Costello's Open-Closed Topological Conformal Field Theory

February 1, 2011

Contents

1 Health warning and intended audience

These notes were written to accompany a journal club for the Mathematical Physics Groups in Hamburg in January 2011. They were primarily meant as a memory jog for the speaker. They are more of an orientation than a full explanation. All formulae and explanations may contain mistakes and misunderstandings. Any corrections would be gratefully received.

2 Overall plan

Old notion that closed strings appear as poles in open string loop amplitudes. Are open strings more fundamental and can we define a closed string theory in terms of a purely open one?

For a class of topological theories the answer is yes. In this lecture we will see how an open topological conformal field theory is defined in terms of an A_∞ algebra (or category) which gives open string amplitudes via its product and tells us how to glue them.

If we consider infinitesimal deformations of that algebra which preserve its A_∞ structure, they turn out to be classified by the cyclic Hochschild cohomology. We can show that this is in 1-to-1 correspondence with the closed string cohomology of the theory and defines for us bulk-boundary amplitudes. Categorifying everything, we can then define amplitudes by gluing elementary disks and punctured disks.

Therefore can define topological closed theories at all genus using open string, such as B model on Calabi-Yau X at all genera from derived category of coherent sheaves on X.

You come to me with an open TCFT, I give you a closed TCFT.

2.1 Sources

Interpolate between Herbst-Lazaroiu-Lerche hep-th/0402110 [3] and Costello math/0412149, math/0601130 [6, 5].

2.2 Notational matters

Blend above. $\mathbb{K}$ is a field of characteristic 0.

2.3 TFT revision

Professor Schweigert defined in first lecture for us a TFT. A TFT is a warm-up for a TCFT.

TFT is functor from the geometric category of cobordisms to linear category of complex vector spaces.

So we have a topological 2d surface interpolating between incoming and outgoing open and closed boundary components, with labelled free boundaries. The objects are the boundary components and the morphisms are the cobordisms. Because it is a category we can compose morphisms (in an associative way).

The functor maps the objects to the open and closed vector spaces $\mathcal{H}_C$ and $\mathcal{H}_O^{\lambda_1\lambda_2}$. It maps the surfaces to a linear operator on the vector spaces.

This gives the TFT the structure of a Frobenius algebra. A finite dimensional, unital (contains identity element), associative algebra A defined over a field k is said to be a Frobenius algebra if A is equipped with a nondegenerate bilinear form $\sigma : A \times A \to k$ that satisfies the following cyclicity condition: $\sigma(ab, c) = \sigma(a, bc)$. This bilinear form is called the Frobenius form of the algebra.

For us a closed string boundary is mapped to a vector space $\mathcal{C}$ over a field k.

Pants diagrams correspond to a product on the vector space.

The product is associative and commutative for a closed string theory.

2 boundaries to nothing gives bilinear form.

For open string label parts of boundary with a new complex vector space $\mathcal{O}$.

$\mathcal{O}$ has a not-necessarily commutative product.

Get 'zipper' between closed and open boundaries, a unital morphism of algebras $i_* : C \to Z(\mathcal{O})$.

Get Cardy formula for annulus, relation between description of annulus $\pi = i_* \circ i^*$.

3 Definition of TCFT

Have a set $\Lambda = \{\lambda_i\}$ of D-branes which label the free boundaries on the worldsheet, i.e. have "boundary changing open strings".

Have a (differential graded symmetric monoidal) category $\mathcal{OC}_\Lambda$ whose objects are open-closed boundary conditions (C, O, s, t) where C, O are integers and $s, t : O \to \Lambda$ are maps. Morphism are now chains on the moduli space of Riemann surfaces with these boundaries.

Category is a collection of objects with morphisms between them. Important point: can compose morphisms.

A TCFT is a symmetric monoidal (h-split) functor from this space to the category of complexes $F : \mathcal{OC}_\Lambda \to \text{Comp}$.

The complexes will give us the usual BRST complexes of string theory.

