocks listed above. One then has to show the compatibility of F with the Reidemeister moves. This follows from the axioms of a ribbon category. $\square$

Remarks 5.3.15.

- In particular, restricting to morphisms between the tensor unit, we get an invariant of $\mathcal{C}$-coloured framed oriented links in S^3:
$$F(L) \in \ \mathrm{End}(\mathbb{I})\,.$$
Since F is a monoidal functor, this invariant is multiplicative, $F(L \otimes L') = F(L) \otimes F(L')$.
- For the special case of the unknot labelled by the object $V \in \ \mathcal{C}$, we get the invariant $\dim V \in \mathrm{End}_{\mathcal{C}}(\mathbb{I})$.

The following generalization will be needed in the construction of topological field theories:

Observation 5.3.16.

1. We define a functor $\mathcal{C}^{\boxtimes n} \to \mathrm{vect}$ by
$$(V_1, \ldots, V_n) \mapsto \mathrm{Hom}_{\mathcal{C}}(\mathbb{I}, V_1 \otimes \cdots \otimes V_n) \tag{14}$$
for any collection $V_1, \ldots, V_n$ of objects of $\mathbb{C}$. We thus consider invariant tensors in the tensor product. The pivotal structure gives functorial isomorphisms
$$z\colon \quad \mathrm{Hom}_{\mathcal{C}}(\mathbb{I}, V_1 \otimes \cdots \otimes V_n) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(\mathbb{I}, V_n \otimes V_1 \otimes \cdots \otimes V_{n-1}) \tag{15}$$
such that $z^n = \mathrm{id}$; thus, up to a canonical isomorphism, the space $\mathrm{Hom}_{\mathcal{C}}(\mathbb{I}, V_1 \otimes \cdots \otimes V_n)$ only depends on the cyclic order of the objects $V_1, \ldots, V_n$.

2. For a spherical category, we can define invariants of tangles where no crossings are allowed in the diagrams. We can also do the more general construction: consider an oriented graph Γ embedded in the sphere S^2, where each edge e is colored by an object $V(e) \in \mathbb{C}$, and each vertex v is colored by a morphism $\varphi_v \in \mathrm{Hom}_{\mathcal{C}}(\mathbb{I}, V(e_1)^{\pm} \otimes \cdots \otimes V(e_n)^{\pm})$, where $e_1, \ldots, e_n$ are the edges adjacent to vertex v, taken in clockwise order, and $V(e_i)^{\pm} = V(e_i)$ if e_i is outgoing edge, and $V^*(e_i)$ if e_i is the incoming edge.

   By removing a point pt from S^2 and identifying $S^2 \setminus pt \simeq \mathbb{R}^2$, we can consider Γ as a planar graph to which our rules assign an element $Z(\Gamma) \in \mathrm{End}_{\mathcal{C}}(\mathbb{I})$. One shows that this number is an invariant of coloured graphs on the sphere.

3. Note that up to this point, only graphs on S^2 without crossings were allowed. We generalize this setup by allowing finitely many non-intersecting edges of a different type, labelled by objects of the Drinfeld double $Z(\mathcal{C})$. These edges are supposed to start and end at the vertices as well. We colour edges of such a graph $\hat{\Gamma}$ with objects in $\mathcal{C}$ and $Z(\mathcal{C})$ respectively and morphisms as before. We get an invariant $Z(\Gamma) \in \mathbb{K}$ for this type of graph as well.

5.4 Topological field theories of Turaev-Viro type

We now discuss the construction of a three-dimensional topological field theory which can be seen as a three-dimensional cousin of the two-dimensional topological field theory constructed in example 4.1.14 on 2-manifolds with a triangulation. Our exposition closely follows [BK1].

We work with a three-layered structure involving manifolds of dimensions 1,2 and 3. It comprises in particular a three-dimensional topological field theory

$$Z_{TV}: \quad \mathrm{Cob}(3,2) \to \mathrm{vect}(\mathbb{C})$$

in the sense of definition 4.1.5. In particular, we have to achieve the following goals:

- To closed oriented three-manifolds, we want to assign an invariant with values in $\mathrm{End}_{\mathcal{C}}(\mathbb{I})$. This invariant should be a topological invariant.
- To closed oriented two-manifolds, we want to assign a finite-dimensional complex vector space. To a three-manifold M with boundary representing a cobordism $\partial_- M \xrightarrow{M} \partial_+ M$, we want to assign a linear map. This expresses a locality property of our invariants.
- We want to go one step further and assign categories to closed oriented 1-manifolds.

Thus we have as a zeroth layer categories associated to closed oriented 1-manifolds. At the first layer, we will have not only closed surfaces, but also 2-manifolds with boundary. After having chosen an object for each boundary circle, we get a vector space which depends functorially on the choice of objects. On the third level, we have three-manifolds with corners relating the 2-manifolds with boundaries. In particular, we obtain invariants of knots and links in three-manifolds generalizing the constructions of the previous subsection and thus to representations of braid groups.

This can be formalized by using bicategories, defining a bicategory $\mathrm{Cob}(1,2,3)$ of 1-2-3 cobordisms, leading to the notion of an extended topological field theory. For an account, see [L, Section 1.2] and for an informal account see [NS].

We stick here to a low key approach.

Definition 5.4.1
Let $\mathbb{K}$ be an algebraically closed field. A fusion category is a rigid, $\mathbb{K}$-linear semisimple tensor category with finitely many isomorphism classes of simple objects and finite dimensional spaces of morphisms. We assume that $\mathrm{End}_{\mathcal{C}}(\mathbb{I}) = \mathbb{K}\mathrm{id}_{\mathbb{I}}$.

We restrict to the ground field $\mathbb{K} = \mathbb{C}$. The following proposition will be used:

Proposition 5.4.2. [ENO, Theorem 2.3]
If $\mathcal{C}$ is a spherical fusion category over the field $\mathbb{C}$, then the so-called global dimension of $\mathcal{C}$ is non-zero:

$$\mathcal{D}^2 := \sum_{i\in I} (\dim V_i)^2 \neq 0 \ .$$

(We do not suppose that a square root $\mathcal{D}$ of the right hand side has been chosen; the notation will just be convenient later.)

Observation 5.4.3.

1. All manifolds are compact, oriented and piecewise linear. We fix as a combinatorial datum a polytope decomposition Δ, in which we allow individual cells to be arbitrary polytopes (rather than just simplices). Moreover, we allow the attaching maps to identify some of

the boundary points, for example gluing polytopes so that some of the vertices coincide. On the other hand, we do not want to consider arbitrary polytope decompositions, since it would make describing the elementary moves between two such decompositions more complicated. We call a piecewise linear manifold M with a polytope decomposition Δ a *combinatorial manifold*.

2. The moves are then:
   - (M1): Removing a (regular) vertex, cf. [BK1, Figure 8].
   - (M2) Removing a (regular) edge, cf. [BK1, Figure 9].
   - (M3) Removing a (regular) 2-cell, cf. [BK1, Figure 10].

Observation 5.4.4.

1. A (simple) labeling l of a combinatorial manifold (M, Δ) is a map that assigns to each edge e of Δ a (simple) object of $\mathcal{C}$ such that $l(\overline{e}) = l(e)^*$ for the edge $\overline{e}$ with opposite orientation.

2. We assign to any 2-cell C with labeling l the vector space of invariant tensors

$$H(C, l) := \mathrm{Hom}_{\mathcal{C}}(\mathbb{I}, l(e_1) \otimes \ldots \otimes l(e_n)) \ ,$$

   cf. [BK1, Figure 15]. Here the edges $e_1, \ldots, e_n$ of the 2-cell C are taken counterclockwise with respect to the orientation of C. One can show that the properties of a spherical category imply:

   - Up to canonical isomorphism, the vector space $H(C, l)$ does not depend on the choice of starting point in the counterclockwise enumeration of the edges $e_1, \ldots, e_n$
   - To the 2-cell with the reversed orientation, we assign the vector space

$$H(\overline{C}, l) := \mathrm{Hom}_{\mathcal{C}}(\mathbb{I}, l(e_n)^* \otimes \ldots l(e_1)^*)$$

     which is canonically in duality with $H(C, l)$.

3. Let now (Σ, Δ) be a combinatorial 2-manifold with labelling l. We assign to a labelled combinatorial surface (Σ, Δ, l) the vector space

$$H(\Sigma, \Delta, l) = \bigotimes_{C \in \Delta} H(C, l) \ ,$$

   i.e. the tensor product over the vector spaces assigned to all faces C of the polytope decomposition Δ of Σ, and then sum over all labelings by simple objects,

$$H(\Sigma, \Delta) := \bigoplus_{l} H(\Sigma, \Delta, l) \ .$$

   - This vector space depends on the choice of polytope decomposition Δ and is therefore *not* the vector space assigned to Σ by the topological field theory we want to construct.
   - The assignment is tensorial: to a disjoint union $\Sigma_1 \sqcup \Sigma_2$ of 2-manifolds with polytope decomposition $\Delta_1 \sqcup \Delta_2$, we assign the vector space

$$H(\Sigma_1 \sqcup \Sigma_2, \Delta_1 \sqcup \Delta_2) = H(\Sigma_1, \Delta_1) \otimes H(\Sigma_2, \Delta_2) \ .$$

- Upon change of orientation, we have

$$H(\overline{\Sigma},\Delta) \cong H(\Sigma,\Delta)^* \ .$$

4. Our next goal is to assign to a 3-cell F with labeling l a vector

$$H(F,l) \in H(\partial F, l)$$

in the vector space associated to the boundary ∂F with the induced labelling ∂l and induced polytope decomposition $\partial\Delta$.

