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LECTURES ON THE ANTIFIELD-BRST FORMALISM FOR GAUGE THEORIES*

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Abstract. The Lagrangian approach to the BRST symmetry based on the antifield formalism is reviewed. First, the concept of "open algebra" is clarified. It is then explicitly indicated how gauge invariance is incorporated in the theory through the BRST cohomology at ghost number zero. This result holds for both the non-gauge fixed and gauge fixed versions of the BRST symmetry in Lagrangian form. The properties of the Lagrangian integration measure are discussed and the role of the Schwinger-Dyson equation is stressed. The problem of spacetime locality of the gauge fixed action is also briefly addressed. The discussion is illustrated in the cases of electromagnetism and of free p-form gauge fields.

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1. INTRODUCTION

It has been recognized for some time now that the BRST method provides one of the most powerful tools for quantizing theories endowed with a local gauge freedom. This method is extremely useful not only in the path integral approach, but also in the operator formalism.

A striking development in the last years has been the emergence of many gauge-theoretical models for which the BRST method appears to be the only satisfactory (covariant) method of quantization. These models are characterized by the fact that the gauge transformations only close on-shell: if one compute the commutator of two infinitesimal gauge transformations -denoted by $\delta_\epsilon \phi^i$ and $\delta_\eta \phi^i$- one finds a transformation of the same type -denoted by $\delta_{[\epsilon,\eta]} \phi^i$ - only modulo the equations of motion,

$$[\delta_\epsilon, \delta_\eta]\phi^i = \delta_{[\epsilon,\eta]}\phi^i + \text{ field equations.}$$

Models with "open gauge algebras" include supergravity theories [**1**], the Green-Schwarz superstring [**2**] and the superparticle [**3**], among others.

If one determines the gauge-fixed action in covariant gauges by means of the standard Faddeev-Popov method [**4**] -designed for true gauge groups-, one gets an incorrect (non-unitary and non-gauge independent) answer in the open algebra case. Hence, it is not enough to add to the gauge invariant action the standard gauge fixing Faddeev-Popov determinant terms.

To determine the correct path integral, two approaches can be used and are both characterized by the fact that they strongly rely on the BRST symmetry. The first one is based on the Hamiltonian formalim [**5**]. The second one starts from the Lagrangian formulation [**6–8**]. This second approach is slightly less general and appears to be formally less precise in what concerns the local measure in the path integral.[1] However, it has the definite advantage of preserving manifest covariance throughout and on this ground, deserves to be studied.

The Hamiltonian construction of the BRST symmetry has been reviewed elsewhere [**9**]. We will therefore analyze here only the Lagrangian antifield formalism of Batalin and Vilkovisky [**6**].

The main goals of these lectures are (i) to derive the correct gauge fixed Lagrangians containing all the necessary ghost vertices; and (ii) to explicitly show how the derivation incorporates throughout gauge invariance. We will in particular indicate why BRST invariance can be used as a substitute for gauge invariance. Both aspects (i) and (ii) reveal interesting algebraic and geometrical features which, we believe, provide the key to the rationale behind the antifield formalism.

[1] It is not unconceivable that this shortcoming could be overcome some day by pure Lagrangian means, without having to resort to the Hamiltonian.
The validity of the Lagrangian path integral also appears to restrict the form of the Lagrangian (see comments in section 8.6).

The gauge fixed Lagrangians obtained by BRST methods generate the correct set of Feynman diagrams. As such, they provide the appropriate starting point for studying the perturbative quantum properties of the theory (renormalization, anomalies). There also, the BRST symmetry proves to be a crucial tool in the analysis. These issues, however, will not be addressed in the lectures.

2. Structure of the gauge symmetries

The structure of the gauge symmetries may appear to be somewhat puzzling in the "open algebra" case, as it may wrongly be felt that the group structure is completely lost.

Our first task, therefore, is to clarify the structure of the gauge symmetries in the general case.

2.1. Action principle, equations of motion.

Our starting point is the action $S_0[\phi^i]$, which we assume to be a local functional of the fields,

$$S_0[\phi^i] = \int d^D x \, \mathcal{L}_0(\phi^i, \partial_\mu \phi^i, \partial_{\mu_1 \mu_2} \phi^i, \dots, \partial_{\mu_1 \mu_2, \dots \mu_k} \phi^i) \tag{1}$$

As it will be seen, the subsequent structure is completely encoded in the action and is not an independent input.

The field equations are

$$\frac{\delta S_0}{\delta \phi^i(x)} \equiv \frac{\delta \mathcal{L}_0}{\delta \phi^i}(x) = 0, \tag{2a}$$

where the "variational derivatives" $\dfrac{\delta \mathcal{L}_0}{\delta \phi^i}$ of the Lagrangian density $\mathcal{L}_0$ are defined by

$$\frac{\delta \mathcal{L}_0}{\delta \phi^i} = \frac{\partial \mathcal{L}_0}{\partial \phi^i} - \partial_\mu \frac{\partial \mathcal{L}_0}{\partial(\partial_\mu \phi^i)} + \partial_\mu \partial_\nu \frac{\partial \mathcal{L}_0}{\partial(\partial_{\mu\nu} \phi^i)} - \dots + (-1)^k \partial_{\mu_1 \mu_2 \cdots \mu_k} \frac{\partial \mathcal{L}_0}{\partial(\partial_{\mu_1 \mu_2 \cdots \mu_k} \phi^i)}. \tag{2b}$$

The derivatives $\dfrac{\delta S_0}{\delta \phi^i(x)}$ are referred to as the "functional derivatives" of S_0.

For notational simplicity, we take the fields to be commuting. Our considerations can straightforwardly be extended to theories with fermions provided the appropriate phases are included.

2.2. Gauge transformations.

A gauge transformation is a transformation (i) that can be prescribed independently at each space-time point and (ii) that leaves the action invariant up to a surface term. So, gauge transformations are parametrized by arbitrary spacetime functions (as opposed to rigid symmetry transformations) and typically take the form

$$\delta_\epsilon \phi^i = \overline{R}^i_\alpha \epsilon^\alpha + \overline{R}^{i\mu}_\alpha \partial_\mu \epsilon^\alpha + \cdots + \overline{R}^{i\mu_1,\dots,\mu_s} \partial_{\mu_1,\dots,\mu_s} \epsilon^\alpha. \tag{3}$$

Here, the coefficients $\overline{R}^i, \overline{R}^{i\mu}, \cdots, \overline{R}^{i\mu_1,\dots,\mu_s}$, depend on ϕ^i and their derivatives up to a finite order and $\epsilon^\alpha(x)$ are arbitrary gauge parameters. Invariance of the acion under (3) means that for any choice of $\epsilon^\alpha(x)$, one has

$$\delta_\epsilon \mathcal{L}_0 = \partial_\mu K^\mu_\epsilon \tag{4}$$

for some local functions $K^\mu_\epsilon(\phi^i, \partial_\nu \phi^i, \partial_{\nu_1\nu_2}\phi^i, \dots, \partial_{\nu_1\nu_2,\dots\nu_t}\phi^i, \epsilon^\alpha, \dots, \partial_{\nu_1\nu_2\dots\nu_n}\epsilon^\alpha)$.

2.3. Noether identities.

It is at this point convenient to adopt De Witt's condensed notations, where the indices i, α also include x (i.e., $i \leftrightarrow (i,x), \alpha \leftrightarrow (\alpha, x)$) and a summation over i, α implies an integration over x [**10**].

In these notations, (3) becomes

$$\delta_\epsilon \phi^i = R^i_\alpha \epsilon^\alpha \left(\Longleftrightarrow \delta_\epsilon \phi^i(x) = \int d^D y \, R^i_\alpha(x,y) \epsilon^\alpha(y) \right) \tag{5a}$$

with

$$R^i_\alpha(x,y) = \overline{R}^i_\alpha(x)\delta(x-y) + \overline{R}^{i\mu}_\alpha(x)\delta_{,\mu}(x-y) + \cdots . \tag{5b}$$

The Noether identities on the field equations are derived by starting from the invariance of the action,

$$\delta_\epsilon S_0 = \frac{\delta S_0}{\delta \phi^i} \delta_\epsilon \phi^i = \frac{\delta S_0}{\delta \phi^i} R^i_\alpha \epsilon^\alpha = 0. \tag{6a}$$

Since (6a) holds for any function ϵ

$$\begin{aligned} \frac{\delta S_0}{\delta \phi^i} R^i_\alpha = 0 &\Longleftrightarrow \int \frac{\delta S_0}{\delta \phi^i(x)} R^i_\alpha(x,y) \, d^D x = 0 \\ &\Longleftrightarrow \left(\frac{\delta \mathcal{L}_0}{\delta \phi^i} \overline{R}^i_\alpha - \left(\frac{\delta \mathcal{L}_0}{\delta \phi^i} \overline{R}^{i\mu}_\alpha\right)_{,\mu} + \dots \right)(y) = 0 \end{aligned} \tag{6b}$$

[Strictly speaking, there could be a surface term in (6a). This terms vanishes if the gauge parameters are zero outside some finite domain, as one can assume. The local identities at y inferred from (6a) are thus certainly valid under this assumption. However, as these identities are local, they do not depend on the behaviour of ϵ^α away from y (actually, in this case, they do not even depend on ϵ^α at all), and so, they are clearly valid without the restriction that ϵ^α should vanish outside some finite domain. This is a useful line of reasoning that is frequently followed in deriving local identities].

One consequence of the Noether identities (6b) is that the field equations are not independent. This is of course all right, as the existence of a gauge symmetry implies the presence of arbitrary functions in the general solution of the equations of motion which must then underdetermine $\phi^i(x)$.

2.4. Gauge group.

For a given action functional, there will be a certain number of gauge transformations. What is the structure of the set containing all the gauge transformations?

One thing that can be said without having to make any calculation is that the infinitesimal gauge transformations form a Lie algebra.[2] There is no escape to that result because (invertible) transformations leaving something (here the action) invariant *always obey the group axioms*. The group of all gauge transformations is denoted by $\overline{\mathcal{G}}$ in the sequel.