[TFT is much simpler object: a functor from topological spaces category $\mathcal{M}_\Lambda$ to category of vector spaces.]

I.e. given a chain α which is a morphism in $\mathcal{OC}_\Lambda$ it gives a map

$$F(\alpha) : F(I) \to F(J) \tag{1}$$

for I incoming and J outgoing boundary conditions. Gluing Riemann surfaces together corresponds to composition of maps and disjoint union corresponds to tensor products.

Furthermore it respects the differential $F(d\alpha) = dF(\alpha)$.

[A *monoidal category* (or tensor category) is a category C equipped with a bifunctor called the tensor or monoidal product

$$\otimes : C \times C \to C \tag{2}$$

which is associative (up to a natural isomorphism).

A monoidal functor between monoidal categories is *split* if the maps $F(a) \otimes F(b) \to F(a \otimes b)$ are isomorphisms (the fact that there is such a relation between the tensor products is the symmetric monoidal part). If the maps are only quasi-isomorphisms then the functor is homologically split, i.e. h-split. So in particular the OTCFT functor satisfies $F(O, s, t) \cong \otimes F(\{s(i), t(i)\})$.

Q: What does quasi-isomorphic mean? There is a morphism $A \to B$ of chain complexes such that the induced morphisms $H_n(A) \to H_n(B)$ of homology groups are isomorphisms for all n.

For us this means disjoint union of surfaces and addition of integers (C, O).]

Subcases: purely open $\mathcal{O}_\Lambda$ and purely closed $\mathcal{C}$.

For open strings we have $F(O = 1, s(1) = \lambda_i, t(1) = \lambda_{i+1}) = C_O^{\lambda_i \lambda_{i+1}}$ (i.e. the complex).

Want to show that an open TCFT $F_O : \mathcal{O}_\Lambda \to \text{Comp}$ is the same as a Calabi-Yau A_∞ category with set of objects Λ (up to quasi-isomorphism).

[Full statement: the category of open TCFTs with branes Λ is homotopy equivalent to the category of (unital) extended Calabi-Yau A_∞ categories with objects Λ.

Categories A and B are homotopy equivalent if there exist functors $A \to B$ and $B \to A$ which are inverse to each other up to quasi-isomorphism.]

Furthermore from this open TCFT we can build an open-closed TCFT $F_{OC} : \mathcal{OC}_\Lambda \to \text{Comp}$ such that it maps a single closed string boundary to a complex whose cohomology [homology for Costello since his degree is defined negative to usual] is the Hochschild cohomology of the CY A_∞ category, i.e. $H^*(F_{OC}(C=1)) = HH^*(F_O)$.

But what does composition of morphisms mean now? How do we glue chains in the moduli space?

For gluing of open string boundaries this is solved by Costello's disk/ribbon graph decomposition of the moduli space.

[Full statement includes twisting coefficients of chains on moduli space by a local system det on the moduli space. Useful for applications.]

3.1 Hidden definition of topological string

$T(z) = \{Q, G(z)\}$, $T(z)$ and fermionic current $G(z)$ both spin 2.

Another important aspect is descent equations that allow you to build up arbitrary forms on the worldsheet from a scalar $\mathcal{O}^{(0)}$

$$\begin{aligned}\{Q, \mathcal{O}^{(0)}\} &= 0 \\ \{Q, \mathcal{O}^{(1)}\} &= d\mathcal{O}^{(0)}\end{aligned} \tag{3}$$

4 Moduli space from kissing disks

In Vito's talk we constructed an orbi-cell complex $D_{g,h,r,s} \hookrightarrow \bar{\mathcal{N}}_{g,h,r,s}$, which is a weak homotopy equivalence. [For map between them $f : D_{g,h,r,s} \to \bar{\mathcal{N}}_{g,h,r,s}$ there is an isomorphism between the homotopy groups - i.e. roughly they are the same shape. For any Riemann surface in $\bar{\mathcal{N}}_{g,h,r,s}$ can construct a deformation retraction onto an orbi-cell in orbi-cell complex $D_{g,h,r,s}$.] $\bar{\mathcal{N}}_{g,h,r,s}$ is a partial compactification, i.e. include open string degeneration nodes, but no bulk kissing nodes; all marked points are distinct from nodes and each other.