The boundary ∂F has the form of a sphere with an embedded graph whose surfaces are faces and thus carry vector spaces $H(C,l)$. Take the dual graph Γ on S^2 as in [BK1, Figure 16]. The vertices of the dual graph are the faces of the original graph and thus labelled by vector spaces $H(C,l)$ which are Hom-spaces of $\mathcal{C}$. Its edges are labeled by (simple) objects, cf. [BK1, Figure 17].

Choose for every face $C \in \partial F$ an element in the dual vector space

$$\varphi_C \in H(C,l)^* \cong \mathrm{Hom}_{\mathcal{C}}(\mathbb{I}, l(e_n)^* \otimes \ldots \otimes l(e_1)^*)$$

It defines a valid labelling of a graph Γ on the sphere S^2 so that observation 5.3.16.2 gives a number $Z(\Gamma)$. We define the vector $H(F,l) \in H(\partial F,l)$ by its values on the vectors in the dual vector space:

$$\langle H(F,l), \otimes_{C\in\partial F}\varphi_C\rangle = Z(\Gamma) \ .$$

5. We now assign to a combinatorial labeled 3-manifold (M,Δ,l) with boundary the combinatorial surface $(\partial F, \partial\Delta)$ a vector

$$H(M,\Delta,l) \in H(\partial M, \partial\Delta, \partial l) \ .$$

We note that

$$\otimes_{F\in\Delta} H(F,l) \in \otimes_F H(\partial F, \partial\Delta, \partial l) = H(\partial M, \partial l) \otimes \bigotimes_c H(c', \partial l) \otimes H(c'', \partial l)$$

where F runs over all 3-cells of M and c runs over all interior 2-cells, which appear for two faces, with opposite orientation. The associated vector spaces are thus in duality and we can contract the corresponding components in $\otimes_{F\in M} H(F,l)$ by applying the evaluation to them. We thus define

$$H(M,\Delta,l) := \mathrm{ev}\Big(\bigotimes_{F\in\Delta} H(F,l)\Big) \in H(\partial M, \partial\Delta, l)$$

6. We have finally to get rid of the labelling. This is done by a summation with weighting factors that are rather subtle:

$$Z_{TV}(M,\Delta) := \mathcal{D}^{-2v(M)} \sum_l \Big(H(M,\Delta,l) \prod_e d_{l(e)}^{n_e}\Big)$$

where

- the sum is taken over all equivalence classes of simple labelings l of Δ,
- the product over e runs over the set of all (unoriented) edges of Δ

- $\mathcal{D}$ is the dimension of the category $\mathcal{C}$ from proposition 5.4.2 and

$$v(M) := \text{number of internal vertices of } M + \frac{1}{2}(\text{number of vertices on } \partial M)$$

- $d_{l(e)}$ is the categorical dimension of $l(e)$ and $n_e = 1$ for an internal edge, and $1/2$ for an edge in the boundary ∂M. Here, we assume that some square root has been chosen for each dimension of a simple object.

7. To get a three-dimensional topological field theory

$$Z_{TV} : \mathrm{Cob}(3,2) \to \mathrm{vect}(\mathbb{C})$$

consider a combinatorial 3-cobordism (M, Δ) between two combinatorial surfaces (N_1, Δ_1) and (N_2, Δ_2), i.e. a combinatorial 3-manifold (M, Δ) with boundary $\partial M = \overline{N_1} \sqcup N_2$ and the induced combinatorial structure $\partial\Delta = \Delta_1 \sqcup \Delta_2$ on the boundary. Then

$$H(\partial M, \partial\Delta) \cong H(N_1, \Delta_1)^* \otimes H(N_2, \Delta_2) \cong \mathrm{Hom}_{\mathbb{K}}(H(N_1, \Delta_1), H(N_2, \Delta_2))$$

so that we have a linear map

$$H(M, \Delta) \colon H(N_1, \Delta_1) \to H(N_2, \Delta_2) \ .$$

One now proves:

Theorem 5.4.5.

1. For a closed PL manifold M, the scalar $Z_{TV}(M, \Delta) \in \mathbb{K}$ does not depend on the choice of polytope decomposition Δ. We write $Z_{TV}(M)$.

2. More generally, if M is a 3-manifold with boundary and Δ, Δ' are two polytope decompositions of M that agree on the boundary, $\partial\Delta = \partial\Delta'$, then

$$H(M, \Delta) = H(M, \Delta') \in H(\partial M, \partial\Delta) = H(\partial M, \partial\Delta') \ .$$

3. For manifolds with boundary, the invariant obeys the gluing condition.

4. For a combinatorial 2-manifold (N, Δ), consider the linear maps associated to the cylinders

$$A_{N,\Delta} := H(N \times [0,1]) : \quad H(N, \Delta) \to H(N, \Delta) \ .$$

The composition of two cylinders is again a cylinder. Thus, as a consequence of 2, the maps are idempotents: $A^2_{N,\Delta} = A_{N,\Delta}$.

5. To a combinatorial 2-manifold (N, Δ) assign the vector space

$$Z_{TV}(N, \Delta) := \mathrm{Im}\,(A_{N,\Delta}) \subset H(N, \Delta) \ .$$

It is an invariant of PL manifolds: for different polytope decompositions, one has *canonical* isomorphisms $Z_{TV}(N, \Delta) \cong Z_{TV}(N, \Delta')$. We write $Z_{TV}(N)$ for the class.

6. We denote this vector space by $Z_{TV}(N)$. For a cobordism $N_1 \xrightarrow{M} N_2$, we denote by $Z_{TV}(M)$ the restriction of the linear map $H(M, \Delta)$ to $Z_{TV}(N)$. This defines a three-dimensional topological field theory $Z_{TV} : \mathrm{Cob}(3,2) \to \mathrm{vect}(\mathbb{K})$.

For the proof of all these statements, we have to refer to [BK1]. The essential step is to show that the properties of a finitely semisimple spherical category imply the independence under the three moves changing the polytope decomposition.

In our construction, we have assigned objects of a category to edges. This raises the question on whether we can assign categorical data to one-dimensional manifolds as well. This will provide us with structure generalizing the skein functors from observation 5.3.12.

Observation 5.4.6.

1. We now allow surfaces with boundaries. To be able to work with closed surfaces, we glue a disc to the boundary circle and work with surfaces with marked discs instead. These discs are supposed to be faces of the triangulation and actually are faces of a new type. For later use, we mark a vertex on the boundary of the disc. We assign to a marked disc an object in the Drinfeld double $Z(\mathcal{C})$.

2. The three-manifolds are now manifolds with corners. Three-manifolds with corners and surfaces with marked discs form an extended cobordism category, actually a bicategory. They contain two types of tubes: open tubes, ending at the boundaries or closed tubes. They will lead to 3-cells of a new type. We suppose that all components of tubes are labelled with objects in $Z(\mathcal{C})$.

3. We aim at extending the TV invariants to such extended surfaces and cobordisms:

   (a) Define, for every labelled extended surface N, a vector space $Z_{TV}(N, \{Y_\alpha\})$ which
      - functorially depends on the colors $Y_\alpha \in Z(\mathcal{C})$,
      - is functorial under homeomorphisms of extended surfaces,
      - has natural isomorphisms $Z_{TV}(\overline{N}, \{Y_\alpha^*\}) = Z_{TV}(N, \{Y_\alpha\})^*$,
      - satisfies the gluing axiom for surfaces.

   (b) Define, for any colored extended 3-cobordism M between colored extended surfaces N_1, N_2, a linear map $Z_{TV}(M)\colon Z_{TV}(N_1) \to Z_{TV}(N_2)$ so that this satisfies the gluing axiom for extended 3-manifolds.

4. We repeat the steps in the previous construction, with the following modifications:

   (a) There is now an additional type of 2-cell corresponding to an embedded disc with label $Y \in Z(\mathcal{C})$. To such a 2-cell, we assign the vector space

   $$\mathrm{Hom}_{\mathcal{C}}(\mathbb{I}, F(Y) \otimes l(e_1) \otimes \ldots \otimes l(e_n)) \ .$$

   Here we applied the forgetful functor $F : Z(\mathcal{C}) \to \mathcal{C}$ from proposition 4.5.22. We needed to specify a point to get a linear order on the objects, because now the position of $F(Y) \in \mathcal{C}$ matters. We then continue to define vector spaces $H(N, \Delta, \{Y_\alpha\})$ as above by summing over labellings of inner edges.

   (b) There are now two different types of 3-cells: tube cells and usual cells. To assign vectors to tube cells, we use observation 5.3.16.3 Then the construction continues as above.

5. A new feature is now the fact that we have a gluing axiom for extended surfaces. We refer to [BK1, Theorem 8.5].

5.5 Quantum codes and Hopf algebras

There are two basic tasks in computing, both for classical and quantum computing:

- Storing information in a medium and transmitting information.
- Doing computations by processing information.

The first question leads to the mathematical notion of codes, the second to the notion of gates. We start our discussion with classical computing.

