Chain-level TCFT – basic features and examples

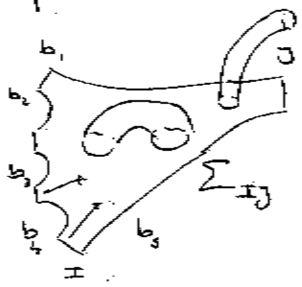
1. <u>What is a (chain-level) TCFT?</u>

Recall: TFT = functor F

$$\bigcirc \longmapsto \mathcal{H}_c$$

$${}^{b_1}_{b_2}I \longmapsto \mathcal{H}_o^{b_1 b_2} \qquad \text{vector sp.}$$

cobordism $\Sigma \longmapsto$ operator O_Σ
(top. surface)



$$\longmapsto O_\Sigma : \mathcal{H}_I \to \mathcal{H}_J$$

$$\mathcal{H}_I = \mathcal{H}_o^{b_1 b_2} \otimes \ldots \otimes \mathcal{H}_o^{b_4 b_5}$$

$$\mathcal{H}_J = \mathcal{H}_o^{b_1 b_5} \otimes \mathcal{H}_c$$

<u>Rem.</u> To define <u>amplitudes</u>, need bil. forms on $\mathcal{H}_c$, $\mathcal{H}_o^{b_1 b_2}$, e.g.

$$Z_\Sigma(v_1, \ldots, v_5) := \langle v_5 , O_\Sigma \cdot v_1 \otimes v_2 \otimes v_3 \otimes v_4 \rangle_{b_1 b_5}$$

Passage to chain-level TCFT:

Replace:

$\mathcal{H}_c$ by complex $\mathcal{E}_c \ldots \to \mathcal{H}_c^{n-1} \xrightarrow{Q_n} \mathcal{H}_c^n \xrightarrow{Q_{n+1}} \ldots$

$\mathcal{H}_o$ " " $\mathcal{E}_o^{b_1 b_2}$

Instead of cobordisms consider chains$^{**)}$ α in $\mathcal{M}_{IJ}^{*)}$

To a chain α associate operator $O_\Sigma^\alpha : \mathcal{E}_I \to \mathcal{E}_J$
$\in \mathrm{Hom}(\mathcal{E}_I \to \mathcal{E}_J)$

*) $\mathcal{M}_{IJ}$: Moduli sp. of Riemann surfaces with incoming bd. I, outgoing bd. J

**) chain : Linear comb. of polyhedra ("integration region")

The assignment $\alpha \to O_\Sigma(\alpha)$ satisfies

$$O_\Sigma(\partial\alpha) = \underbrace{(Q \cdot O_\Sigma)(\alpha)}_{:= O_\Sigma(\alpha) Q_I - (-)^{P_{IJ}} Q_J O_\Sigma(\alpha)} \qquad (B)$$

acts on $\otimes_k \mathcal{H}_k$ by graded Leibniz-rule

This is nothing but an abstract string (field) theory

$\mathcal{E}_c$: (off-shell) closed string states

$\mathcal{E}_o$: " open " "

Q : BRST-operator

cohomology of Q : on-shell states

$$Z^k_\Sigma(v_J, v_I) = \langle v_J, O_\Sigma(\alpha) \cdot v_I \rangle_{\mathcal{E}_J}$$

If v_I, v_J el. of Q-cohomology then

$Z^k_\Sigma(v_J, v_I)$: on-shell amplitude.