Space must always have boundary $h > 0$; exclude unstable curves (without finite automorphism group), which go to a point ($D_{0,1,\{0,1,2\},0}$, $D_{0,2,0,0}$ and $D_{0,1,0,1}$). [The unstable ones we will put back in later with moduli spaces as just points, so we can use them as identities, etc.]

$D_{g,h,r,s}$ is generated by disks, $D_{0,1,n,0}$ for $n \geq 3$ and $D_{0,1,n,1}$ for $n \geq 1$.

The way the disks are glued corresponds to a ribbon graph $\gamma \in \Gamma_{g,h,r,s}$, where vertices are split $V(\gamma) = V_0(\gamma) \sqcup V_1(\gamma)$, where these is an isomorphism $V_1(\gamma) \cong \{1, \dots s\}$. Edges are nodes. Valencies in V_0 are ≥ 3 and in V_1 are ≥ 1.

For each graph $\gamma \in \Gamma_{g,h,r,s}$ get an orbi-cell $\prod_{v \in V(\gamma)} X(v)/\text{Aut}(\gamma)$ [NB: original $X(\gamma)$ are cells]

Orbi-cell is quotient of a cell by a finite group, $\text{Aut}(\gamma)$, which preserves the labelling.

Chains correspond to linear combinations of polyhedra together with an orientation on each orbi-cell (order set of vertices and germs of each edge).

The boundary operator corresponds to degenerating surfaces to allow more nodes.

The boundary in this chain complex is given by summing over all ways of splitting a vertex in V_0 into two legal vertices in V_0, and splitting a vertex in V_1 into a legal vertex in V_1 and a legal vertex in V_0.

It respects the composition of maps.

[NB: have rational homology for orbi-cell complex but integer homology for ordinary cell complex.]

5 A_∞ and disk amplitudes

5.1 Generators and relations for $\mathcal{D}_{\Lambda,open}$

Use above ribbon graph model to construct a category $\mathcal{D}_{\Lambda,open}$, whose objects are the same as for $\mathcal{O}_\Lambda$ but whose morphisms are the chains defined above. As dgsm categories they are quasi-equivalent.

$\mathcal{D}_{\Lambda,open}$ is freely generated by $D^+_{\Lambda,open}$ (disks with all incoming except for one outgoing) and the disks with two incoming or two outgoing boundaries [provide metric], modulo relation that gluing these two disks gives identity, and cyclicity.

$D(\lambda_0,\dots\lambda_{n-1})$ has only incoming boundaries, corresponds to the object in $\mathcal{D}_{open}(\alpha,\beta)$ with $\alpha = (O = n, C = 0, s(i) = \lambda_i, t(i) = \lambda_{i+1})$ and $\beta = 0$. They must be cyclic (up to a sign).

D^+_{open} is freely generated by the disks $D^+(\lambda_0,\dots\lambda_{n-1})$, which have all marked points incoming except that between λ_{n-1} and λ_0, modulo relation that $D^+(\lambda_i)$ kills it when glued at λ_i,λ_i point. [This will correspond to A_∞ unital condition later.]

The boundary operator is

$$dD(\lambda_0,\dots\lambda_{n-1}) = \sum_{\substack{0\le i\le j\le n-1\\ j-i\ge 2}} \pm D(\lambda_i,\dots\lambda_j) * D(\lambda_j,\dots\lambda_i) \tag{4}$$

where $*$ means glue along open marked points between λ_i and λ_j.

5.2 A_∞ categories

For Sebastian the A_∞ algebra was defined in terms of n-linear maps $m_n : A^{\otimes n} \to A$ for a $\mathbb{Z}$-graded vector space A, with a condition on the products.