5.5.1 Classical codes

Implicitly, assumptions made on storage devices and manipulation of information in classical information theory is based on classical physics, as opposed to quantum mechanics.

Information is stored in the form of binary numbers, hence in terms of elements of the standard vector space $\mathbb{F}_2^n$ over the field $\mathbb{F}_2 = \{0,1\}$ of two elements. (Note that the standard basis of $\mathbb{F}_2^n$ plays a distinguished role.) We identify the elements of $\mathbb{F}_2 = \{0,1\}$ with either on/off or with the truth values 0 = false and 1 = true. If we are dealing with an element of $\mathbb{F}_2^n$, we say that we have n bits of information.

For storing and transmitting information, it is important that errors occurring in the transmission or by the dynamics of the storage device can be corrected. For this reason, only a subset $C \subset \mathbb{F}_2^n$ should correspond to valid information.

Definition 5.5.1

1. *A subset $C \subset (\mathbb{F}_2)^n$ is called a code. The number n is called the length of the code. One says that a code word $c \in C$ is composed of n bits.*
2. *A code $C \subset (\mathbb{F}_2)^n$ is called linear, if C is a vector subspace. Then $\dim_{\mathbb{F}_2} C =: k$ is called the dimension of the code.*

One frequently allows instead of the field $\mathbb{F}_2$ an arbitrary finite field. We will not discuss this in more detail.

To deal with error correction, one defines:

Definition 5.5.2
Let $\mathbb{K} = \mathbb{F}_2$ and $V = \mathbb{K}^n$. The map

$$d_H : V \times V \to \mathbb{N}$$
$$d_H(v,w) := |\{j \in \{1,\dots,n\} |\, v_j \neq w_j\}|$$

is called Hemming distance. It equals the number of components (bits) in which the two code words v and w differ.

Lemma 5.5.3.
The Hemming distance has the following properties:

1. $d_H(v,w) \geq 0$ for all $v,w \in V$ and $d_H(v,w) = 0$, if and only $v = w$
2. $d_H(v,w) = d_H(w,v)$ for all $v,w \in V$ (symmetry)
3. $d_H(u,w) \leq d_H(u,v) + d_H(v,w)$ for all $u,v,w \in V$ (triangle inequality)

4. $d_H(v, w) = d_H(v + u, w + u)$ for all $u, v, w \in V$ (translation invariance)

Definition 5.5.4
For $\lambda \in \mathbb{N}$, a subset $C \subset (\mathbb{F}_2)^n$ is called a λ–error correcting code, if

$$d_H(u, v) \geq 2\lambda + 1 \quad \textit{for all } u, v \in C \quad \textit{with } u \neq v \; .$$

The reason for this name is the following

Lemma 5.5.5.
Let $C \subset V$ be a λ-error correcting code. Then for any $v \in V$, there is at most one $w \in C$ with $d_H(v, w) \leq \lambda$.

Proof.
Suppose we have $w_1, w_2 \in C$ with $d_H(v, w_i) \leq \lambda$ for $i = 1, 2$. Then the triangle inequality yields

$$d_H(w_1, w_2) \leq d_H(w_1, v) + d_H(v, w_2) \leq 2\lambda \; .$$

Since the code C is supposed to be λ-error correcting, we have $w_1 = w_2$. □

Remarks 5.5.6.

1. It is important to keep in mind the relative situation: a code C is a subspace of $\mathbb{F}_2^n$. The Hemming distance gives an indication to what extend the subspace C of code words is spread out in V.

2. We say that information is stored in the code, if an element $c \in C$ is selected.

3. If $C \subset \mathbb{F}_2^n$, we say that a codeword of C is composed of n bits. If C is a linear code with $\dim_{\mathbb{F}_2} C = k$, we refer to a $[n, k]$ code. Denote by

   $$d := \min_{c \in C \setminus \{0\}} d_H(c, 0)$$

   the minimal distance of a code. We refer to an $[n, k, d]$ code. In practice, the length n of the code has to be kept small, because this causes costs for storing and transmitting. The minimal distance d has to be big, since by lemma 5.5.5 this allows to many correct errors. The dimension k of the code has to be big enough to allow enough code words. From elementary linear algebra, one derives the singleton bound

   $$k + d \leq n + 1$$

   which shows that these goals are in competition.

4. Classical storage devices are typically localized, either in space (e.g. an electron or a nuclear spin) or in momentum space (e.g. a photon polarization).

5. Many storage devices are magnetic, i.e. a collection of coupled spins. The Hamiltonian is such that it favours the alignment of spins. So if one spin is kicked out by thermal fluctuation, the Hamiltonian tends to push it back in the right position. Thus errors in the storage device are corrected by the dynamics of the system. This idea will also enter in the construction of quantum codes.

5.5.2 Classical gates

To process information, we need logical gates: A logical gate takes as an input n bits of information an yields m bits as an output.

Definition 5.5.7
Let $\mathbb{K} = \mathbb{F}_2$.

1. *A gate is map $f : \mathbb{K}^n \to \mathbb{K}^m$. Typically, one requires a gate to act only on few, two or three) bits, i.e. to act as the identity on all except for a few summands of $\mathbb{K}^n$.*

2. *A gate is called linear, if the map f is $\mathbb{K}$-linear.*

3. *If the map f is invertible, the gate is called reversible.*

4. *A finite set of gates is called a library of gates. One then applies to $\mathbb{F}_2^n$ a sequence of gates in the library acting on any subset of summands in $\mathbb{F}^n$. The composition of such maps is called a circuit.*

5. *A library of gates is called universal, for any Boolean function $f(x_1, x_2, ..., x_m)$, there is a circuit consisting of gates in the library which takes $x_1, x_2, \ldots, x_m$ and some extra bits set to 0 or 1 and outputs $x_1, x_2, \ldots, x_m, f(x_1, x_2, \ldots, x_m)$, and some extra bits (called garbage). Essentially, this means that one can use the gates in the library to build systems that perform any desired Boolean function computation.*

We wish to use gates to implement the basic Boolean operations:

Examples 5.5.8.

1. Basic gates include negation NOT, AND and OR:

A	$\neg A$
t	f
f	t

A	B	$A \wedge B$
t	t	t
t	f	f
f	t	f
f	f	f

A	B	$A \vee B$
t	t	t
t	f	t
f	t	t
f	f	f

These are gates acting on one bits resp. mapping two bits to one bit.

2. Also in use are the following gates acting on two bits:

A	B	NAND
t	t	f
t	f	t
f	t	t
f	f	t

A	B	NOR
t	t	f
t	f	f
f	t	f
f	f	t

A	B	XOR
t	t	f
t	f	t
f	t	t
f	f	f

3. It is an important theoretical question whether a library of gates is universal. For example, the NAND gate is universal:

   - To get the NOT gate, double the input and feed it into a NAND gate.
   - To get the AND gate, take a NAND gate, followed by a NOT gate, which can be constructed from a NAND gate.
   - To get an OR gate, use de Morgan's law: apply NOT gates to both inputs and feed it into a NAND gate.

4. The Toffoli gate is the linear map

$$T: \mathbb{F}^3 \to \mathbb{F}^3$$

given by the truth table

INPUT			OUTPUT		
0	0	0	0	0	0
0	0	1	0	0	1
0	1	0	0	1	0
0	1	1	0	1	1
1	0	0	1	0	0
1	0	1	1	0	1
1	1	0	1	1	1
1	1	1	1	1	0

It is the identity on the first two bits. If the first two bits are both one, then the last bit is flipped. It thus acts

$$\begin{array}{ll} \mathbb{F}^3 & \to \mathbb{F}^3 \\ (a,b,c) & \mapsto (a,b,c+ab) \end{array}$$

It is not linear, but universal: one can use Toffoli gates to build systems that will perform any desired boolean function computation in a reversible manner.

5. To add two bits A and B, double the bits and feed them into a XOR gate to get the last digit S of the sum and into an AND gate to get a carry-on bit C:

A	B	S=XOR	C=AND
t=1	t=1	f=0	t=1
t=1	f=0	t=1	f=0
f=0	t=1	t=1	f=0
f=0	f=0	f=0	f=0

In such a way, one realizes the arithmetic operations on natural numbers.

5.5.3 Quantum computing

Quantum computation is based on quantum mechanical systems. Now states can be superposed, which leads to a richer structure. On the other hand, the uncertainty principle introduces new limitations, e.g. quantum information cannot be copied: there is no complete set of observables characterizing a state completely that can be measured simultaneously.

We use the simplest possible quantum mechanical system: The state of the system is now not a vector in $\mathbb{F}_2^n$, but rather a vector in the following space: denote by $H = \mathbb{C}\mathbb{Z}_2$ the *complex* group algebra of the cyclic group $\mathbb{Z}_2$. It will be essential that this is a Hopf algebra H with a two-sided integral. As a vector space, $H \cong \mathbb{C}^2$, with a selected basis. We can interpret this system as a non-interacting spin 1/2-particle with basis vectors $|\uparrow\rangle$ and $|\downarrow\rangle$.

The analogue of the ambient vector space $\mathbb{F}_2^n$ is the tensor power $V := H^{\otimes n}$ which we can think of as n coupled spins. A quantum code is a linear subspace $C \subset H^{\otimes n}$ on which the dynamics should afford an error correction. We will say that a code vector $v \in C$ is composed of n qubits. We should mention that H has a natural unitary scalar product by declaring the vectors of the canonical basis to be orthonormal. In general, one has to require on the complex Hopf algebra H the additional structure of a $*$-involution.