The unconvinced reader may easily check that if $\delta_\eta \phi^i$,

$$\delta_\eta \phi^i = S^i_A \eta^A$$

is another gauge transformation ($\delta_\eta S_0 = 0$), then both $\lambda \delta_\epsilon \phi^i + \mu \delta_\eta \phi^i$, ($\lambda, \mu \in \mathbb{R}$) and $[\delta_\epsilon, \delta_\eta]\phi^i = \delta_\epsilon(\delta_\eta \phi^i) - \delta_\eta(\delta_\epsilon \phi^i)$ obey (6a). Furthermore, $[\lambda \delta_\epsilon, \delta_\eta] = \lambda[\delta_\epsilon, \delta_\eta]$, $\lambda \in \mathbb{R}$. So, one clearly has a Lie algebra structure.

What is then meant by "open gauge algebras"? In order to precisely answer this question, it is necessary to introduce some new concepts. The guiding line of the next developments is to determine the minimum number of independent Noether indentities.

2.5. Trivial gauge transformations.

Consider the following transformations,

$$\delta_\mu \phi^i = \mu^{ij} \frac{\delta S_0}{\delta \phi^j} \tag{7a}$$

where μ^{ij} is an arbitrary antisymmetric function

$$\mu^{ij} = -\mu^{ji} \tag{7b}$$

Exercise: write explicitly (7a) and (7b) for

$$\mu^{ij}(x,y) = k_1^{ij}(x)\,\delta(x,y) + k_2^{ij\mu}(x)\,\delta_{,\mu}(x,y) + \ldots + k_s^{ij\mu_1\ldots\mu_{s-1}}(x)\,\delta_{,\mu_1\ldots\mu_{s-1}}(x,y).$$

The arbitrary functions $k_1, \ldots, k_s$ may involve the fields and their derivatives up to some finite order.

It is easy to see that (7) leaves the action S_0 invariant

$$\delta S_0 = \frac{\delta S_0}{\delta \phi^j} \frac{\delta S_0}{\delta \phi^i} \mu^{ij} = 0. \tag{8}$$

[2] Accordingly, the finite gauge transformations formally form a Lie group (formally because the gauge Lie algebras are infinite dimensional continuous).

This is because the product $\frac{\delta S_0}{\delta \phi^i}\frac{\delta S_0}{\delta \phi^j}$ is symmetric in i,j while μ^{ij} is antisymmetric. So, (7) defines a gauge transformation.[3]

The commutator of any gauge transformation of the type (7a) with an arbitrary gauge transformation is a transformation of the type (7a). Indeed, if $\delta S_0 = 0$ for $\delta_t \phi^i = t^i$,

$$\frac{\delta S_0}{\delta \phi^i} t^i = 0 \tag{9}$$

then one finds, using (9),

$$[\delta_\mu, \delta_t]\phi^i = \left(\frac{\delta t^i}{\delta \phi^k}\mu^{kj} - \frac{\delta t^j}{\delta \phi^k}\mu^{ki} - t^k \frac{\delta \mu^{ij}}{\delta \phi^k}\right)\frac{\delta S_0}{\delta \phi^j} \tag{10}$$

which is of the form (7).

We can thus conclude that the set of all gauge transformations (7) form a normal (i.e. invariant) subgroup $\mathcal{N}$ of the full gauge group $\overline{\mathcal{G}}$.

How significant are the transformations (7)? Are they really new symmetry transformations? It is easy to convince oneself that these transformations are of no physical significance. This is because:

(1) They exist idependently of what the action is; in other words, they do not restrict at all the form of the Lagrangian and there is indeed no non trivial Noether identities associated with them.

(2) They thus imply no degeneracy of the action and in the Hamiltonian formalism, there is no corresponding constraint. Actually, the conserved charges associated with (7), when rewritten as phase space functions by using the equations of motion if necessary, vanish identically.

(3) The transformations (7) vanish on-shell, i.e., do not map solutions of the equations of motion on new, different solutions.

(4) There is accordingly no need for a "gauge fixing" of (7). This is fortunate, as it is impossible to gauge-fix (7), which exists for any action!

On these grounds, it is legitimate to disregard the transformations (7). The relevant invariance group of the action is thus given by the factor group $\mathcal{G} = \overline{\mathcal{G}}/\mathcal{N}$ of all gauge transformations modulo the transformations (7), a concept that is mathematically well-defined as (7) form a normal subgroup.

For this reason, the transformations (7) are usually not even mentioned in standard textbooks on mechanics or field theory. These transformations have never been a source of concern for theories without gauge invariance, even though they are already present there.

Before closing this section, we mention the following useful theorem.

[3]Let us insist that one does not need to use the equations of motion to prove (8).

THEOREM. *Under suitable regularity assumptions on the functions $\frac{\delta S_0}{\delta \phi^i}$ to be precised below, any gauge transformation that vanishes on-shell can be written as in (7),*

$$\delta\phi^i \approx 0 \text{ and } \delta\phi^i \frac{\delta S_0}{\delta\phi^i} = 0 \Rightarrow \delta\phi^i = \epsilon^{ij}\frac{\delta S_0}{\delta\phi^j} \tag{11}$$

for some $\epsilon^{ij} = -\epsilon^{ji}$

The theorem will be proved below.

Exercises

1. Consider the action $S_0[A_\mu] = \int d^3x\, \epsilon^{\mu\nu\rho}F_{\mu\nu}A_\rho$ for pure Chern-Simons (abelian) Yang-Mills theory in three dimensions [**11**]. This action is invariant under ordinary gauge transformations $\delta A_\mu = \partial_\mu \Lambda$ and diffeomorphisms, $\delta A_\mu = \xi^\rho\partial_\rho A_\mu + \partial_\mu \xi^\rho A_\rho$. Show that the diffeomorphisms differ from the ordinary gauge transformations by a trivial gauge transformation.

2. Consider the action $S[q,p] = \int(p\dot{q} - H)dt$. Show that it is invariant under $\delta q = \epsilon(\dot{q} - \frac{\partial H}{\partial p}), \delta p = \epsilon(\dot{p} + \frac{\partial H}{\partial q})$. Check that the algebra of these trivial gauge transformations is isomorphic with the algebra of diffeomorphisms in one dimension. Observe that when $H = 0$, these transformations actually reduce to standard diffeomorphisms "along the wordline".

2.6. Factorization of $\mathcal{N}$.

If the gauge group $\overline{\mathcal{G}}$ is the semi-direct product of $\mathcal{N}$ by $\mathcal{G}$, i.e., if the quotient group $\mathcal{G} \equiv \overline{\mathcal{G}}/\mathcal{N}$ can be realized as a subgroup of $\overline{\mathcal{G}}$ "complementary" to $\mathcal{N}$, then, it is easy to disregard the transformations of $\mathcal{G}$. One simply works with the gauge transformations of $\mathcal{G}$ (viewed as a subgroup of $\overline{\mathcal{G}}$) and forgets about $\mathcal{N}$. This is permissible, as the commutator of two gauge transformations of $\mathcal{G}$ is again in $\mathcal{G}$ and does not generate a trivial transformation.

However, it may turn out that $\overline{\mathcal{G}} \neq \mathcal{N} \times_\sigma \mathcal{G}$. In other words, the gauge algebra may symbolically read

$$[\text{trivial, trivial}] = \text{trivial}, \tag{12a}$$

$$[\text{trivial, non trivial}] = \text{trivial}, \tag{12b}$$

$$[\text{non trivial, non trivial}] = \text{non trivial} + \text{trivial}, \tag{12c}$$

where the trivial piece in (12c) does not identically vanish and cannot be removed by redefinition of the "non trivial" transformations (compatible with locality, covariance, etc...).

In that case, one cannot forget about the trivial transformations as they are generated through the commutators (12c). One must work with all the gauge transformations and build the formalism so that the addition of trivial transformations to any transformation is ultimately irrelevant.

2.7. Independent Noether Identities.

The factorization of the trivial gauge transformations was motivated by the fact that they imply no Noether identity and hence, lead by themselves to no independent degeneracy of the equations of motion and do not need any gauge fixing condition.

This is not the end of the story, however. Indeed, the remaining gauge transformations do not all lead to independent Noether identities. This can be seen as follows.

Let $\delta_\epsilon \phi^i$ be gauge transformations

$$\delta_\epsilon \phi^i = R^i_{\ \alpha} \epsilon^\alpha, \tag{13a}$$

leading to the Noether identities

$$\frac{\delta S_0}{\delta \phi^i} R^i_\alpha = 0. \tag{13b}$$

Consider next the transformations

$$\delta_\eta \phi^i = (R^i_\beta M^\beta_A) \eta^A$$

where M^β_A is some matrix *that is allowed to depend on the fields*, $M^\beta_A(\phi^i)$. These transformations also leave the action invariant. From the point of view of Lie algebra theory, the transformations $\delta_\eta \phi^i$ are linearly independent from the transformation R^i_α as one cannot write $\delta_\eta \phi^i$ as a combination of R^i_α with coefficients that belong to the ground field, i.e., that are real (or complex) numbers.

However, the Noether identities that follow from $\delta_\eta S_0 = 0$,

$$\frac{\delta S_0}{\delta \phi^i} R^i_\beta M^\beta_A = 0 \tag{14b}$$

are clearly not independent from the Noether identities (13b), as (14b) is a consequence of (13b). Hence, there is no new information in $\delta_\eta \phi^i$.

To take the example of electromagnetism, the gauge transformations

$$\delta A_\mu = \partial_\mu \Lambda \tag{15}$$

with $\Lambda(x, A_\nu)$ a functional of A_ν, are independent from the Lie algebraic point of view from the transformations $\delta A_\mu = \partial_\mu \Lambda$ with $\Lambda = \Lambda(x)$. However, if one completely freezes the gauge freedom associated with the second set of transformations (e.g., $\vec{\nabla} \cdot \vec{A} = 0, A_0 = 0$), one automatically freezes the gauge freedom associated with the first set. So, there is nothing new in (15).