Here we generalise it to take account of boundary condition changing sectors, generalising the to a A_∞ category. The objects are the branes Λ and for each pair we have a complex of ($\mathbb{K}$) vector spaces $C_o^{\lambda_0\lambda_1}$, which is the morphism of the A_∞ category.

So maps are for $n \ge 2$

$$m_n : C_o^{\lambda_0,\lambda_1} \otimes \cdots \otimes C_o^{\lambda_{n-1},\lambda_n} \to C_o^{\lambda_0,\lambda_n} \tag{5}$$

[degree $2-n$ like Sebastian and physics literature, different from Costello's $n-2$]. Differential on complex $C_o^{\lambda_0\lambda_1}$ is $m_1 \equiv d \equiv Q$.

Maps satisfy A_∞ condition

$$\sum_{0\le i\le j\le n-1} \pm m_{n-j+i}(\psi_0,\dots,\psi_{i-1}, m_{j-i+1}(\psi_i\otimes\cdots\otimes\psi_j),\psi_{j+1},\dots,\psi_{n-1}) = 0 \tag{6}$$

Will abbreviate to $\sum \pm m(\cdots m \cdots)$ in future.

For a Calabi-Yau A_∞ category require in addition a closed non-degenerate bilinear pairing

$$\langle\cdot,\cdot\rangle : C_o^{\lambda_0,\lambda_1} \otimes C_o^{\lambda_1,\lambda_0} \to \mathbb{K} \tag{7}$$

This is the BPZ inner product on open string states. It satisfies a cyclic identity

$$\langle\psi_0 \dots \psi_n\rangle_\alpha \equiv \langle m_n(\psi_0 \otimes \dots \otimes \psi_{n-1}), \psi_n\rangle = \pm \langle m_n(\psi_1 \otimes \dots \otimes \psi_n), \psi_0\rangle \tag{8}$$

We've introduced a physicist style correlation function $\langle\psi_0 \dots \psi_n\rangle_\alpha$ for ease of notation; α refers to the chain in moduli space corresponding to the disk with $(n+1)$ marked boundary points.

Require to be unital too, i.e. element $\mathbb{I}_{\lambda_i}$ for each λ_i, which is identity for $n = 2$ and vanishes in products m_n for $n \geq 3$.

So we see that, if we include $Q = m_1$ then the condition of compatibility with the differential $dF(\alpha) = F(d\alpha)$

$$\langle[Q, \psi_0 \dots \psi_n]\rangle_\alpha = \langle\psi_0 \dots \psi_n\rangle_{d\alpha} \tag{9}$$

is the same as the A_∞ product condition (6) if we separate out the terms containing $m_1 = Q$.

For $j - i = 0$ get Leibniz m_1 on the α_i inside an m_n. For $j = n-1, i = 0$ get $m_1(m_n(\cdots))$ which we can cycle around to transfer to $\langle m_n(\cdots), m_1(\cdots)\rangle$.

[In HLL terms the open string operators are already on-shell, i.e. $m_1(\alpha_i) = 0$ so only have A_∞ conditions for $n \geq 2$ - this is a *minimal* A_∞ algebra. With $n \geq 1$ get a strong A_∞ algebra, which is what we have here.]

[I think this needs a sum over the $\psi_j \omega^{\psi_j, \psi_i} \psi_i$ as intermediate states with an inverse of the metric ω for the gluing?]

Lemma 7.3.1: A unital CY A_∞ category with objects Λ is the same as a split symmetric monoidal functor $F : \mathcal{D}_{\Lambda,open} \to \mathrm{Comp}_{\mathbb{K}}$.

Since $\mathcal{D}_{\Lambda,open}$ is quasi-isomorphic to $\mathcal{O}_\Lambda$ we're done.

6 Hochschild to closed strings

$\mathcal{D}_\Lambda$ is a category with incoming open boundaries and outgoing open and closed boundaries.

The punctured disk $A(\lambda_0, \dots \lambda_{n-1})$ has $\alpha = (O = n, C = 0, s(i) = \lambda_i, t(i) = \lambda_{i+1})$ and $\beta = (O = 0, C = 1)$.