To get a framework for quantum computing, we need to set up:

- Codes, i.e. interesting subspaces of $H^{\otimes n}$. To make quantum computing fault tolerant, these subspaces should have special properties. In particular, in a physical realization, the dynamics of the system should suppress errors.
- Unitary operators acting on $H^{\otimes n}$ that preserve these subspaces.

5.5.4 Quantum gates

First let us discuss quantum gates: for quantum computation, we need unitary operators $H^{\otimes n} \to H^{\otimes n}$ to be realized by some time evolution. Unitarity implies reversibility.

Definition 5.5.9

1. *A* <u>*quantum gate*</u> *on $H^{\otimes n}$ is a unitary map $H^{\otimes n} \to H^{\otimes n}$ that acts as the identity on at least $n-2$ tensorands.*
2. *Consider a fixed finite set $\{U_i\}_{i\in I}$ of quantum gates, i.e. $U_i \in \mathrm{U}(H)$ or $U_i \in \mathrm{U}(H \otimes H)$, called a* <u>*library*</u> *of quantum gates. Denote by $U_i^{\alpha\beta}$ the gate U_i acting on the α and β tensorand resp. U_i^{α} acting on the α tensorand of H^n. A* <u>*quantum circuit*</u> *based on this library is a finite product of U_i^{α} and $U_i^{\alpha\beta}$. It is a unitary endomorphism of $H^{\otimes n}$.*
3. *A library of quantum gates is called* <u>*universal*</u>*, if for any n, the subgroup of $\mathrm{U}(H^{\otimes n})$ generated by all circuits is dense.*

Examples 5.5.10.

1. An important examples of a gate is the CNOT gate (controlled not gate) which acts on two qubits: $H^{\otimes 2} \to H^{\otimes 2}$. The CNOT gate flips the second qubit (called the target qubit), if and only if the first qubit (the control qubit) is 1. Here we write $1 = |\uparrow\rangle$ and $0 = |\downarrow\rangle$.

Before		After	
Control	Target	Control	Target
0	0	0	0
0	1	0	1
1	0	1	1
1	1	1	0

   The resulting value of the second qubit corresponds to the result of a classical XOR gate while the control qubit is unchanged.

   An experimental realization of the CNOT gate was afforded by a single Beryllium ion in a trap in 1995 with a reliability of 90%. The two qubits were encoded into an optical state and into the vibrational state of the ion.

2. The relative phase gate $H \to H$ acting on one qubit, a popular choice of which is in the selected basis $|\uparrow\rangle, |\downarrow\rangle$:
$$\begin{pmatrix} 1 & 0 \\ 0 & \exp(2\pi \mathrm{i}/5) \end{pmatrix}$$
   Similarly, the three Pauli matrices give rise to so-called Pauli gates acting on a single qubit.

3. The library consisting of the CNOT gate and the relative phase gate can be shown to be universal.

4. A universal gate is the Deutsch gate which depends on an angular parameter θ

$$\begin{array}{rcl} H^3 & \to & H^3 \\ (a,b,c) & \mapsto & \begin{cases} \mathrm{i}\cos\theta(a,b,c)+\sin\theta(a,b,1-c) & \text{if } a=b=1 \\ (a,b,c) \text{ else} \end{cases} \end{array}$$

   For $\theta = \frac{\pi}{2}$, we recover the classical Toffoli gate. This is taken as an argument that all operations possible in classical computing are possible in quantum computing.

5.5.5 Quantum codes

We can now define quantum codes. For a general reference, see [FKLW].

Definition 5.5.11
Denote by H again the complex group algebra of $\mathbb{Z}_2$. We call a tensor product $V := H^{\otimes n}$ a discrete quantum medium. (Think of a system composed of n spin 1/2 particles.)

1. *A quantum code is a linear subspace $W \subset V$ of a quantum medium V. Sometimes, a quantum code is also called a protected space.*

2. *Let $0 \leq k \leq n$. A k-local operator is a linear map $O : V \to V$ which is the identity on $n-k$ tensorands of V. (Quantum gates are thus at least 2-local.)*

3. *Denote by $\pi_W : V \to W$ the orthogonal projection. A quantum code $W \subset V$ is called a k-code, if the linear operator*
$$\pi_W \circ O : W \to W$$
   is multiplication by a scalar for any k-local operator O.

One can show the following analogue of a lemma 5.5.5:

Lemma 5.5.12.
If W is a k-code, then information cannot be degraded from errors operating on less than $\frac{k}{2}$ of the n particles.

Remarks 5.5.13.

1. A first attempt to realize qubits might be to take *isolated* trapped particles, individual atoms, trapped ions or quantum dots. Such a configuration is fragile and one has to minimize any external interaction. On the other hand, external interaction is need to write and read off information.

   The idea of topological quantum computing is to use non-local degrees of freedom to produce fault tolerant subspaces. Concretely, one needs non-abelian anyons in quasi two-dimensional systems.

2. Storage devices are typically effectively two-dimensional. Thus the complex vector space of qubits should be the space of states of a three-dimensional topological field theory. Maps describing gates and circuits are obtained from colored cobordisms, i.e. three-manifolds containing links. For example, the quantum analogue of the XOR gate, the CNOT gate can be realized to arbitrary precision by braids.

3. A theorem of Freedman, Kitaev and Wang asserts that quantum computers and classical computers can perform exactly the same computations. But their efficiency is different, e.g. for problems like factoring integers into primes.

5.5.6 Topological quantum computing and Turaev-Viro models

We now present a toy model for a system providing a quantum code where the Hamiltonian describes a dynamics which tends to correct errors. It generalizes Kitaev's toric code (which is literally not suitable for quantum computing since it does not allow for universal gates for which one needs more complicated, nonabelian representations of the braid group).

Since storage devices are (quasi)two-dimensional, we take a compact oriented surface Σ on which the physical degrees of freedom are located. To get a discrete structure, take a polytope decomposition Δ of Σ.

Our input is a complex semisimple finite-dimensional $*$-Hopf algebra H. Note that by the theorem of Larson-Radford 3.3.19, then $S^2 = \mathrm{id}_H$. We allocate degrees of freedom to edges e of Δ and consider as the discrete quantum medium the $\mathbb{K}$-vector space

$$V(\Sigma, \Delta) := \otimes_{e \in \Delta} H \ .$$

Here we should first choose an orientation of the edges, and identify $x \mapsto S(x)$ if the orientation is reversed. Since $S^2 = \mathrm{id}$, this isomorphism is well defined. It is clear that the discrete quantum medium depends on the choice of a polytope decomposition.

To construct subspaces for quantum codes $W \subset V$, we need linear endomorphisms on V.

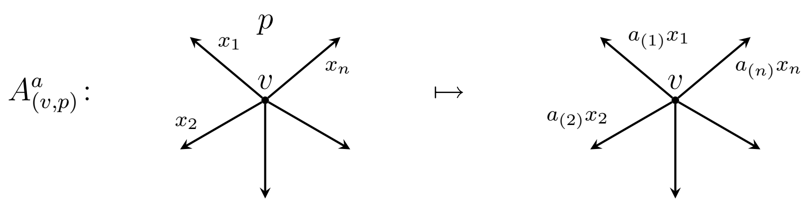
Definition 5.5.14

1. *Let Σ be a two-dimensional manifold with a polytope decomposition Δ. A <u>site</u> of Δ is a pair (v, p), consisting of a face p and a vertex v adjacent to p.*

2. *For every site (v, p) of (Σ, Δ) and every element $h \in H$, we define an endomorphism*

   $$A_{(v,p)}(h) : V(\Sigma, \Delta) \to V(\Sigma, \Delta)$$

   by a multiple coproduct and the left action of H on itself:

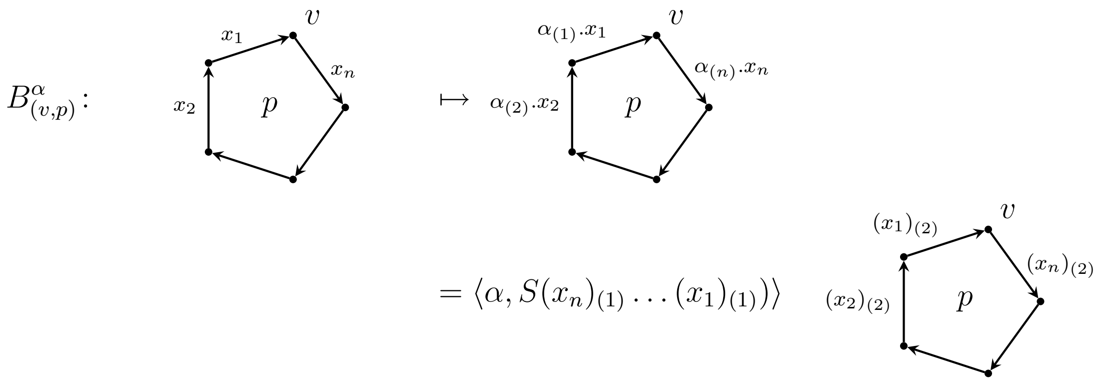
   where the edges incident to the vertex v are indexed counterclockwise starting from p. Here all edges incident to the vertex v are assumed to point away from v. Using the antipode to change orientation, we see that for edges oriented towards the vertex v, the left regular action has to be replaced by the following left action: instead of $a_{(i)}x_i$, we have $S\left(a_{(i)}S(x_i)\right) = x_i S(a_{(i)})$.