2.8. Generating sets.

This leads one to the concept of generating set. A set G of gauge transformations

$$\delta_\epsilon \phi^i = R^i_{\ \alpha} \epsilon^\alpha \tag{16a}$$

is a "generating set" if it contains all the information about the Noether identities.

More precisely, G is a generating set if any gauge transformation can be written in terms of the elements of G as

$$\delta\phi^i \frac{\delta S_0}{\delta\phi^i} = 0 \Rightarrow \delta\phi^i = \lambda^\alpha R^i_\alpha + M^{ij} \frac{\delta S_0}{\delta\phi^j}, \qquad M^{(ij)} = 0 \tag{16b}$$

with coefficients λ^α and M^{ij} *that may involve the fields.* Given a generating set, the Lie algebra of all the gauge transformations is spanned by (16b). Note that a generating set is in general *not* a basis in the Lie algebraic sense.

As the bracket of two elements of the generating set is a gauge transformation, it must be expressible as in (16b). So one has

$$R^j_\alpha \frac{\delta R^i_\beta}{\delta\phi^j} - R^j_\beta \frac{\delta R^i_\alpha}{\delta\phi^j} = C^\gamma_{\alpha\beta}(\phi)\, R^i_\gamma + M^{ij}_{\alpha\beta}(\phi)\, \frac{\delta S_0}{\delta\phi^j}. \tag{17}$$

From the physical point of view, it is enough to consider only generating sets. This is because generating sets contain all the information about the Noether identities, about the degeneracy of the action principle and about the number of required gauge conditions.

The situation is analogous to the following finite-dimensional geometrical setting. Consider a manifold A in $\mathbb{R}^n$. The vector fields tangent to A form an infinite-dimensional Lie algebra. However, for describing functions that are constant along A, the number of relevant vector fields is really finite and equal to the dimension n_a of A. If $X_a\,(a = 1, \ldots, n_a)$ provides at each point of A a basis of tangent vectors (Y tangent $\Rightarrow Y = y^a(x)X_a$), then the n_a equations $X_a f = 0$ imply the infinite number of equations $Yf = 0$ for all vectors tangent to A.

The vector fields X_a obey

$$[X_a, X_b] = C^c_{ab}(x) X_c \tag{18}$$

and may not form a Lie subalgebra.

To select a point on A, it is enough to impose n^a coordinate conditions. One does not need an infinite number of them.

2.9. "Open algebras".

It is now clear what the terminology "open algebra" means. It really applies to the generating sets and not to the gauge groups $\mathcal{G}$ or $\overline{\mathcal{G}}$.

So, one says that a given generating set is "open" if $M^{ij}_{\alpha\beta}(\phi)$ in (17) is different from zero. It is closed if $M^{ij}_{\alpha\beta}(\phi) = 0$. It defines a Lie algebra if, in addition, $C^\gamma_{\alpha\beta}$ does not depend on the fields.

This last case includes the usual gauge theories (Yang-Mills, gravity), but not all the interesting ones.

Gauge theories with a generating set G that forms a Lie algebra are very special

in that one can think of the transformations of G "abstractly ", i.e., independently of what the dynamics or the field content are. Furthermore, the BRST construction is then much simpler.

However, this is a very lucky instance, which misses some of the important ingredients of the general case.

2.10. Reducible generating sets.

Although generating sets should be complete, they can contain some redundancy. This occurs when there are some relations among the generators, i.e., when there exist some non trivial λ^α such that the following identities hold,

$$\lambda^\alpha R^i_\alpha = N^{ij} \frac{\delta S_0}{\delta \phi^j}. \tag{19}$$

The coefficients N^{ij} are antisymmetric as the right hand side of (19) should be a gauge transformation. One says that the generating set is reducible. A generating set is irreducible otherwise.[4]

The consideration of reducible generating sets is permissible within the formalism. However, the ghost spectrum associated with a reducible set is more complicated: besides the usual ghosts, one needs ghosts for ghosts.

An example of a reducible theory is given by p-form gauge fields. For a 2-form $A_{\mu\nu}$ with field strength $F_{\mu\nu\rho} = \partial_{[\mu} A_{\nu\rho]}$, the gauge transformations read

$$\delta A_{\mu\nu} = \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu \tag{20a}$$

$$\Rightarrow R^i_\alpha \sim R^\lambda_{\mu\nu}(x,y) = -\delta^\lambda_\mu \delta_{,\nu}(x-y) + \delta^\lambda_\nu \delta_{,\mu}(x-y). \tag{20b}$$

Then one finds

$$\lambda^\alpha R^i_\alpha = 0 \tag{20c}$$

with

$$\lambda_\lambda(x,y) = \delta_{,\lambda}(x,y). \tag{20d}$$

2.11. Relation between different generating sets.

Although the gauge groups $\overline{\mathcal{G}}, \mathcal{N}$ and $\mathcal{G}$ are entirely determined by the action S_0 itself, there is clearly an enormous freedom in the choice of the generating sets.

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[4]So, an irreducible theory is such that the only solution of (19) reads $\lambda^\alpha = M^{\alpha i} \frac{\delta S_0}{\delta \phi^i}$ i.e., vanishes on shell. [One then finds $\lambda^\alpha R^i_\alpha = M^{\alpha j} \frac{\delta S_0}{\delta \phi^j} R^i_\alpha = (M^{\alpha j} R^i_\alpha - M^{\alpha i} R^j_\alpha) \frac{\delta S_0}{\delta \phi^j}$ since $R^j_\alpha \frac{\delta S_0}{\delta \phi^j} = 0$. Accordingly, $\lambda^\alpha R^i_\alpha$ indeed defines a trivial gauge transformation]. In the irreducible group case, one says that the group has a "free action".

Two generating sets R^i_α and R^i_A ($\alpha = 1, \ldots, m; A = 1, \ldots, M \geq m$) are related as

$$R^i_\alpha = t^A_\alpha R^i_A + M^{ij}_\alpha \frac{\delta S_0}{\delta \phi^j}, \quad M^{ij}_\alpha = -M^{ji}_\alpha \tag{21a}$$

$$R^i_A = \bar{t}^\alpha_A R^i_\alpha + M^{ij}_A \frac{\delta S_0}{\delta \phi^j}, \quad M^{ij}_A = -M^{ji}_A \tag{21b}$$

where t^A_α and $\bar{t}^\alpha_A$ are of maximum rank m.

The requirement of covariance and locality in spacetime does, however, narrow down the choice of available generating sets. [One could otherwise always find one that is abelian [**12,13**] – but usually not covariant, not local in spacetime or not globally defined. A geometrical proof of abelianization is given in the appendix].

It turns out that the BRST methods incorporate not only gauge invariance, but also, formal independence on the choice of generating set. On this ground, all the generating sets – which, of course, describe the same gauge symmetry – are equivalent.

2.12. Generating sets and gauge orbits.

On the stationary surface where the equations of motion $\frac{\delta S_0}{\delta \phi^i} = 0$ hold, the transformations generated by the elements of any generating set are integrable, i.e., obey Frobenius integrability condition (the Lie bracket $[X_i, X_j]$ is proportional to X_k). Therefore, these transformations generate well-defined surfaces, the "gauge orbits". The gauge orbits do not depend on the choice of generating set on account of (21).

The number of elements in an irreducible generating set is equal to the dimension of the gauge orbits on the stationary surface. This gives a geometrical explanation of why generating sets are so relevant. By contrast, the dimension of the Lie algebra $\bar{\mathcal{G}}$ containing all the gauge transformations is much greater: $\bar{\mathcal{G}}$ is far from having a free action on the gauge orbits.

The above observation yields a criterion more practical than (16b) for deciding whether a set of gauge transformations is complete, i.e., generating. The set $\delta_\epsilon \phi^i = R^i_\alpha \epsilon^\alpha$ is complete if it accounts for all degeneracies in the general solution of the equations of motion. In that case, any two solutions ϕ^i and $\bar{\phi}^i$ fulfilling the same initial conditions are related by iteration of $\delta_\epsilon \phi^i = R^i_\alpha \epsilon^\alpha$.

This will be the case if all the null eigenvectors of the matrix $\frac{\delta^2 S_0}{\delta \phi^i \delta \phi^j}$ are spanned on-shell by R^i_α, i.e.,

$$\frac{\delta^2 S_0}{\delta \phi^i \delta \phi^j} \xi^j \approx 0 \Rightarrow \xi^j \approx \lambda^\alpha R^j_\alpha. \tag{22}$$

Here, $\approx$ means "modulo field equations". Indeed, if $\delta \phi^i$ is a gauge transformation, one finds by differentiation of $\frac{\delta S_0}{\delta \phi^i} \delta \phi^i = 0$ and upon use of (22) that $\delta \phi^i$ is equal to $\lambda^\alpha R^i_\alpha$

on the sationary surface. Hence, $\delta\phi^i - \lambda^\alpha R^i_\alpha$ is a gauge transformation that vanishes on-shell and thus, it is a trivial gauge transformation by the theorem of section 2.5.

A constructive method for getting a complete set of gauge transformations from any given action S_0 is given in **[14]**. This method is based on the Hamiltonian formalism of Dirac **[15]**.

2.12 Why ghosts are ghosts fields.

While the dimension of the Lie algebra of the gauge transformations is huge (it is given by the number of functionals $\lambda^\alpha(\phi)$ and $M^{ij}(\phi)$ in (16b)), the number of elements in a standard generating set is "smaller"and parametrized by spacetime fields: the index α in (16a) ranges over both $\mathbb{R}^n$ (it contains x) and a discrete set.

This is very important because, as we shall see, the number of ghosts in the BRST formalism is determined by the number of elements in the chosen generating set. So, the ghosts are *ghost fields* and one can apply the usual methods of local field theory for analyzing the gauge fixed action. It is therefore of crucial importance that the BRST construction is based on generating sets and not on the full group $\overline{\mathcal{G}}$ containing all the gauge transformations.