For once-punctured disk (n incoming open, 1 outgoing closed) the boundary operator is

$$dA(\lambda_0, \dots \lambda_{n-1}) = \sum_{\substack{0 \leq i \leq j \leq n-1 \\ \text{legal}}} \pm A(\lambda_i, \dots \lambda_j) * D(\lambda_j, \dots \lambda_i) \pm D(\lambda_i, \dots \lambda_j) * A(\lambda_j, \dots \lambda_i) \tag{10}$$

where $*$ means glue along open marked points between λ_i and λ_j; we restrict to legal components.

Deformation of A_∞ products to other A_∞ products given by cyclic Hochschild cohomology. This corresponds to BRST cohomology for closed string state.

Deform product by infinitesimal with $t^2 = 0$ for $n \geq 1$

$$m_n^t = m_n + t\Phi_n \tag{11}$$

Φ_n is same type of object as m_n as in a map (5), and obeys the cyclicity constraint

$$\langle\Phi_n(\psi_0, \dots \psi_{n-1}), \psi_n\rangle = \pm \langle\Phi_n(\psi_1, \dots \psi_n), \psi_0\rangle \tag{12}$$

We now impose the condition that m_n^t satisfies A_∞ relations

$$0 = \sum \pm m^t(\cdots m^t \cdots)$$
$$= \sum \pm m(\cdots m \cdots) + t\sum \pm m(\cdots \Phi \cdots) + t\sum \pm \Phi(\cdots m \cdots) \tag{13}$$

The first term follows from the fact that m are products from an A_∞ category. The second term defines a differential on Φ with $\delta^2 = 0$

$$\delta(\Phi)_n(\psi_0, \cdots \psi_{n-1}) = \sum \pm m(\cdots \Phi \cdots) + \sum \pm \Phi(\cdots m \cdots) \tag{14}$$

Elements of homology mean that m_n^t satisfies A_∞.

[Q: What does δ map from/to? $\Phi = \sum_{n=0}^\infty \Phi_n \in \oplus_{n=0}^\infty CC^n(H_O)$ where $CC^n(H_O) = \mathrm{Hom}(C_o^{\lambda_0\lambda_1} \otimes \cdots \otimes C_o^{\lambda_{n-1}\lambda_n}, C_o^{\lambda_0\lambda_n})$, the latter being the Hochschild complex. δ always maps from $CC^{\leq n}$ to CC^n, i.e. it mixes. Easier to say for case $m_n = 0$ for $n \neq 2$ where no boundary changing conditions and $\delta : CC^{n-1}(A) \to CC^n(A)$ where

$$\delta f(a_1, \ldots a_n) = m_2(a_1, f(a_2, \ldots a_n)) + (-1)^{n+1} m_2(f(a_1, \ldots a_{n-1})a_n) + \sum_{k=1}^{n-1} f(a_1, \ldots m_2(a_k, a_{k+1}), \ldots a_{n-1}) \tag{15}$$

Note that in Costello [6] page 45 have only $m_2 = \circ$ and $m_1 = d$ turned on, which adds extra terms. This inclusion of the differential $\delta + Q$ also is there for differential-graded algebra in Kapustin-Rozansky [1] and Möller-Sachs [2] who both have $m_{n\geq 3}$ turned off.]

Now compare with differential on the punctured disk (10). It has exactly the same form. If we identify Φ with the bulk-boundary correlator with only one closed string insertion ϕ

$$F(\alpha)(F(I)) = \langle \Phi(\psi_0, \ldots \psi_{n-1}), \psi_n \rangle = \langle \psi_0 \cdots \psi_n \phi \rangle_\alpha \tag{16}$$

How is this consistent with our assertion that it is compatible with the differential

$$\langle [Q, \psi_0 \cdots \psi_n \phi] \rangle_\alpha = \langle \psi_0 \cdots \psi_n \phi \rangle_{d\alpha} \tag{17}$$