3. *Given a site $s = (v, p)$ of the polytope decomposition Δ and every element $\alpha \in H^*$, the plaquette operator*

   $$B_{(v,p)}(h) : V(\Sigma, \Delta) \to V(\Sigma, \Delta)$$

   is defined by a multiple coproduct in H^* and a left action where $\alpha.x = \alpha \rightharpoonup h$ of H^* on

H.



We need the following

Lemma 5.5.15.
Let X be a representation of H, and Y a representation of H^*. For $h \in H$, $\alpha \in H^*$, define the endomorphisms $p_h, q_\alpha \in \mathrm{End}(H \otimes X \otimes Y \otimes H)$ by

$$p_h(u \otimes x \otimes y \otimes v) = h_{(1)}u \otimes h_{(2)}x \otimes y \otimes vS(h_{(3)})$$
$$q_\alpha(u \otimes x \otimes y \otimes v) = \alpha_{(3)}.u \otimes x \otimes \alpha_{(2)}.y \otimes \alpha_{(1)}.v$$

Then these endomorphisms satisfy the straightening formula of $D(H)$. Then the map

$$D(H) \to \mathrm{End}(H \otimes X \otimes Y \otimes H)$$
$$h \otimes \alpha \mapsto p_h q_\alpha$$

is a morphism of algebras.

Proof.
It is obvious that we have actions $a \mapsto p_a$ and $\alpha \mapsto p_\alpha$ of H and H^*. It remains to show that these endomorphisms satisfy the straightening formula of $D(H)$. This is done in a direct, but tedious calculation in [BMCA, Lemma 1, Theorem 1]. $\square$

This allows us to show:

Theorem 5.5.16.

1. If v, w are distinct vertices of Δ, then the operators $A^a_{(v,p)}$, $A^b_{(w,p')}$ commute for any pair $a, b \in H$.

2. Similarly, if p, q are distinct plaquettes, then the operators $B^\alpha_{(v,p)}$, $B^\beta_{(v',q)}$ commute for any pair $\alpha, \beta \in H^*$.

3. If the sites are different, then the operators $A^h_{(v,p)}$ and $B^\alpha_{(v',p')}$ commute.

4. For a given site $s = (v,p)$, the operators $A^h_{(v,p)}$ and and $B^\alpha_{(v,p)}$ satisfy the commutation relations of the Drinfeld double $Z(H)$: the map

$$\rho_s \colon D(R) \to \mathrm{End}(V(\Sigma, \Delta)) \tag{16}$$
$$a \otimes \alpha \mapsto A^a_{(v,p)} B^\alpha_{(v,p)} \tag{17}$$

is an algebra morphism.

Proof.

1. The operators $A^-_{(v,-)}$, $A^-_{(w,-)}$ obviously commute if the edges incident to the vertex v and those incident to the vertex w are disjoint. We therefore assume that the vertices v and w are adjacent, i.e. at least one edge connects them. Clearly, we need only to check that the actions of $A^-_{(v,-)}$ and $A^-_{(w,-)}$ commute on their common support. Suppose such an edge e is oriented so that it points from the vertex v to the vertex w. Then $A^-_{(v,-)}$ acts on the corresponding copy of H via the left regular representation, and $A^-_{(w,-)}$ acts on the copy of H associated the edge e via the right regular representation. These are obviously commuting actions.

2. This statement is dual to 1.

3. Follows by the same type of argument.

4. Follows from lemma 5.5.15.

□

Observation 5.5.17.

1. Let $h \in H$ be a cocommutative element, i.e. $\Delta(h) = \Delta^{opp}(h)$. Then the multiple coproduct $\Delta^{(n)}(h) \in H^{\otimes n}$ is cyclically invariant. As a consequence, the endomorphism $A_{(s,p)}(h)$ is independent of the plaquette p which was previously used to construct a linear order on the edges incident to the vertex v. We denote the endomorphism by $A_s(h)$. Similarly, $B_p(f)$ for a cocommutative element $f \in H^*$ is independent on the vertex.

2. Let $\Lambda \in H$ and $\lambda \in H^*$ be two-sided integrals, normalized such that $\epsilon(\Lambda) = 1$. (A normalized two-sided integral is also called a Haar integral.) Two-sided integrals are cocommutative. We thus get an endomorphism $A_v := A_v(\Lambda)$ for each vertex and $B_p := B_p(\lambda)$ for each plaquette.

Lemma 5.5.18.
All endomorphisms A_v and B_p commute with each other and are idempotents,

$$A_v^2 = A_v \quad \text{and} \quad B_p^2 = B_p \ .$$

Proof.
For a normalized integral, we have $\Lambda \cdot \Lambda = \epsilon(\Lambda)\Lambda = \Lambda$. Theorem 5.5.16 now implies that the endomorphisms are idempotents. A two-sided integral is central, $\Lambda \cdot h = \epsilon(h)\Lambda = h \cdot \Lambda$ for all $h \in H$, which implies again with theorem 5.5.16 that the endomorphisms commute. □

One shows that with respect to the scalar product on the quantum medium $H^{\otimes n}$, these endomorphisms are hermitian. We now define as a Hamiltonian the sum of these commuting endomorphisms:

$$H := \sum_v (\mathrm{id} - A_v) + \sum_p (\mathrm{id} - B_p) \ .$$

As a sum over commuting hermitian endomorphisms, the Hamiltonian is Hermitian and diagonalizable.

Definition 5.5.19

The ground state or protected subspace is the zero eigenspace of H:

$$K(\Sigma, \Delta) := \{v \in H^{\otimes n} : Hv = 0\}$$

It is a quantum code.

Remarks 5.5.20.

1. One shows that $x \in K(\Sigma, \Delta)$, if and only if $A_v x = x$ and $B_p x = x$ for all vertices v and all plaquettes p.

2. Up to canonical isomorphism, the ground space does not depend on the choice of polytope decomposition Δ of Σ.

3. In the case of a group algebra of a finite group, $H = \mathbb{C}[G]$, we use the distinguished basis of H consisting of group elements of G. (Kitaev's toric code uses the cyclic group $\mathbb{Z}_2$.) A basis of $V(\Sigma, \Delta)$ is given by assigning to any edge of Δ a group element g. We interpret the group elements g as the holonomy of a connection along the edge.

   - The projection by the operator A_v implements gauge invariance at the vertex v by averaging with respect to the Haar measure.
   - The projection by the operator B_p implements that locally on the face p the field strength vanishes, i.e. that the connection is locally flat. Indeed, integrals project to invariants and thus for the holonomy around a plaquette, we have $\epsilon(g_1 \cdot g_2 \cdot \ldots \cdot g_n) = 1$ which amounts to the flatness condition $g_1 \cdot g_2 \cdot \ldots \cdot g_n = e$.

4. One can then easily modify at single sites the projection condition: instead of requiring invariance under the action of the double associated to the site, one only keeps states transforming in a specific representation of the double. In this way, again the category of $D(H{-}\mathrm{mod})$ appears in the description of degrees of freedoms at insertions.

One can now show:

Theorem 5.5.21 (Balsam-Kirillov)**.**
Let H be a finite-dimensional semisimple Hopf algebra. The vector spaces of ground states constructed from the Hopf algebra H are canonically isomorphic to the vector spaces of the Turaev-Viro topological field theory based on H.

For the proof, we refer to [BK2].

5.6 Modular tensor categories

We want to discuss yet another construction of three-dimensional extended topological field theories, this time starting from a class of braided monoidal categories. Again it provides in particular invariants of links and three-manifolds.

Definition 5.6.1
Let $\mathbb{K}$ be a field.

1. *A tensor category $\mathcal{C}$ is called additive with ground ring $\mathbb{K}$ if the following conditions hold:*

   - *All Hom sets $\mathrm{Hom}_{\mathcal{C}}(U, V)$ are endowed with the structure of a $\mathbb{K}$-vector space.*
   - *Composition and tensor product are $\mathbb{K}$-bilinear.*

- *The map*

$$\begin{array}{rcl} \mathbb{K} & \to & \mathrm{End}_{\mathcal{C}}(\mathbb{I}) \\ \lambda & \mapsto & \lambda \, \mathrm{id}_{\mathbb{I}} \end{array}$$

  is an isomorphism of $\mathbb{K}$*-vector spaces.*

2. *More generally, an object* V *of an additive category* $\mathcal{C}$ *is called* <u>*absolutely simple*</u>*, if the map*

$$\begin{array}{rcl} \mathbb{K} & \to & \mathrm{End}_{\mathcal{C}}(V) \\ \lambda & \mapsto & \lambda \, \mathrm{id}_{V} \end{array}$$

   is an isomorphism of $\mathbb{K}$*-vector spaces.*

3. *Let* $\{V_i\}_{i\in I}$ *be a family of objects in a* $\mathbb{K}$*-linear category* $\mathcal{C}$*. An object* V *in this category is said to be* <u>*dominated*</u> *by the family* $\{V_i\}$*, if the image of the map*

$$\begin{array}{rcl} \mathrm{Hom}_{\mathcal{C}}(V, V_i) \otimes \mathrm{Hom}_{\mathcal{C}}(V_i, V) & \to & \mathrm{End}_{\mathcal{C}}(V) \\ (f, g) & \to & g \circ f \end{array}$$

   additively generates $\mathrm{End}_{\mathcal{C}}(V)$*.*

Definition 5.6.2
A <u>*modular category*</u> *is pair* $(\mathcal{C}, \{V_i\}_{i\in I})$ *consisting of*

- *an additive ribbon category* $\mathcal{C}$ *with ground field* $\mathbb{K}$
- a finite *family* $\{V_i\}_{i\in I}$ *of simple objects*

such that

(A1) There exists an index $0 \in I$ *with* $V_0 = \mathbb{I}$*.*

(A2) For any index $i \in I$ *there is an index* $i^* \in I$ *such that* $V_{i^*} \cong V_i^*$*.*

(A3) The family $\{V_i\}$ *dominates all objects of* $\mathcal{C}$*.*

(A4) The $|I| \times |I|$*-matrix with entries*

$$S_{ij} = \mathrm{Tr}\, c_{V_j,V_i} \circ c_{V_i,V_j} \in \ \mathrm{End}\ (\mathbb{I}) \cong \mathbb{K}$$

is invertible over $\mathbb{K}$*.*

Remarks 5.6.3.