From now on, we will therefore exclusively deal with generating sets. These may have to be open and reducible in order to define transformations that are local in spacetime or covariant. The group structure of the set $\overline{\mathcal{G}}$ of all the gauge transformations is only of marginal interest in the BRST context.

3. Gauge Invariance and BRST invariance– Basic requirements

The derivation of the gauge-fixed Lagrangian is performed in two steps. First one replaces the original local gauge invariance by an equivalent global symmetry, the "BRST" symmetry. The replacements is done in such a manner that BRST invariance can be used as a substitute for gauge invariance. The first step is completely intrinsic and does not require any gauge fixing condition. Second, one chooses appropriate gauge fixing conditions and works out the correspondings gauge-fixed action in a way that incorporates BRST invariance.

The key requirements for constructing the BRST symmetry are the following:

i) The BRST symmetry $\mathfrak{s}$ acts as a graded odd derivation on the original fields ϕ^i and on some extra fields to be determined, i.e., for any A, B with B of definite Grassmann parity ϵ_B, one finds[5]

$$\mathfrak{s}(AB) = A(\mathfrak{s}B) + (-1)^{\epsilon_B}(\mathfrak{s}A)B \qquad \text{(Leibnitz rule)} \tag{23a}$$

$$\mathfrak{s}^2 = 0 \qquad \text{(nilpotency).} \tag{23b}$$

[5] We choose an action from the rigth for $\mathfrak{s}$. Also, when we say "fields ϕ^i", we really mean "field histories $\phi^i(x)$" (condensed notation and terminology!)

The grading of $\mathfrak{s}$ is called the ghost number and one has

$$\mathrm{gh}(\mathfrak{s}A) = \mathrm{gh}(A) + 1 \tag{23c}$$

$$\epsilon(\mathfrak{s}A) = \epsilon_A + 1 \qquad (\mathrm{mod}\ 2). \tag{23d}$$

(ii) The zeroth cohomological group $H^0(\mathfrak{s}) \equiv (\ker \mathfrak{s} / \operatorname{Im} \mathfrak{s})^0$ is isomorphic with the set of gauge invariant observables

$$H^0(\mathfrak{s}) = \{\text{gauge invariant observables.}\} \tag{24}$$

In other words, if one identifies two BRST invariant functions that differ by a BRST exact one,

$$\mathfrak{s}A = 0, \quad \mathfrak{s}A' = 0, \quad A \sim A' \Longleftrightarrow A - A' = \mathfrak{s}B \tag{25}$$

one just finds, at ghost number zero, the gauge invariant functions.

iii) The BRST symmetry is a canonical transformation in an appropriate bracket structure (,) to be defined below. Hence,

$$\mathfrak{s}A = (A, S) \tag{26}$$

where S is the canonical generator of $\mathfrak{s}$.

These three requirements completely determine S up to a canonical transformation, at least in the so-called "minimal sector" (see below). Accordingly, they completely capture the BRST symmetry.

4. Relativistic description of gauge invariant functions

In order to construct a nilpotent symmetry obeying (24), it is necessary to recall first how gauge invariant functions ("observables") are described. As we want to develop a manifestly covariant formalism, we need a manifestly relativistic description.

4.1. Covariant phase space in the absence of gauge invariance.

Let us first assume for a moment that there is no gauge invariance. The observables are then usually realized as the phase space functions $F(q,p)$. This is, however, not fully satisfactory as a phase space point refers to the state of the system at a given instant of time.

As (q,p) at $t = t_0$ completely determines $(q(t), p(t))$ through the Hamiltonian equations, one can alternatively view phase space as the space of all solutions of the equations of motion. One can then drop reference to the momenta and consider the solutions $q(t)$ of the equations of motion for q obtained by eliminating p from the Hamiltonian equations. These equations for q usually take a manifestly covariant form.

The space of all solutions of the equations of motion is known as the "covariant phase space". Its consideration goes back to the work of Peierls who showed how to determine the Poisson bracket structure directly in the covariant phase space [**6, 10**]. More recent work includes [**17–19**].

The same idea applies of course to field theory, where observables can be viewed as functions[6] $f(\phi^i)$ of the solutions ϕ^i to the equations of motion $\frac{\delta S_0}{\delta \phi^i} = 0$.

As the explicit description of the solutions to the equations of motion may be involved, it is convenient to push the reformulation of the concept of observables one step further. This is done as follows.

Denote by I the (infinite-dimensional functional) space of all possible field history. So, a point of I is an arbitrary entire history that may not solve $\frac{\delta S_0}{\delta \phi^i} = 0$. In I, the equations of motion $\frac{\delta S_0}{\delta \phi^i} = 0$ determine a submanifold Σ which we call the stationary surface. This submanifold is just the covariant phase space (in the absence of gauge invariance).

The observables are the functions defined on Σ, i.e., the elements of $C^\infty(\Sigma)$ ("smooth" functions on Σ). Now, any function f on Σ can be extended off Σ to a function $F(\phi^i)$ defined on I, i.e., to an element of $C^\infty(I)$ ("smooth" functions on I). Two different extensions F and F' differ by a function that vanishes on Σ. These functions form an ideal $\mathcal{N}$ as FG vanishes on Σ whenever F (or G) does. The algebra $C^\infty(\Sigma)$ of the smooth functions on Σ is thus the quotient algebra $C^\infty(I)/\mathcal{N}$ of the smooth functions on I by the functions that vanish on Σ.

It should be stressed that our considerations based on the use of the equations of motion can be extended to cover quantum mechanics. This is because the observables can still be identified with the (operator-valued) functions of $\hat{q}$ and $\hat{p}$ at a given instant of time. These functions are again in bijective correspondence with -and hence can be realized as- the functions on the space of solutions $\hat{q}(t)$, $\hat{p}(t)$ of the equations of motion. The absence of conflict with the principles of quantum mechanics is particularly obvious in the manifestly covariant Heisenberg picture, where the fields operators obey the appropriately ordered equations of motion $\frac{\delta S_0}{\delta \phi^i} = 0$.

4.2. Boundary conditions.

In order to contain all the solutions of the equations of motion, and not just the one corresponding to a definite set of initial data, the space I of histories should not be restricted by boundary conditions at the "initial" and "final" times t_i and t_f. The stationary surface Σ contains then all the possible dynamical states of the system.

For this reason, the space I is not the space $I_{i\to f}$ over which one integrates in the path integral representation of a definite quantum mechanical amplitude between given in- and out-states (I is too large). The space I is actually the union over all possible pairs of in- and out-states of the spaces $I_{i\to f}$.

[6]The words "functions" and "functionals" are used interchangeably in the sequel. The suggestive terminology and notations of finite dimensional manifold theory will also be adopted, without any analysis of the (complicated) functional aspects.

Furthermore, as one does not vary the boundary data at t_1 and t_2 in the action principle, the functional derivatives $\frac{\delta S_0}{\delta \phi^i}$ in the field equations $\frac{\delta S_0}{\delta \phi^i} = 0$ do not refer to the derivatives of S_0 with respect to the boundary data. More precisely, if we write

$$\phi^i(\vec{x}, t) = \overline{\phi}^i(\vec{x}, t) + f^i[\alpha_1, \alpha_2](\vec{x}, t)$$

where (i) $f^i[\alpha_1, \alpha_2](\vec{x}, t)$ is, for given α_1, α_2, a fixed history such that $f^i[\alpha_1, \alpha_2](\vec{x}, t_1) = \alpha_1^i(\vec{x}), f^i[\alpha_1, \alpha_2](\vec{x}, t_2) = \alpha_2^i(\vec{x})$; and (ii) $\overline{\phi}^i(\vec{x}, t_1) = \overline{\phi}^i(\vec{x}, t_2) = 0$; then, the field equations are $\frac{\delta S_0}{\delta \overline{\phi}^i} = 0$. This will always be implicitly understood in the sequel even though the above decomposition will never be used.

4.3. Covariant phase space in the presence of a gauge freedom.

If there is a gauge freedom, the observables should be in addition gauge invariant.

We have pointed out that the gauge transformations $\delta_\epsilon \phi^i = R^i_\alpha \epsilon^\alpha$ are integrable when the equations of motion hold. Accordingly, they generate well defined orbits on Σ, the dimension of which is equal to the number of independent R^i_α. The gauge invariant functions are constant along the gauge orbits and hence, induce definite functions on the quotient space Σ/G of the stationary surface by the gauge orbits. Formally, one can thus write the space of observables as $C^\infty(\Sigma/G)$, i.e., as the space of "smooth" functions on Σ/G [In general, Σ/G is not a smooth manifold but we will nervertheless use this suggestive notation].

The gauge invariant observables are thus reached in two steps. First, one goes from I to Σ; then, from Σ to Σ/G. To solve (24), one must find a nilpotent operator $\mathfrak{s}$ that implements these two steps through its cohomology,

$$H^0(\mathfrak{s}) = C^\infty(\Sigma/G). \tag{27}$$

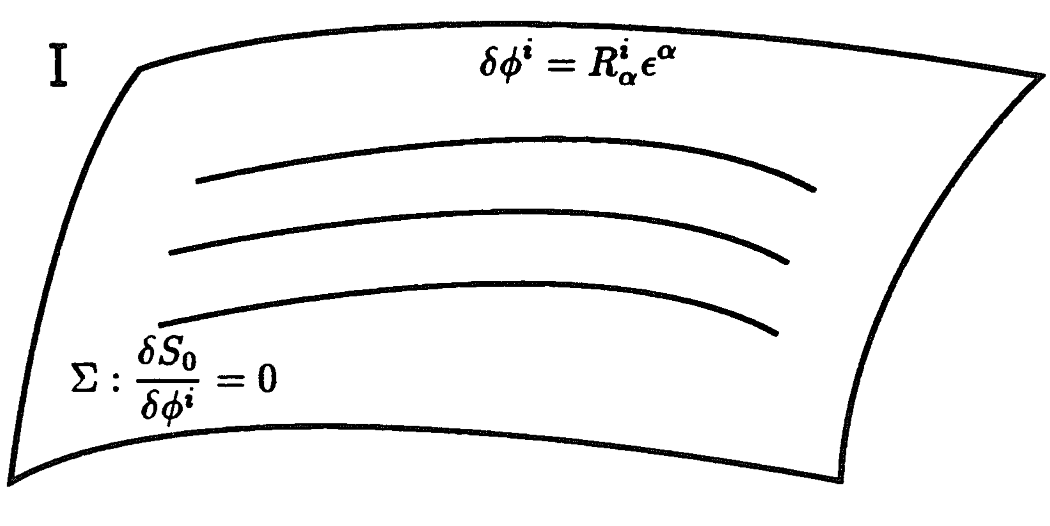

Fig. 1

The gauge orbits are obtained by integrating $\delta_\epsilon \phi^i = R^i_\alpha \epsilon^\alpha$ on Σ.