Well, we're almost there because separating out the m_1 terms, which is Q on open strings states,

$$0 = t\sum \pm m(\cdots \Phi \cdots) + t\sum \pm \Phi(\cdots m \cdots)$$
$$= \langle \psi_0 \cdots \psi_n \phi \rangle_{d\alpha} - \langle [Q, \psi_0 \cdots \psi_n] \phi \rangle_\alpha \tag{18}$$

Thus the differential is only compatible if

$$\langle \psi_0 \cdots \psi_n [Q, \phi] \rangle_\alpha = 0 \tag{19}$$

i.e. if $[Q, \phi] = 0$, which shows that ϕ is closed under Q; bit of extra work to show it's in cohomology of Q.

Thus closed string cohomology $\leftrightarrow$ cyclic Hochschild cohomology. In functor language $H^*(F_{OC}(C = 1)) = HH^*(F_O)$.

7 Closed string amplitudes

For a single disk with closed string insertion, this is defined by allowed Hochschild deformation, i.e. choose direction in deformation of A_∞ algebra, and this linear perturbation gives you the bulk-boundary amplitude.

For more general surface just get the result by gluing, inherited from categorical construction.

This is essentially what is hopefully encoded in Costello's functorial language.

8 Category theory magic

Have the embedding maps $i : \mathcal{O}_\Lambda \to \mathcal{OC}_\Lambda$ and $j : \mathcal{C} \to \mathcal{OC}_\Lambda$.

Given an open TCFT (Λ, F) it is possible to push forward the functor $F : \mathcal{O}_\Lambda \to \mathrm{Comp}_\mathbb{K}$ to $\mathbb{L}i_* F : \mathcal{OC}_\Lambda \to \mathrm{Comp}_\mathbb{K}$, then pull back to $j^* \mathbb{L}i_* F : \mathcal{C} \to \mathrm{Comp}_\mathbb{K}$.

9 Examples

Topological Landau-Ginzburg models [1]: $A_\infty \sim$ CDG-modules over a certain commutative CDG algebra.

Topological A model: $A_\infty \sim$ Fukaya category.

Topological B model: $A_\infty \sim$ derived category of coherent sheaves on manifold X.

10 From TCFT to open string field theory

See [2] for recent work in OSFT on relating deformations of A_∞ to closed string cohomology.

Q: What allows you to go from full moduli space integral in OSFT to integration over orbi-cell complex in TCFT which is only homotopy equivalent to the moduli space?

Complexes in OSFT are infinite dim, whereas for Costello they are finite dimensional. In OSFT must take semi-relative cohomology for physical states, which in addition satisfy $(b_0 - \bar{b}_0) |\Phi\rangle = 0$.

In OSFT include propagators to hit ALL points in moduli space with 3-valent vertices. Cover it once, where higher vertices correspond to collapsing vertices to zero length.

Factorisation over the open string poles is manifest, it corresponds to propagator lengths T_i going to infinity.

Can also formally factorize over closed string poles, by adding an extra closed string vertex overlap.

References

[1] A. Kapustin and L. Rozansky, "On the relation between open and closed topological strings," Commun. Math. Phys. **252** (2004) 393 [arXiv:hep-th/0405232].

[2] N. Moeller and I. Sachs, "Closed String Cohomology in Open String Field Theory," arXiv:1010.4125 [hep-th].

[3] M. Herbst, C. I. Lazaroiu and W. Lerche, "Superpotentials, A(infinity) relations and WDVV equations for open topological strings," JHEP **0502** (2005) 071 [arXiv:hep-th/0402110].

[4] M. Penkava and A. S. Schwarz, "A(infinity) algebras and the cohomology of moduli spaces," arXiv:hep-th/9408064.

[5] K. Costello, "A dual point of view on the ribbon graph decomposition of moduli space," [arXiv:math/0601130].

[6] K. Costello, "Topological conformal field theories and Calabi-Yau categories," [arXiv:math/0412149].

[7] C. Hofman, "On the open-closed B-model," JHEP **0311** (2003) 069 [arXiv:hep-th/0204157].