1. The symmetry of the trace implies that the matrix S is symmetric, $S_{ij} = S_{ji}$.
2. The matrix element S_{ij} equals the invariant of the Hopf link with the two components coloured by the objects V_i and V_j.

Observation 5.6.4.

1. We now restrict to algebraically closed fields $\mathbb{K}$. We can then find an element $\mathcal{D} \in \mathbb{K}$ such that
$$\mathcal{D}^2 = \sum_{i\in I} (\dim V_i)^2 \ .$$
We will fix such an element from now on.

2. We assume that all simple objects are absolutely simple. In particular, on the simple object V_i, the twist acts as a scalar multiple of the identity, $\theta_{V_i} = v_i \mathrm{id}_{V_i}$ with $v_i \in \mathbb{K}^\times$.

3. Let
$$\Delta = \Delta_{\mathcal{C}} = \sum_{i\in I} v_i^{-1} (\dim V_i)^2 \, .$$
We assume that Δ is invertible in $\mathbb{K}$.

Proposition 5.6.5.
Let H be a complex semi-simple ribbon factorizable Hopf algebra. Then the category $H-\mathrm{mod}_{fd}$, together with a set of representatives of its simple objects, is a modular tensor category.

Proof.
It remains to show that the S-matrix is non-degenerate. Using proposition 5.1.18, we find
$$\begin{array}{rcl} S_{ij} &=& \mathrm{Tr}\ c_{V_j V_i} \circ c_{V_i V_j} \\ &=& \mathrm{Tr}_{V_j \otimes V_i}\ u\nu^{-1} \circ c \circ c \\ &=& \chi_j \otimes \chi_i (u\nu^{-1} \otimes u\nu^{-1} R_{21} R_{12}) \end{array}$$
where we used the Drinfeld element u, the ribbon element ν and the fact that the pivot $u\nu^{-1}$ is group like.

One checks that $(\chi \leftharpoonup (u\nu^{-1})) \in C(H)$ is a central form. By theorem 4.4.9, its image under the Drinfeld map F_R is in the center of H, i.e. $F_R(\chi \leftharpoonup (u\nu^{-1})) \in Z(H)$.

Since H is required to be semisimple, we can choose as a basis for $C(H)$ the irreducible characters $(\chi_i \leftharpoonup (u\nu^{-1}))$. Choose as a basis of $Z(H)$ the primitive idempotents, which are again in bijection to simple H-modules. Since the H is factorizable, the Drinfeld map is invertible. We thus find
$$F_R(\chi_j \leftharpoonup (u\nu^{-1})) = \sum_i T_{ij} e_i$$
with an invertible matrix T_{ij}.

Thus
$$\begin{array}{rcl} S_{ij} &=& \chi_i(u\nu^{-1} F_R(\chi_j \leftharpoonup (u\nu^{-1}))) \\ &=& \sum_k T_{kj}\ \chi_i(u\nu^{-1} e_k) \\ &=& \dim V_i\ \chi_i(u\nu^{-1})\ T_{ij} \ . \end{array}$$
Thus S is the product of the invertible matrix T with the diagonal matrix $(\dim V_i \chi_i(u\nu^{-1}))$. Since the pivot is an invertible central element, the claim follows. □

Corollary 5.6.6.
Let H be a semi-simple complex Hopf algebra. Then the category of finite-dimensional modules over its Drinfeld double $D(H)$-mod_{fd} is modular.

Proof.
From corollary 4.5.14, we know that the Drinfeld double is semisimple and from proposition 4.5.11 that it is factorizable. □

5.7 Topological field theories of Reshetikhin-Turaev type and invariants of 3-manifolds and knots

Observation 5.7.1.
We need the following description of closed oriented three-manifolds.

- Let $L \subset S^3$ be a framed link with components L_i. Choose around all components L_i a solid torus U_i which is small enough not to intersect any other component of the link. Choose a parametrization of these solid tori:

  $$\varphi_i : U_i \to S^1 \times D^2 ,$$

  where D^2 is a standard disc in $\mathbb{R}^2$. The parametrization should be such that the link L_i itself is mapped to the center of the discs,

  $$\varphi_i(L_i) = S^1 \times \{0\}$$

  and that the normal vector field of the framing is mapped to a constant vector field of $\mathbb{R}^2$.

- Consider now a solid four-ball B^4 with boundary a three-sphere, $\partial B^4 = S^3$.

  Consider four-dimensional handles $D^2 \times D^2$ which are four-dimensional manifolds with boundary
  $$\partial(D^2 \times D^2) = S^1 \times D^2 \cup D^2 \times S^1$$
  For any component L_i, glue a handle to $S^3 = \partial B^4$ by identifying the boundary of the handle with the parametrization φ_i to the solid torus $U_i \subset S^3$.

  This gives a new oriented four-dimensional manifold W_L with boundary $M_L = \partial W_L$ a closed oriented three-manifold. We say that three-manifold M_L has been obtained from S^3 by <u>surgery</u> on the link $L \subset S^3$.

  We can view M_L as the complement $S^3 \setminus \bigcup_i U_i$, with other solid tori glued in instead of the removed solid tori.

- A theorem of Lickorish and Wallace asserts that any closed oriented three-manifold can be obtained in this way.

- For a given modular category $\mathcal{C}$, we can now define invariants of three-manifolds. Let L be a surgery link for M. Denote by $\mathrm{coll}(L)$ the set of colourings of the link L by objects in the dominating family $\{V_i\}_{i \in I}$. We then define

  $$\tau(M) = \Delta^{\sigma(L)} \mathcal{D}^{-\sigma(L)-m-1} \sum_{\lambda \in \mathrm{coll}(L)} \Big(\prod_{i=1}^{m} \dim V_{\lambda(i)} \Big) F(L, \lambda) \in k \, ,$$

  where m is the number of components of the link L and $\sigma(L)$ is another integer associated to the link L.

Theorem 5.7.2.
The element $\tau(M) \in \mathbb{K}$ is a topological invariant of M.

Proof.
The invariance on the choice of the surgery link L follows from the invariance under the so-called Kirby moves. □

Remark 5.7.3.

- Given a modular tensor category $\mathcal{C}$, the invariants can be explicitly computed. We find, e.g.
$$\tau(S^1 \times S^2) = 1 \qquad \tau(S^3) = \mathcal{D}^{-1} \,.$$

- One can consider invariants of a $\mathcal{C}$-coloured link L in an arbitrary three-manifold M:
$$\tau(M, L) \in \mathbb{K} \,.$$
They are topological as well.

  In the special case $M = S^3$, we find the link invariant
$$\tau(S^3, L) = \mathcal{D}^{-1} F_{\mathcal{C}}(L) \,,$$
with $F_{\mathcal{C}}$ as in 5.3.14. If one takes a modular category constructed from representations of a deformation of the universal enveloping algebra of the Lie algebra sl(2) and labels the link with the fundamental representation, one obtains the Jones polynomial from definition 5.3.8. This is shown by checking the Kauffman relations.

- We can even construct an extended three-dimensional topological field theory. For modular tensor category extracted from deformed universal enveloping algebras $\mathrm{U}_q(\mathfrak{g})$ with $\mathfrak{g}$ a semisimple complex Lie algebra, the theory obtained is closely related to three-dimensional Chern-Simons gauge theories. This is, unfortunately, beyond the scope of these lectures.

The following theorem relates the Turaev-Viro and the Reshetikhin-Turaev invariant:

Theorem 5.7.4 (Balsam-Kirillov, Turaev-Virelizier)**.**
As extended three-dimensional topological field theories, the Turaev-Viro TFT based on a semisimple ribbon category $\mathcal{C}$ and the Reshetikhin-Turaev TFT based on its Drinfeld center $Z(\mathcal{C})$ are isomorphic.

Two independent proofs can be found in [BK1] and [TV].

A Facts from linear algebra

A.1 Free vector spaces

Let $\mathbb{K}$ be a fixed field. To any $\mathbb{K}$-vector space, we can associate the underlying set. Any $\mathbb{K}$-linear map is in particular a map of sets. We thus have a so-called forgetful functor $U : \mathrm{vect}(\mathbb{K}) \to \mathrm{Set}$. We also need a functor from sets to $\mathbb{K}$-vector spaces.