It is not surprising that the searched-for nilpotent $\mathfrak{s}$ contains actually two nilpotent operators. Each of these two "differentials" implements one of the steps.

The first differential, δ, provides what is known as a "Koszul-Tate resolution" of $C^\infty(\Sigma)$, i.e., is such that $H_0(\delta) \equiv \left(\frac{\ker \delta}{\operatorname{Im} \delta}\right)_0 = C^\infty(\Sigma)$. It implements the first step, from I to Σ. The second differential, d, is the "vertical exterior derivative along the gauge orbits" and implements the second step, from Σ to Σ/G, $H^0(d) = C^\infty(\Sigma/G)$. The BRST derivative $\mathfrak{s}$ is formally the sum of δ and d.

4.4. Regularity conditions – A useful theorem.

In order to develop the formalism, it is necessary to make some regularity assumptions on the derivatives $\delta S_0/\delta\phi^i$. One assumes that one can split on Σ the derivatives $\delta S_0/\delta\phi^i$ into "independent" functions y^a and "dependent ones" z^α, $(\delta S_0/\delta\phi^i) = (y^a, z^\alpha)$ in such a way that (i) the equations $\delta S_0/\delta\phi^i = 0$ are completely equivalent to $y^a = 0$, i.e., $z^\alpha = 0$ is a consequence of $y^a = 0$; and (ii) the exterior form $\bigwedge_a dy^a$ does not vanish on Σ. This means that zero is a regular value of the map defined by y^a and that one can locally take the y^a's as first coordinates of a new regular coordinate system on I, $(\phi^i) \leftrightarrow (y^a, \tau^\alpha)$.

To illustrate these conditions in a simpler finite-dimensional situation, let us consider a two dimensional space with coordinates (u,v) instead of I. If the action S_0 is $\frac{1}{2}(u-v)^2$, one finds $\frac{\partial S_0}{\partial u} = u - v$, $\frac{\partial S_0}{\partial v} = -(u-v)$. One can take as independent equation the first one; the second one is clear consequence of the first. Hence, $y \equiv \frac{\partial S_0}{\partial u}$, $z \equiv \frac{\partial S_0}{\partial v}$ and $y = 0 \Rightarrow z = 0$. The gradient $dy = d(u-v)$ does not vanish on Σ. The regularity conditions are fulfilled and $y \equiv u - v$ can be taken as first coordinate of a new regular coordinate system (y, τ), e.g. with $\tau = v$.

If S_0 were to be replaced by $\overline{S}_0 \equiv \frac{1}{6}(u-v)^3$ the regularity conditions would not be fulfilled because both $d\left(\frac{\partial \overline{S}_0}{\partial u}\right) = u - v$ and $d\left(\frac{\partial \overline{S}_0}{\partial v}\right) = v - u$ vanish on Σ. The function $y \equiv \frac{\partial \overline{S}_0}{\partial u} = \frac{1}{2}(u-v)^2 = \frac{\partial \overline{S}_0}{\partial v}$ cannot be taken as first coordinate of a new regular coordinate system in the vicinity of Σ, as the inverse transformation $u - v = \sqrt{2y}$ is not smooth on Σ.

When the regularity conditions hold, the following theorem is immediate.

THEOREM. *Any smooth function $F(\phi^i)$ vanishing on Σ can be written as*

$$F(\varsigma^i) = \lambda^i(\phi) \frac{\delta S_0}{\delta \phi^i} \tag{29}$$

with smooth coefficients $\lambda^i(\phi)$.

PROOF: The proof is standard. In the (y, τ) coordinate system, one finds

$$\begin{aligned} F(\phi^i) =& F(y,\tau) = F(y=0,\tau) + \int_0^1 d\mu \, \frac{dF}{d\mu}(\mu y, \tau) \\ &= y^a \int_0^1 d\mu \, \frac{\partial F}{\partial y^a}(\mu y, \tau) = \lambda^i(\phi) \, \frac{\delta S_0}{\delta \phi^i}. \end{aligned}$$

as $F(y=0,\tau)$ vanishes by assumption and as y^a are some of the field equations.

This proof is local in field space because the coordinates (y, τ) are usually only defined locally. However, it is easy to see, using for instance partitions of unity, that it can be extended to cover the whole of I.

In algebraic terms, the theorem expresses that the ideal $\mathcal{N}$ of the functions that vanish on Σ is the same as the ideal $\mathcal{N}'$ of the functions that are combinations $\lambda^i(\phi)\dfrac{\delta S_0}{\delta \phi^i}$ of the field equations.

We have been a bit cavalier with the functional aspects of the theorem and have proceeded as if the space I of all histories were finite-dimensional. So, our discussion is rather formal. However, things are not as bad as one may think at first sight. This is because spacetime locality comes as a help. Indeed, only local functionals occur in practice in the constructions, i.e., functionals that take the form

$$F = \int k(\phi^i, \phi^i{}_{,\mu}, \ldots, \phi^i{}_{,\mu_1 \ldots \mu_s}) \, d^D x$$

where k involves a finite number of derivatives. One can then reformulate the question in terms of $k(\phi^i, \phi^i{}_{,\mu}, \ldots, \phi^i{}_{,\mu_1 \ldots \mu_s})$ as a finite-dimensional problem. This enables one to prove as a bonus that $\lambda^i(\phi)$ in (29) is also local in spacetime [20]. But it should be immediately added that in spite of the spacetime locality of the gauge fixed action and of the generator of the BRST transformation, gauge invariant functions that are not local in spacetime can be of great interest. So non local functionals should also be considered. For these, the above formal derivations must be supplemented by appropriate functional analysis arguments which will not be given here.

The regularity conditions are usually fulfilled by all models of physical interest. The only exception that I know is given by the Siegel model for chiral bosons [21] where, as pointed out in [22], some of the relevant field equations only vanish quadratically on the stationary surface. This seems to prevent a consistent, physically meaningful Lagrangian BRST formulation of the model, already at the classical level. This difficulty does not appear to have been fully appreciated in the literature. [By contrast, the Hamiltonian formulation is straightforward. For more information, see [23] and references therein].

5. The Koszul-Tate Resolution

5.1. The problem.

The first step in the BRST construction is to implement the restriction from I to Σ. So, one needs to define a "differential" δ (i) that acts as a (nilpotent) graded derivative on polynomials in some generators (to be specified) with coefficients that are functions on I (just like d in the standard exterior calculus acts on polynomials in dx, dy, dz, ... with coefficients that are functions on the manifold); and (ii) that computes $C^\infty(\Sigma)$ through its homology.

The grading of δ is called the antighost number. As δ decreases the antighost number by one unit, it behaves like a boundary operator.

The requirement that δ computes $C^\infty(\Sigma)$ through its homology reads

$$H_0(\delta) \equiv \left(\frac{\ker \delta}{\operatorname{Im} \delta}\right)_0 = C^\infty(\Sigma) = \frac{C^\infty(I)}{\mathcal{N}} \tag{30}$$

We will actually ask more, namely that (30) contains *all* the homology of δ. In other words, we require

$$H_k(\delta) = 0 \qquad k \neq 0. \tag{31}$$

This requirement turns out to be essential not only for guaranteeing that the BRST cohomology at ghost number zero is given by the gauge invariant functions, but also, for being able to prove the existence, in general, of the BRST symmetry itself.

A differential complex with the properties (30) and (31) is said to provide a "resolution" of the quotient algebra $\dfrac{C^\infty(I)}{\mathcal{N}}$. In the present context, the relevant resolution is due to Koszul [**24**], Borel [**25**] and Tate [**26**].

One can describe δ in an intrinsic manner, without having to work with a specific representation of the surface Σ. However, our ultimate goal is to derive the gauge fixed action with a definite set of fields. For this reason, we will not strive for intrinsiqueness.

5.2. Actions without gauge invariance.

In the absence of gauge invariance, the construction of δ is very simple. Because $\left(\dfrac{\ker \delta}{\operatorname{Im} \delta}\right)_0$ should be equal to $\dfrac{C^\infty(I)}{\mathcal{N}}$, we simply define δ so that

$$(\ker \delta)_0 = C^\infty(I) \tag{32a}$$

$$(\operatorname{Im} \delta)_0 = \mathcal{N}. \tag{32b}$$

Consequently, we set

$$\delta\phi^i = 0. \tag{33a}$$

Using the Leibnitz rule, this implies $\delta F(\phi^i) = 0$ for any functions on I and hence, $(\ker\delta)_0 = C^\infty(I)$.

To implement $(\mathrm{Im}\,\delta)_0 = \mathcal{N}$, we observe that due to our regularity assumptions, the elements of $\mathcal{N}$ are given by the combinations of the field equations.

$$G(\phi^i) \in \mathcal{N} \Leftrightarrow G(\phi^i) = \lambda^j(\phi^i)\frac{\delta S_0}{\delta\phi^j}.$$

Therefore, we introduce as many new generators ϕ_i^* as there are field equations and simply set

$$\delta\phi_i^* = -\frac{\delta S_0}{\delta\phi^i}$$

The minus sign is inserted for later convenience. This implies $G = \delta(-\lambda^i\phi_i^*)$ and $(\mathrm{Im}\,\delta)_0 = \mathcal{N}$. With this definition, our first goal is achieved, namely, equation (30) holds.