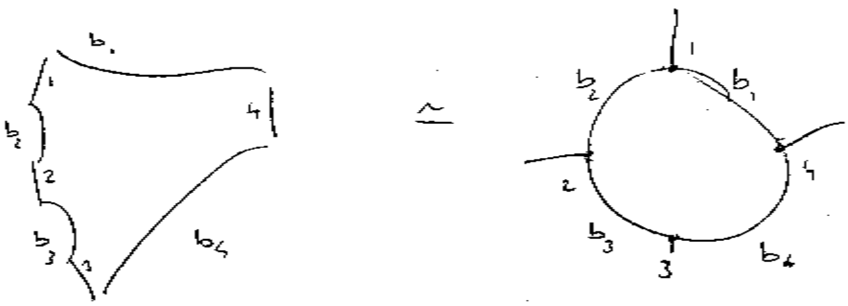
Amplitude version of (B):

Decoupling of BRST-trivial states up to contrib. from $\partial \mathcal{M}_{IJ}$

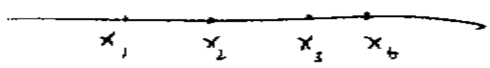
$$\mathcal{Z}^{\kappa}_{\Sigma}(Q_J v_J, v_I) + (-)^{P_{IJ}} \mathcal{Z}^{\kappa}_{\Sigma}(v_J, Q_I v_I) = \mathcal{Z}^{\partial\kappa}_{\Sigma}(v_J, v_I)$$

Examples:

1) Σ:



$\cong$

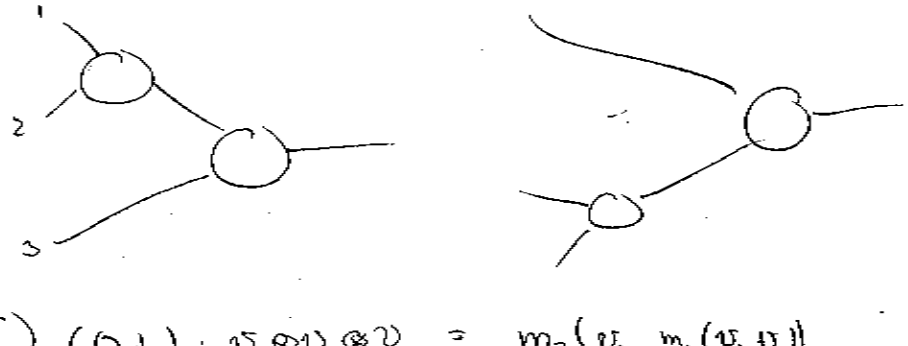


$$\mathcal{M}_{IJ} \supset \alpha := \{(x_1, x_2, x_3, x_4) \in (\mathbb{R} \cup \infty)^4 ;\ x_1 < x_2 < x_3 < x_4 ;\} / SL(2, \mathbb{R})$$

$$\simeq \{ x \in \mathbb{R} \setminus \{0,1\} \} \qquad x = \frac{x_{12} x_{34}}{x_{13} x_{24}}$$

$$\partial\alpha = (\partial\alpha)_{12} \cup (\partial\alpha)_{23}$$

$x_1 = x_2 \qquad x_2 = x_3$



$$O_{\Sigma}((\partial\alpha)_{12}) \cdot v_3 \otimes v_2 \otimes v_1 = m_2(v_3, m_2(v_2, v_1))$$

$$O_{\Sigma}((\partial\alpha)_{23}) \cdot \quad " \quad = m_2(m_2(v_3, v_2), v_1)$$

$$O_{\Sigma}(\alpha) \cdot v_3 \otimes v_2 \otimes v_1 =: m_3(v_3, v_2, v_1)$$

Let also $m_1(v) := Q \cdot v$

$$\Rightarrow \bigcirc_{\Sigma}(x) \cdot Q_I \cdot v_3 \otimes v_2 \otimes v_1$$

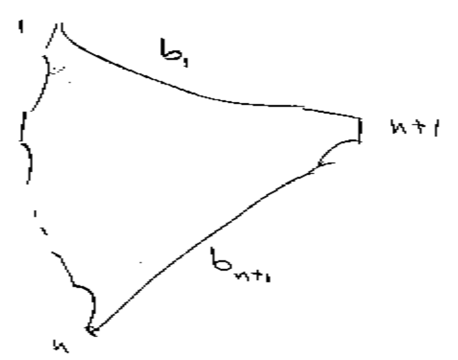
$$= m_3(m_1(v_3), v_2, v_1) + (-)^{\cdot\cdot} m_3(v_3, m_1(v_2), v_1) + (-)^{\cdot\cdot} m_3(v_3, v_2, m_1(v_1))$$