Definition A.1.1
Let S be a set and $\mathbb{K}$ be a field. A $\mathbb{K}$-vector space $V(S)$, together with a map of sets $\iota_S : S \to V(S)$, is called the free vector space on the set X, if for any $\mathbb{K}$-vector space W and any map $f : X \to W$ of sets, there is a unique *$\mathbb{K}$-linear map $\tilde{f} : V(S) \to W$ such that $\tilde{f} \circ \iota_S = f$. As a commuting diagram:*

$$\begin{tikzcd} S \arrow[r, "\iota_S"] \arrow[dr, "f"'] & V(S) \arrow[d, dashed, "\exists! \tilde{f}"] \\ & W \end{tikzcd}$$

Remarks A.1.2.

1. A free vector space, if it exists, is unique. Suppose that (V', ι') is another free vector space on the same set S. Consider the commutative diagram

$$\begin{tikzcd} & V(S) \arrow[d, dashed, "\phi"] \\ S \arrow[ur, "\iota_S"] \arrow[r, "\iota'"] \arrow[dr, "\iota_S"] & V' \arrow[d, dashed, "\phi'"] \\ & V(S) \end{tikzcd}$$

   By the defining property of $V(S)$, applied to the map ι' of sets, we find a (unique) $\mathbb{K}$-linear map $\phi : V(S) \to V'$ such that the upper triangle commutes. By the defining property of V', applied to the map ι_S of sets, we find a (unique) $\mathbb{K}$-linear map $\phi' : V(S) \to V'$ such that the lower triangle commutes. Thus the outer triangle commutes, $\iota_S = \phi' \circ \phi \circ \iota_S$. On the other hand, $\iota_S = \mathrm{id}_{V(S)} \circ \iota_S$, and by the defining property of $V(S)$, such a map is unique. Thus $\phi' \circ \phi = \mathrm{id}_{V(S)}$. Exchanging the roles of $V(S)$ and V', we find $\phi \circ \phi' = \mathrm{id}_V$. Thus $V(S)$ and V are isomorphic.

2. The free vector space exists: take $V(S)$ the set of maps $S \to \mathbb{K}$ which take value zero $0 \in \mathbb{K}$ almost everywhere. Adding the values of two maps $(f+g)(s) := f(s) + g(s)$ and taking scalar multiplication on the values $(\lambda f)(s) := \lambda \cdot f(s)$ endows $V(S)$ with the structure of a $\mathbb{K}$-vector space. To define the map ι_S, let $\iota_S(s)$ for $s \in S$ be the map which takes value 1 on s and zero on all other elements,

$$\iota_S(s)(s) = 1 \qquad \iota_S(s)(t) = 0 \quad \text{for } t \neq s \ .$$

   Then the set $\iota_S(S) \subset V(S)$ is a $\mathbb{K}$-basis. The condition $\tilde{f} \circ \iota_S(s) = f(s)$ then fixes $\tilde{f}$ on a basis which is possible and unique. This shows that we have constructed the free vector space on the set S.

3. Let $f : S \to S'$ be any map of sets. Applying in the diagram

$$\begin{tikzcd} S \arrow[r, "\iota_S"] \arrow[d, "f"'] & V(S) \arrow[d, dashed, "V(f)"] \\ S' \arrow[r, "\iota_{S'}"'] & V(S') \end{tikzcd}$$

   the defining property of the free vector space $V(S)$ to the map $\iota_{S'} \circ f$ of sets, we find a unique $\mathbb{K}$-linear map $V(f) : V(S) \to V(S')$.

   One checks that this defines a functor $V : \mathrm{Set} \to \mathrm{vect}(\mathbb{K})$. For any $\mathbb{K}$-vector space W and any set S, we have a bijection of morphism spaces:

$$\mathrm{Hom}_{\mathbb{K}}(V(S), W) \cong \mathrm{Hom}_{\mathrm{Set}}(S, U(W))$$

   that is compatible with morphism of sets and $\mathbb{K}$-linear maps. The functor V assigning to a set the vector space generated by the set is thus a left adjoint to the forgetful functor U, cf. definition 2.5.19.

A.2 Tensor products of vector spaces

We summarize some facts about tensor products of vector spaces over a field $\mathbb{K}$.

Definition A.2.1
Let $\mathbb{K}$ be a field and let V, W and X be $\mathbb{K}$-vector spaces. A $\mathbb{K}$-bilinear map is a map

$$\alpha : V \times W \to X$$

that is $\mathbb{K}$-linear in both arguments, i.e. $\alpha(\lambda v + \lambda' v', w) = \lambda\alpha(v, w) + \lambda'\alpha(v', w)$ and $\alpha(v, \lambda w + \lambda' w') = \lambda\alpha(v, w) + \lambda'\alpha(v, w')$ for all $\lambda, \lambda' \in \mathbb{K}$ and $v, v' \in V$, $w, w' \in W$.

Given any $\mathbb{K}$-linear map $\varphi : X \to X'$, the map $\varphi \circ \alpha : V \times W \to X'$ is $\mathbb{K}$-bilinear as well. This raises the question of whether for two given $\mathbb{K}$-vector spaces V, W, there is a "universal" $\mathbb{K}$-vector space with a universal bilinear map such that *all* bilinear maps out of $V \times W$ can be described in terms of linear maps out of this vector space.

Definition A.2.2
The tensor product of two $\mathbb{K}$-vector spaces V, W is a pair, consisting of a $\mathbb{K}$-vector space $V \otimes W$ and a bilinear map

$$\begin{array}{rccc} \kappa : & V \times W & \to & V \otimes W \\ & (v, w) & \mapsto & v \otimes w \end{array}$$

with the following universal property: for any $\mathbb{K}$-bilinear *map*

$$\alpha : V \times W \to U$$

there exists a unique linear *map $\tilde{\alpha} : V \otimes W \to U$ such that*

$$\alpha = \tilde{\alpha} \circ \kappa \, .$$

As a diagram:

$$\begin{tikzcd} V \times W \arrow[rr, "\alpha"] \arrow[dr, "\kappa"'] & & U \\ & V \otimes W \arrow[ur, dashed, "\exists! \tilde{\alpha}"'] & \end{tikzcd}$$

Remarks A.2.3.

1. This reduces the study of bilinear maps to the study of linear maps.

2. We first show that the tensor product, if it exists, is unique up to unique isomorphism. Suppose we have two maps

$$\kappa : V \times W \to V \otimes W \qquad \tilde{\kappa} : V \times W \to V \tilde{\otimes} W .$$

   having each the universal property.

   Using the universal property of κ for the specific bilinear map $\tilde{\kappa}$, we find a unique linear map $\Phi_{\tilde{\kappa}} : V \otimes W \to V \tilde{\otimes} W$ with $\Phi_{\tilde{\kappa}} \circ \kappa = \tilde{\kappa}$.

   Exchanging the roles of κ and $\tilde{\kappa}$, we obtain a linear map $\Phi_\kappa : V \tilde{\otimes} W \to V \otimes W$ with $\Phi_\kappa \circ \tilde{\kappa} = \kappa$. The maps $\kappa = \mathrm{id}_{V\otimes W} \circ \kappa$ and $\Phi_\kappa \circ \Phi_{\tilde{\kappa}} \circ \kappa$ describe the same bilinear map $V \times W \to V \otimes W$. The uniqueness statement in the universal property implies $\Phi_\kappa \circ \Phi_{\tilde{\kappa}} = \mathrm{id}_{V\otimes W}$. Similarly, we conclude $\Phi_{\tilde{\kappa}} \circ \Phi_\kappa = \mathrm{id}_{V\tilde{\otimes} W}$.

3. To show the existence of the tensor product, chose a basis $\mathcal{B} := (b_i)_{i\in I}$ of V and $\mathcal{B}' := (b'_i)_{i\in I'}$ of W. Since a bilinear map is uniquely determined by its values on all pairs $(b_i, b'_j)_{i\in I, j\in I'}$, we need a vector space with a basis indexed by these pairs. Thus define $V \otimes W$ as the vector space freely generated by the set of these pairs. We denote by $b_i \otimes b'_j$ the corresponding element of the basis of $V \otimes W$.

   The bilinear map κ is then defined by $\kappa(b_i, b'_j) := b_i \otimes b'_j$. It has the universal property: to any bilinear map $\alpha : V \times W \to X$, we associate the linear map $\tilde{\alpha} : V \otimes W \to X$ with $\tilde{\alpha}(b_i \otimes b'_j) = \alpha(b_i, b'_j)$.

4. As a corollary, we conclude that for finite-dimensional vector spaces V, W, the dimension of the tensor product is $\dim V \otimes W = \dim V \cdot \dim W$.

5. The elements of $V \otimes W$ are called <u>tensors</u>; elements of the form $v \otimes w$ with $v \in V$ and $w \in W$ are called <u>simple tensors</u>. The simple tensors span $V \otimes W$, but there are elements of $V \otimes W$ that are not tensor products of a vector $v \in V$ and $w \in W$.