The generators ϕ_i^* are known as the antifields associated with the original fields ϕ^i. They are in equal number as the ϕ^i's, since the number of field equations is equal to the number of fields (the fields equations set the gradient of S_0 equal to zero).

To preserve the grading properties of δ one must impose

$$\epsilon(\phi_i^*) = 1 \tag{34a}$$

(as we assume the fields to be bosonic) and

$$\mathrm{antigh}\,\phi_i^* = 1 \tag{34b}$$

(of course, $\mathrm{antigh}\,\phi^i = 0$). The action of δ on a general polynomial in ϕ^i, ϕ_i^* is obtained by using the Leibnitz rule and one easily checks nilpotency, $\delta^2 = 0$.

To see whether δ provides a resolution of $\dfrac{C^\infty(I)}{\mathcal{N}}$, it remains to compute $H_k(\delta)$. It is here that the assumed absence of gauge invariance plays a key role. Indeed, the equations of motion are then independent, so that the number of new objects ϕ_i^* in degree one is exactly equal to the number of independent equations of motion. Using this property, one easily proves [**24, 25, 9**],

$$H_k(\delta) = 0 \qquad k \neq 0 \tag{35}$$

5.3. Actions with a gauge freedom (irreducible case).

The above definition (33), (34) of δ can still be used if there is a gauge freedom, and one still finds $H_0(\delta) = C^\infty(\Sigma)$.

However, it is no longer true that $H_k(\delta) \neq 0$. Because of the Noether identity,

$$\frac{\delta S_0}{\delta \phi^i} R^i_\alpha = 0, \tag{36}$$

one actually finds non trivial δ-closed polynomials in degree one. These are given by

$$R^i_\alpha \phi^*_i. \tag{37a}$$

Indeed, one checks that the $R^i_\alpha \phi^*_i$ are δ-closed

$$\delta(R^i_\alpha \phi^*_i) = -R^i_\alpha \frac{\delta S_0}{\delta \phi^i} = 0 \tag{37b}$$

and exhaust all non-trivial δ-closed polynomials of degree one,

$$\delta(\lambda^i \phi^*_i) = 0 \Rightarrow \lambda^i \phi^*_i = \mu^\alpha R^i_\alpha \phi^*_i + \delta\left(\frac{1}{2}\epsilon^{ij}\phi^*_i \phi^*_j\right). \tag{37c}$$

Furthermore, the $R^i_\alpha \phi^*_i$ are non exact. So, $H_1(\delta) \neq 0$.

To understand how this problem can be remedied, let us first assume that the gauge transformations are independent, so that all the non trivial cycles in (37a) are independent.

The way in which we can then recover $H_1(\delta) = 0$ and at the same time, also $H_k(\delta) = 0$ for all $k \neq 0$, is extremely elegant and was devised by Tate [**26**]. One simply adds one new generator ϕ^*_α for each cycle in (37a) and defines

$$\delta \phi^*_\alpha = R^i_\alpha \phi^*_i. \tag{38a}$$

Because $\delta(R^i_\alpha \phi^*_i) = 0$, one has $\delta^2 \phi^*_\alpha = 0$. Furthermore, by taking

$$\text{antigh}(\phi^*_\alpha) = 2, \qquad \epsilon(\phi^*_\alpha) = 0 \tag{38b}$$

(recall that $\epsilon(\phi^*_i) = 1$) and extending δ as a graded derivation to any polynomial in ϕ^i, ϕ^*_i and ϕ^*_α one maintains $\delta^2 = 0$.

With the introduction of the antifields ϕ^*_α, the cycles $R^i_\alpha \phi^*_i$ that were non exact become exact. So, $H_1(\delta)$ is now zero. Furthermore, using the assumed irreducibility of the gauge transformations, one easily shows that $H_k(\delta) = 0$ for all $k > 0$ [**26, 27, 22**].

5.4. Reducible case.

The construction of δ in the reducible case proceeds along the same lines as in the irreducible one. First, one observes that with (33) and (38), the homology group $H_2(\delta)$ is non-zero, even though $H_1(\delta) = 0$. This is because the polynomials

$$Z^{\alpha}_{A}\phi^{*}_{\alpha} + \frac{1}{2}C^{ij}_{A}\phi^{*}_{i}\phi^{*}_{j} \tag{39}$$

are closed but not exact. Here, the Z's form a complete set of reducibility functions, i.e., they are such that

$$\mu^{\alpha}R^{i}_{\alpha} = N^{ij}\frac{\delta S_0}{\delta\phi^j} \Rightarrow \mu^{\alpha} = \nu^{A}Z^{\alpha}_{A} + M^{\alpha i}\frac{\delta S_0}{\delta\phi^i}. \tag{40a}$$

One has for the Z's

$$Z^{\alpha}_{A}R^{i}_{\alpha} = C^{ij}_{A}\frac{\delta S_0}{\delta\phi^j}, \qquad C^{ij}_{A} = -C^{ji}_{A} \tag{40b}$$

and this property guarantees that the polynomials (39) are closed. These polynomials are not exact, because the Z^{α}_{A}'s cannot all vanish on Σ when the gauge transformations are truly reducible.

One therefore introduces new generators ("antifields") ϕ^{*}_{A} at antighost number three, and sets

$$\delta\phi^{*}_{A} = -Z^{\alpha}_{A}\phi^{*}_{\alpha} - \frac{1}{2}C^{ij}_{A}\phi^{*}_{i}\phi^{*}_{j} \tag{41a}$$

$$\text{antigh}\,\phi^{*}_{A} = 3, \qquad \epsilon(\phi^{*}_{A}) = 1. \tag{41b}$$

This kills the homology at degree two, i.e., $H_2(\delta)$ is now zero. The minus sign in (41a) is again inserted for later convenience.

If the reducibility equations (40b) are all independent, this is the end of the story. Not only are $H_1(\delta)$ and $H_2(\delta)$ zero, but also all the higher homology groups $H_k(\delta), k = 3, 4, \dots$ [**26, 27, 22**].

But if the reducibility equations (40b) are not independent, the analysis is not finished and one has to keep going. This is because $H_3(\delta) \neq 0$ with (33), (38), (41). For each non-trivial relation

$$Z^{A}_{(i)}\,Z^{\alpha}_{A} = C^{\alpha j}_{(i)}\,\frac{\delta S_0}{\delta\phi^j} \tag{42}$$

on the reducibility functions, one must therefore introduce one antifield at antighost number four. This kills $H_3(\delta)$.

One can, if desired, introduce more antifields, i.e., one can consider an overcomplete set of reducibility relations (42). But one must then compensate at the next order by adding antifields of antighost number five that take into account the relations on the $Z^{A}_{(i)}$. The general idea of passing from order k to order $k+1$ is always the same.

We refer the reader to the reference [27] where the construction of δ is more explicitely analyzed and where the complete proofs of its properties are given. These proofs are explicitly worked out within the Hamiltonian formalism. But, as briefly indicated below, the algebraic features of that approach to the BRST symmetry are identical. For this reason, we only list here the salient facts:

i) δ can be constructed recursively antighost level by antighost level and the spectrum of antifields can be chosen at each step so that $\delta^2 = 0$, $H_k(\delta) = 0$, $k > 0$ (and of course, $H_0(\delta) = C^\infty(\Sigma)$).

ii) The explicit expression for δ becomes awkward when the coefficients C^{ij}_A in (40b), $C^{\alpha j}_{(i)}$ in (42) etc... are non-zero ("on-shell reducibility") but this affects neither $\delta^2 = 0$, nor $H_k(\delta) = 0$, $k > 0$ (it only technically complicates the proofs).

iii) The requirement of acyclicity of δ at antighost number $k > 0$ turns out to be equivalent to the "proper solution" requirement of Batalin and Vilkovisky [6], as both demands lead to the same spectrum of fields and antifields.

iv) Let λ^i be a gauge transformation that vanishes on-shell,

$$\lambda^i \frac{\delta S_0}{\delta \phi^i} = 0, \qquad \lambda^i = \lambda^{ij} \frac{\delta S_0}{\delta \phi^j}.$$

A priori, λ^{ij} may not be antisymmetric in (i,j). However, λ^{ij} is not uniquely defined by λ^i as one can replace λ^{ij} by $\lambda^{ij} + \mu^{ijk}\dfrac{\delta S_0}{\delta \phi^k}$ with $\mu^{ijk} = -\mu^{ikj}$. One can use this freedom to set $\lambda^{ij} = -\lambda^{ji}$. Indeed, one has $\delta(\lambda^i \phi^*_i) = 0$ and hence $\lambda^i \phi^*_i = \delta(\lambda^\alpha \phi^*_\alpha + \nu^{ij} \phi^*_i \phi^*_j)$ with $\nu^{ij} = -\nu^{ji}$. Since $\lambda^i \approx 0$, the λ^α's must be such that $\lambda^\alpha R^i_\alpha \approx 0$, i.e., $\lambda^\alpha = Z^\alpha_A t^A + \sigma^{\alpha i}\dfrac{\delta S_0}{\delta \phi^i}$. This implies $\lambda^\alpha \phi^*_\alpha = Z^\alpha_A t^A \phi^*_\alpha + \sigma^{\alpha i} \phi^*_\alpha \dfrac{\delta S_0}{\delta \phi^i} = \delta(t^A \phi^*_\alpha + \sigma^{\alpha i} \phi^*_\alpha \phi^*_i) + \epsilon^{ij} \phi^*_i \phi^*_j$ for some $\epsilon^{ij} = -\epsilon^{ji}$. Accordingly, $\lambda^i \phi^*_i = \delta[(\nu^{ij} + \epsilon^{ij}) \phi^*_i \phi^*_j]$, i.e., $\lambda^i = \lambda^{ij} \left(\dfrac{\delta S_0}{\delta \phi^j}\right)$ with $\lambda^{ij} = -\lambda^{ji} (= \nu^{ij} + \epsilon^{ij})$ as required.

v) For definiteness, we will explicitly develop the subsequent formalism in the case of reducible gauge theories of the first order, i.e., of gauge theories with reducibility functions Z^α_A that are independent. The antifield spectrum is then given by ϕ^*_i, ϕ^*_α and ϕ^*_A. The general case is treated along the lines of these lectures in [22, 27].