So (B) is rewritten as

$$m_2(m_2(v_3, v_2), v_1) - m_2(v_3, m_2(v_2, v_1))$$
$$= m_1(m_3(v_3, v_2, v_1))$$
$$+ m_3(m_1(v_3), v_2, v_1) + (-)^{|v_3|} m_3(v_3, m_1(v_2), v_1)$$
$$+ (-)^{|v_3|+|v_2|} m_3(v_3, v_2, m_1(v_1))$$

" m_2 is associative up to homotopy "

2) Σ :



uses cell decomp. of $\mathcal{M}_{0,n}$?

(B) $\Rightarrow \ldots \Rightarrow$

$$\sum_{\substack{k+l=n+1 \\ i+j=k-1}} (-)^{S_{i,l}} m_k(v_1, \ldots, v_i, m_l(v_{i+1}, \ldots, v_{i+l}), v_{i+l+1}, \ldots, v_n) = 0$$

Defining relations of an A_∞ - algebra/category

(2) Examples of TCFT ?

(S) Any bosonic String Theory, i.e.

(1) CFT *) with $c = 0$.

Vir-module $\mathcal{H}$, $[L_n, L_m] = (n-m) L_{n+m}$

(2) $Q : \mathcal{H} \to \mathcal{H}$ "BRST-operator"

$Q^2 = 0$

(3) Fields $G(z), \bar{G}(\bar{z})$,

$$G(z) = \sum_k G_k z^{-k-2}$$

s.t. $\{Q, G_n\} = L_n$, $\{Q, \bar{G}_n\} = \bar{L}_n$

$[L_n, G_m] = (n-m) G_{n+m}$

*) CFT : Functor $\mathcal{F}$

$$\bigcirc \longmapsto \mathcal{H}$$

cobordism Σ_Γ (cplx. str., μ?) $\longmapsto$ op. O_{Σ_Γ}

$$O_{\Sigma_\Gamma} : \mathcal{H}_I \longrightarrow \mathcal{H}_J$$

Amplitudes $Z_{\Sigma_\Gamma}(v_J, v_I) = \langle v_J, O_{\Sigma_\Gamma} v_I \rangle_{\mathcal{H}_J}$

Given (1)–(3), there ex. standard constr. of differential forms on $\mathcal{M}_{I,J}$

$$\Omega_\Sigma(v_J, v_I) \in H^*(\mathcal{M}_{IJ})$$

Amplitudes $\mathcal{Z}^{\alpha}_{\Sigma}(v_J, v_I) \equiv \int_X \Omega_\Sigma(v_J, v_I)$.

Main problem: Construction of op. $\mathcal{O}_{\Sigma_\gamma}$?

TA Top. str. – A model / Gromov - Witten inv.

X: Calabi - Yau mfd.

$H^*(X) \longleftrightarrow$ Q-cohom.

Consider case – no open bd.

– $v_J = \bigotimes_{k=m+1}^{n} \phi_k \quad v_I = \bigotimes_{k=1}^{m} \phi_k$

$\phi_k \in H^*(X)$

Claim: $\mathcal{Z}^{M_{IJ}}_{\Sigma}(v_J, v_I)$: generating fctn. of Gromov - Witten inv.

TB Top. Str. – B model

X: Calabi - Yau

$H^p_{\bar\partial}(X, \Lambda^q T^{(1,0)} X) \hookrightarrow$ Q-cohom.

Aim (Costello)

Construct A/B - model via corresp. open TCFT

A: Fukaya categ.

B: der. cat. of coh. sheaves on X.