Observation A.2.4.
Given $\mathbb{K}$-linear maps

$$\varphi : V \to V' \qquad \psi : W \to W'$$

we obtain a $\mathbb{K}$-linear map

$$\varphi \otimes \psi : V \otimes W \to V' \otimes W'$$

on the tensor products. To this end, consider the commuting diagram:

$$\begin{tikzcd} V \times W \arrow[r, "\otimes"'] \arrow[d, "\varphi\times\psi"'] & V \otimes W \arrow[d, dashed, "\varphi\otimes\psi"] \\ V' \times W' \arrow[r, "\otimes"'] & V' \otimes W' \end{tikzcd}$$

Since the map $\otimes \circ (\varphi \times \psi)$ is bilinear, the universal property of the tensor product implies the existence of a map $\varphi \otimes \psi$ for which the identity

$$(\varphi \otimes \psi)(v \otimes w) = \varphi(v) \otimes \psi(w)$$

holds.

Remarks A.2.5.

1. The bilinearity of κ implies that the tensor product of linear maps is bilinear:

$$\begin{aligned}(\lambda_1\varphi_1 + \lambda_2\varphi_2) \otimes \psi &= \lambda_1\varphi_1 \otimes \psi + \lambda_2\varphi_2 \otimes \psi \\ \varphi \otimes (\lambda_1\psi_1 + \lambda_2\psi_2) &= \varphi \otimes \lambda_1\psi_1 + \varphi \otimes \lambda_2\psi_2\end{aligned}$$

2. Similarly, one deduces the following compatibility with direct sums:

$$(V_1 \oplus V_2) \otimes W \cong (V_1 \otimes W) \oplus (V_2 \otimes W)\,,$$

and analogously in the second argument.

3. There are canonical isomorphisms

$$\begin{array}{rcrcl} a_{U,V,W} & : & (U \otimes V) \otimes W & \xrightarrow{\sim} & U \otimes (V \otimes W) \\ & & u \otimes (v \otimes w) & \mapsto & (u \otimes v) \otimes w \end{array}$$

which allow to identify the $\mathbb{K}$-vector spaces $U \otimes (V \otimes W)$ and $(U \otimes V) \otimes W$. With this identification, the tensor product is strictly associative.

One verifies that the following diagram commutes:

$$\begin{tikzcd}
(U \otimes V) \otimes (W \otimes X) \arrow[rr, equal] & & (U \otimes V) \otimes (W \otimes X) \arrow[d, "\alpha_{U,V,W\otimes X}"] \\
((U \otimes V) \otimes W) \otimes X \arrow[u, "\alpha_{U\otimes V,W,X}"] \arrow[d, "\alpha_{U,V,W}\otimes \mathrm{id}_X"'] & & U \otimes (V \otimes (W \otimes X)) \\
(U \otimes (V \otimes W)) \otimes X \arrow[rr, "\alpha_{U,V\otimes W,X}"'] & & U \otimes ((V \otimes W) \otimes X) \arrow[u, "\mathrm{id}_U\otimes\alpha_{V,W,X}"']
\end{tikzcd}$$

One can show that, as a consequence, any bracketing of multiple tensor products gives canonically isomorphic vector spaces.

4. There are canonical isomorphisms

$$\begin{array}{rcl} \mathbb{K} \otimes V & \xrightarrow{\sim} & V \\ \lambda \otimes v & \mapsto & \lambda \cdot v \end{array}$$

with inverse map $v \mapsto 1 \otimes v$ which allow to consider the ground field $\mathbb{K}$ as a unit for the tensor product. There is also a similar canonical isomorphism $V \otimes \mathbb{K} \cong V$. We will tacitly apply the identifications described in 3. and 4. This shows that $\mathrm{vect}(\mathbb{K})$ is a monoidal category.

5. For any pair U, V of $\mathbb{K}$-vector spaces, there are canonical isomorphisms

$$\begin{array}{rcrcl} c_{U,V} & : & U \otimes V & \to & V \otimes U \\ & & u \otimes v & \mapsto & v \otimes u\ , \end{array}$$

which allow to permute the factors. One has $c_{V,U} \circ c_{U,V} = \mathrm{id}_{U\otimes V}$ as well as the identity

$$(c_{V,W} \otimes \mathrm{id}_U) \circ (\mathrm{id}_V \otimes c_{U,W}) \circ (c_{U,V} \otimes \mathrm{id}_W) = (\mathrm{id}_U \otimes c_{V,W}) \circ (c_{U,W} \otimes \mathrm{id}_V) \circ (\mathrm{id}_W \otimes c_{U,V})$$

Note that in this identify, we have tacitly used the identification $(U \otimes V) \otimes W \cong U \otimes (V \otimes W)$ from 3. This shows that $\mathrm{vect}(\mathbb{K})$ is a braided monoidal category.

6. For any pair of $\mathbb{K}$-vector spaces vector spaces V, W, the canonical map

$$\begin{array}{lcl} V^* \otimes W^* & \to & (V \otimes W)^* \\ \alpha \otimes \beta & \mapsto & (v \otimes w \mapsto \alpha(v) \cdot \beta(w)) \end{array}$$

is an injection. If both V and W are finite-dimensional, this is an isomorphism. (Give an example of a pair of infinite-dimensional vector spaces and an element that is not in the image!)

7. For any pair of $\mathbb{K}$-vector spaces V, W, the canonical map

$$\begin{array}{lcl} V^* \otimes W & \to & \mathrm{Hom}_{\mathbb{K}}(V, W) \\ \alpha \otimes w & \mapsto & (v \mapsto \alpha(v)w) \end{array}$$

is an injection. If both V and W are finite-dimensional, this is an isomorphism. (Give again an example of a pair of infinite-dimensional vector spaces and an element that is not in the image!)

B Glossary German-English

For the benefit of German speaking students, we include a table with German versions of important notions.

English	German
abelian Lie algebra	abelsche Lie-Algebra
absolutely simple object	absolut einfaches Objekt
additive tensor category	additive Tensorkategorie
adjoint functor	adjungierter Funktor
alternating algebra	alternierende Algebra
antipode	Antipode
associator	Assoziator
augmentation ideal	Augmentationsideal
autonomous category	autonome Kategorie
Boltzmann weights	Boltzmann-Gewichte
braid	Zopf
braid group	Zopfgruppe
braided tensor category	verzopfte Tensorkategorie
braided tensor functor	verzopfter Tensorfunktor
braided vector space	verzopfter Vektorraum
braiding	Verzopfung
character	Charakter
class function	Klassenfunktion
coaction	Kowirkung
code	Code
coevaluation	Koevaluation
coinvariant	Koinvariante
commutativity constraint	Kommutatitivitätsisomorphismus
convolution product	Konvolutionsprodukt, Faltungsprodukt
coopposed coalgebra	koopponierte Algebra
counitality	Kounitarität

English	German
derivation	Derivation
distinguished group-like element	ausgezeichnetes gruppenartiges Element
dominating family	dominierende Familie
Drinfeld center	Drinfeld Zentrum
Drinfeld double	Drinfeld-Doppel
enriched category	angereicherte Kategorie
enveloping algebra	einhüllende Algebra
error correcting code	fehlerkorrigierender Code
essentially small category	wesentlich kleine Kategorie
evaluation	Evaluation
exterior algebra	äußere Algebra
factorizable Hopf algebra	faktorisierbare Hopf-Algebra
fibre functor	Faserfunktor
forgetful functor	Vergissfunktor
free vector space	freier Vektorraum
fundamental groupoid	Fundamentalgruppoid
fusion category	Fusionskategorie
gate	Gatter
gauge transformation	Eichtransformation
group-like element	gruppenartiges Element
groupoid	Gruppoid
Hamming distance	Hamming-Abstand
Haar integral	Haarsches Maß
hexagon axioms	Hexagon-Axiome
Hopf algebra	Hopf-Algebra
isotopy	Isotopie
knot	Knoten
left adjoint functor	linksadjungierter Funktor
left integral	Linksintegral
left module	Linksmodul
link	Verschlingung
linking number	Verschlingungszahl
modular category	modulare Kategorie
modular element	modulares Element
monodromy element	Monodromie-Element
opposite algebra	opponierte Algebra
pentagon axiom	Pentagon-Axiom
pivotal category	pivotale Kategorie
primitive element	primitives Element
projective module	projektiver Modul
pullback of a representation	Pullback einer Darstellung

English	German
quantum circuit	Quantenschaltkreis
quantum code	Quantenkode
quantum gate	Quantengatter
quantum group	Quantengruppe
quasi-triangular bialgebra	quasitrianguläre Bialgebra
Reidemeister move	Reidemeister-Bewegung
representation	Darstellung
right adjoint functor	rechtsadjungierter Funktor
right integral	Rechtsintegral
semisimple algebra	halbeinfache Algebra
semisimple module	halbeinfacher Modul (der)
separable algebra	separable Algebra
simple module	einfacher Modul (der)
skew antipode	Schiefantipode
spherical category	sphärische Kategorie
surgery	Chirurgie
tensor unit	Tensoreins
trace	Spur
trefoil knot	Kleeblattknoten
triangle axiom	Dreiecksaxiom
trivial module	trivialer Modul (der)
unimodular Hopf algebra	unimodulare Hopf-Algebra
universal R-matrix	universelle R-Matrix
universal enveloping algebra	universelle einhüllende Algebra
universal quantum gate	universelles Quantengatter
unknot	trivialer Knoten, Unknoten
Yang-Baxter equation	Yang-Baxter-Gleichung
Yetter-Drinfeld module	Yetter-Drinfeld-Modul (der)

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