6. The Exterior Derivative along the Gauge Orbits

6.1. Definition.

As the orbits generated by the gauge transformations $\delta_\epsilon \phi^i = R^i_\alpha \epsilon^\alpha$ are integrable on the stationary surface Σ, one can define, on Σ, an exterior derivative operator d that takes antisymmetrized derivatives along the gauge orbits. This operator acts on p-forms "along the gauge orbits".

Borrowing the terminology of fiber bundle theory where the "fibers" (here, the

"orbits") are drawn vertically, one can call a vector tangent to the orbits a "vertical vector". A p-form along the gauge orbits in then named "a vertical p-form" and d is the "vertical exterior derivative".

So, if F is a function on Σ, dF is the vertical 1-form defined by

$$dF(X) = \mathcal{L}_X F \tag{43a}$$

for all vertical vectors X. dF vanishes iff F is invariant along the orbits. The exterior derivative of a vertical 1-form α is given by

$$(d\alpha)(X,Y) = -\mathcal{L}_Y \alpha(X) + \mathcal{L}_X \alpha(Y) - \alpha([X,Y]) \tag{43b}$$

where X and Y are vertical vector fields. The Lie bracket $[X,Y]$ of X and Y in (43b) is also a vertical vector field as the gauge transformations are integrable. Similar formulas hold for higher-rank p-forms and one finds $d^2 = 0$ (on the stationary surface Σ, where d is defined). We follow the exterior calculus conventions of [27].

Because d only takes derivatives along the gauge orbits, it is clear that $H^0(d) \equiv (\ker d / \operatorname{Im} d)^0$ is isomorphic with the algebra of gauge invariant functions. Note that the higher order cohomology groups $H^k(d)$, $k > 0$, may be non trivial.

6.2. Representation (irreducible case).

In the irreducible case, the dimension of the gauge orbits is equal to the number of gauge transformations $\delta_\epsilon \phi^i = R^i_\alpha \epsilon^\alpha$. A basis at each point $\{X_\alpha\}$ of vertical vectors is thus given by

$$X_\alpha F \equiv \frac{\delta F}{\delta \phi^i} R^i_\alpha \tag{44a}$$

and one has

$$[X_\alpha, X_\beta] = C^\gamma_{\alpha\beta}(\phi^i) X_\gamma \qquad (\text{on } \Sigma). \tag{44b}$$

If $\{C^\alpha\}$ stands for the basis dual to $\{X_\alpha\}$, one can represent the vertical exterior derivative along the gauge orbits as

$$dF = (X_\alpha F) C^\alpha \tag{45a}$$

$$dC^\alpha = \frac{1}{2} C^\alpha_{\beta\gamma} C^\beta C^\gamma \tag{45b}$$

The dual one-forms C^α's are anticommuting and will be interpreted as the ghosts below. The form degree will therefore be called already now the "pure ghost number". The operator d increases the pure ghost number by one unit.

6.3. Representation (reducible case).

In the reducible case, the vertical vectors X_α associated with the gauge transformations no longer form a basis. Rather, they form an overcomplete set, subject to the following reducibility equations,

$$Z^\alpha_A X_\alpha = 0 \qquad (on \Sigma). \tag{46a}$$

Accordingly, the "components" $\alpha_{\alpha_1 \dots \alpha_p} \equiv \alpha(X_{\alpha_1}, \cdots, X_{\alpha_p})$ of a vertical p-form in the overcomplete set $\{X_\alpha\}$ are also subject to the same conditions

$$Z_A^{\alpha_i} \alpha_{\alpha_1 \ldots \alpha_i \ldots \alpha_p} = 0 \qquad (on \Sigma). \tag{46b}$$

We will assume for simplicity that there is no relation on the Z_A^α's, i.e., that these functions are all independent.

Although there is no dual basis to $\{X_\alpha\}$ one can introduce formal objects C^α. By saturating the indices of the "components" $\alpha_{\alpha_1 \ldots \alpha_p}$ with $C^{\alpha_1} \ldots C^{\alpha_p}$ one can identify vertical p-forms with polynomials of order p in C^α whose coefficients obey the algebraic condition (46b). One can then compute d using the same formulas (45) as in the irreducible case.

It is, however, more convenient to relax the algebraic condition (46b) as follows. Introduce as many additional objects C^A as there are conditions on X_α and modify dC^α as

$$dC^\alpha = \frac{1}{2} C^\alpha_{\beta\gamma} C^\beta C^\gamma + Z_A^\alpha C^A. \tag{47}$$

The new term in the right-hand side of (47) does not affect the vertical derivative of a vertical p-form because of (46b). To preserve the grading properties of d, the generators C^A should be even and of pure ghost number two. These generators will be identified later with the ghosts of ghosts.

It is possible to define dC^A so that $d^2 = 0$ on arbitrary polynomials in C^α, C^A (on Σ). The proof will not be given here (see [**27**]). The mathematical structure defined by d, ϕ^i, C^α and C^A is known as a free differential algebra.

What is the effect of the new term added to dC^α? Its effect is to enforce the algebraic condition (46b) through the closedness relation. So, for instance, a one-form $\alpha = \alpha_\alpha c^\alpha$ is closed iff

$$d\alpha = 0 \Longleftrightarrow d^{\text{old}}\alpha + \alpha_\alpha Z_A^\alpha C^A = 0. \tag{48}$$

This implies both the algebraic equation $\alpha_\alpha Z_A^\alpha = 0$ and the requirement $d^{\text{old}}\alpha = 0$.

Thus, one can represent the vertical exterior derivative d in the space of arbitrary polynomials in C^α and C^A. The algebraic condition (46b) is automatically enforced when passing to the cohomology.

This is made possible through the introduction of the ghosts of ghosts.

We refer again the reader to [**27**] for the details. This reference analyses also the case when the reducibility equations are not independent, which requires further "ghosts of ghosts of ghosts".

7. BRST Symmetry– Master equation

7.1. The problem.

With the Koszul-Tate operator and the vertical exterior derivative at hand, all the building blocks of the BRST symmetry have been constructed. What is required now is to put these ingredients together in a manner that preserves the crucial nilpotency.

To that end, we tentatively first define

$$\mathfrak{s}\phi^i \stackrel{?}{=} d\phi^i, \quad \mathfrak{s}C^\alpha \stackrel{?}{=} dC^\alpha, \quad \mathfrak{s}C^A \stackrel{?}{=} dC^A \tag{49a}$$

$$\mathfrak{s}\phi^*_i \stackrel{?}{=} \delta\phi^*_i, \quad \mathfrak{s}\phi^*_\alpha \stackrel{?}{=} \delta\phi^*_\alpha, \quad \mathfrak{s}\phi^*_A \stackrel{?}{=} \delta\phi^*_A \tag{49b}$$

where d is extended off Σ by using the same formulas (45a) and (45b) (47) as before. This makes sense as the gauge transformations and the reducibility functions are well defined for all histories, and not just for those histories that obey the equations of motion.

So, in (49), the BRST transformation $\mathfrak{s}$ reduces to d in the sector containing the fields ϕ^i and the ghosts C^α (and the ghosts of ghosts C^A if any). It reduces to δ in the antifield sector.

The grading of $\mathfrak{s}$ is called the "ghost number" and is given by the difference between the pure ghost number and the antighost number: $\mathfrak{s}$ increases the ghost number by one unit.

The problem with the simple definition (49) is that it does not yield a nilpotent operator. This is because:

i) $\delta d + d\delta \neq 0$ off Σ, except when the gauge transformations form an Abelian group;

ii) $d^2 \neq 0$ off Σ in the open algebra case (the gauge transformations $\delta_\epsilon \phi^i = R^i_{\ \alpha}\epsilon^\alpha$ are then non integrable off Σ).

In order to remedy this situation, one needs to improve $\mathfrak{s}$ by terms of higher antighost number,

$$\mathfrak{s} = \delta + d + \text{“more”}. \tag{50a}$$

More precisely, the appropriate definitions read

$$\mathfrak{s}\phi^i = d\phi^i + S^i \tag{50b}$$
$$\mathfrak{s}C^\alpha = dC^\alpha + S^\alpha \tag{50c}$$
$$\mathfrak{s}C^A = dC^A + S^A \tag{50d}$$
$$\mathfrak{s}\phi^*_i = \delta\phi^*_i + T_i \tag{50e}$$
$$\mathfrak{s}\phi^*_\alpha = \delta\phi^*_\alpha + T_\alpha \tag{50f}$$
$$\mathfrak{s}\phi^*_A = \delta\phi^*_A + T_A, \tag{50g}$$

with

$$\mathrm{antigh}(S^i) \geq \mathrm{antigh}(d\phi^i) + 1 = 1 \tag{50h}$$
$$\mathrm{antigh}(S^\alpha) \geq \mathrm{antigh}(dC^\alpha) + 1 = 1 \tag{50i}$$
$$\mathrm{antigh}(S^A) \geq \mathrm{antigh}(dC^A) + 1 = 1 \tag{50j}$$
$$\mathrm{antigh}(T_i) \geq \mathrm{antigh}(\delta\phi^*_i) + 1 = 1 \tag{50k}$$
$$\mathrm{antigh}(T_\alpha) \geq \mathrm{antigh}(\delta\phi^*_\alpha) + 1 = 2 \tag{50l}$$
$$\mathrm{antigh}(T_A) \geq \mathrm{antigh}(\delta\phi^*_A) + 1 = 3. \tag{50m}$$

The improvement terms are determined by requiring $\mathfrak{s}^2 = 0$. [Note: the extra terms in (50) are in general not of definite antighost number. The inequalities (50h–m) should therefore be understood as inequalities on the componentes of $S^i, S^\alpha, S^A, T_i, T_\alpha$ and T_A of minimum antighosts number].

7.2. Homological perturbation theory.

It turns out that it is always possible to find $S^i, S^\alpha, S^A, T_i, T_\alpha$ and T_A with the above properties such that $\mathfrak{s}^2 = 0$. This appears to be a general theorem of "homological perturbation theory" [28], a subject of algebraic topology.

Furthermore, the cohomology of $\mathfrak{s}$ is given by the cohomology of d on Σ. As anticipated, the role of the antifield is to enforce $\frac{\delta S_0}{\delta\phi^i} = 0$ when one passes to the cohomology . The operator $\mathfrak{s}$ then reduces to d and one finds $H^0(\mathfrak{s}) = H^0(d) =$ {gauge invariant functions}. More generally, one gets

$$H^g(\mathfrak{s}) = \begin{cases} 0, & \text{if } g < 0 \\ H^g(d), & \text{if } g \geq 0 \end{cases} \tag{51}$$

Here the cohomologies of $\mathfrak{s}$ and d can be either the local cohomologies, or the cohomologies in the space of all local and nonlocal functionals. In this latter case, the arguments leading to (51) are more formal. Note that the local cohomology may be non trivial for $g > 0$ even if the corresponding cohomology in the space of all functionals is trivial.

The proof of (51) which is immediate in the Abelian case for which (49) is correct is given in [**29, 27, 30, 20**] and will not be reproduced here. Rather, we will only prove the existence of $\mathfrak{s}$.

Because of (51), one can conclude that *the BRST symmetry completely incorporates gauge invariance at ghost number zero.* This is a general feature valid even if the elements R^i_α of the generating set under consideration do not form a group. The group structure is actually never used and one can thus say that the *properties of the BRST symmetry rely on a more primitive structure.* It is the author's opinion that the works on the BRST symmetry that overemphasize the group structure are sometimes misleading.

The same primitive structure is also encountered in the Hamiltonian formalism, where again, the description of the gauge invariant observables involves two steps:

first, the restriction to the "constraint surface"; second, the passage to the quotient of the constraint surface by the gauge orbits. These same ingredients lead to the same algebraic construction of the BRST symmetry **[9, 27, 30]**.[7] The techniques of homological perturbation theory were actually first rediscovered by physicists in the Hamiltonian context.

In order to prove the existence of the extra higher order terms $S^i, S^\alpha, S^A, T_i, T_\alpha$ and T_A needed to secure nilpotency, we will take advantage of one extra property of the BRST symmetry. This extra property is that the BRST transformation preserves a natural bracket structure in the space of fields, ghosts and antifields. Accordingly, rather than trying to work out individually $S^i, S^\alpha, S^A, T_i, T_\alpha$ and T_A it is more economical to construct directly the BRST generator for the BRST symmetry, which is a single function(al).

7.3. Antibracket.

What is the natural bracket structure in the Lagrangian formalism?

It is well known that the Hamiltonian Poisson bracket does not induce a Poisson bracket in the Lagrangian space I of all field histories: a physically meaningful symplectic structure is only defined on the stationary surface Σ, modulo G.

However, because the stationary surface has the property of being obtained by equating to zero the gradient $\dfrac{\delta S_0}{\delta \phi^i}$ of a single function S_0 it turns out that one can nevertheless define a (rather odd) bracket structure among the variables of the Lagrangian BRST complex. This bracket structure possesses strange features and is named "antibracket". It is very useful when developing the formalism but disappears when one fixes the gauge and does not seem to have a direct quantum-mechanical analog.

The definition of the antibracket suggests itself once it is realized that there is a remarkable symmetry between the fields and the ghosts on the one hand, and the antifields on the other hand. This symmetry, in turn, is a consequence of the fact that it is the same functional S_0 which determines both the ghost spectrum (through the gauge symmetries) and the antifield spectrum (through the Noether identities).

Taking again for definiteness a reducible gauge theory without reducibility equation on the coefficients Z^α_A one finds:

$$\begin{array}{cccccccl} -3 & -2 & -1 & 0 & 1 & 2 & & \\ \vert & \vert & \vert & \vert & \vert & \vert & \longrightarrow & \\ \phi^*_A & \phi^*_\alpha & \phi^*_i & \phi^i & C^\alpha & C^A & & \text{ghost number} \end{array} \tag{52}$$

[7]The major difference between the Lagrangian and Hamiltonian constructions lies in the bracket structure that is naturally defined. While the Lagrangian bracket ("antibracket") to be defined below does not appear to be realized quantum-mechanically, the Hamiltonian bracket ("Poisson bracket") becomes the physical, graded commutator (times $(i\hbar)^{-1}$) in the quantum theory.

So, it is natural to declare that the pairs $\phi^i, \phi^*_i; C^\alpha, \phi^*_\alpha$; and $C^A, \phi^*{}_A$ are conjugate,

$$(\phi^i, \phi^*_j) = \delta^i_j \tag{53a}$$
$$(C^\alpha, \phi^*_\beta) = \delta^\alpha_\beta \tag{53b}$$
$$(C^A, \phi^*_B) = \delta^A_B \tag{53c}$$

[Recall that the indices i, α, A stand both for a discrete index and a continuous one. Explicitly, $(\phi^i(x), \phi^*_j(y)) = \delta^i_j \delta^D(x-y)$. The expressions (53) are manifestly covariant in spacetime]. The Lagrangian antibracket (,) is extended to arbitrary functionals A, B of the fields, the ghosts and the antifields as follows,

$$\begin{aligned}(A,B) =& \frac{\delta^r A}{\delta\phi^i}\frac{\delta^l B}{\delta\phi^*_i} - \frac{\delta^r A}{\delta\phi^*_i}\frac{\delta^l B}{\delta\phi^i} \\ &+ \frac{\delta^r A}{\delta C^\alpha}\frac{\delta^l B}{\delta\phi^*_\alpha} - \frac{\delta^r A}{\delta\phi^*_\alpha}\frac{\delta^l B}{\delta C^\alpha} \\ &+ \frac{\delta^r A}{\delta C^A}\frac{\delta^l B}{\delta\phi^*_A} - \frac{\delta^r A}{\delta\phi^*_A}\frac{\delta^l B}{\delta C^A}\end{aligned} \tag{53d}$$

The striking features of the antibracket are:

i) the antibracket carries ghost number + 1, i.e.,

$$\mathrm{gh}\,((A,B)) = \mathrm{gh}\,A + \mathrm{gh}\,B + 1; \tag{54a}$$

ii) it is odd, i.e.

$$\epsilon\,((A,B)) = \epsilon_A + \epsilon_B + 1 \tag{54b}$$

iii) it obeys symmetry properties that are opposite to the usual ones,

$$(A,B) = -(-1)^{(\epsilon_A+1)(\epsilon_B+1)}(B,A). \tag{54c}$$

So, in particular,

$$(\mathrm{Boson}_1, \mathrm{Boson}_2) = (\mathrm{Boson}_2, \mathrm{Boson}_1) \tag{54d}$$
$$(\mathrm{Fermion}, \mathrm{Boson}) = -(\mathrm{Boson}, \mathrm{Fermion}) \tag{54e}$$
$$(\mathrm{Fermion}_1, \mathrm{Fermion}_2) = -(\mathrm{Fermion}_2, \mathrm{Fermion}_1). \tag{54f}$$

A further important property of the antibracket, which is an immediate consequence of its definition, is the Jacobi identity.

$$(-1)^{(\epsilon_A+1)(\epsilon_C+1)}(A,(B,C)) + \text{“cyclic”} = 0. \tag{55}$$

Also, the antibracket acts as a derivation

$$(AB,C) = A(B,C) + (-1)^{\epsilon_B(\epsilon_C+1)}(A,C)B \tag{56a}$$

$$(A,BC) = (A,B)C + (-1)^{\epsilon_B(\epsilon_A+1)}B(A,C) \tag{56b}$$

Because of (54d), it is in general not true that an arbitrary bosonic functional A has vanishing antibracket with itself. If $\epsilon_A = 0$ one may have $(A,A) \neq 0$. However, by the Jacobi identity, $((A,A),A) = 0$.

The antibracket is easily defined along exactly the same lines for more general reducible theories requiring ghosts of ghosts. We leave this problem as an exercise to the reader. One finds the same features as in the case explicitly investigated here. In particular, the conjugate to a variable A of parity ϵ_A and ghost number g_A has itself parity $\epsilon_A + 1$ and ghost number $-g_A - 1$.

7.4. The master equation.

As we mentioned above, the BRST symmetry is a canonical transformation in the antibracket. So, the BRST variation $\mathfrak{s}A$ of an arbitrary functional A is given by

$$\mathfrak{s}A = (A,S). \tag{57a}$$

On account of the parity and ghost number properties of the antibracket, the BRST generator S should be even and have ghost number zero

$$\epsilon(S) = 0 \qquad \mathrm{gh}(S) = 0. \tag{57b}$$

Furthermore, the nilpotency of $\mathfrak{s}$ is equivalent to

$$(S,S) = 0 \tag{57c}$$

as it follows from the Jacobi identity and the fact that there is no c-number of ghost number one.

The first few terms in S should generate δ and d. This means that in the expansion of S according to antighost number,

$$S = \sum_{n\geq 0} \overset{(n)}{S} \tag{58a}$$

$$\mathrm{antigh}\, \overset{(n)}{S} = n \tag{58b}$$

one should have:

$$\text{(i)} \ \overset{(0)}{S} = S_0 \tag{58 c}$$

so that $(\phi_i^*, S) = \delta\phi_i^* + \text{more}$;

$$\text{(ii)} \ \overset{(1)}{S} = \phi_i^* R^i_\alpha C^\alpha \tag{58d}$$

so that $(\phi^*_\alpha, S) = \delta\phi^*_\alpha$ + more, and $(\phi^i, S) = d\phi^i$ + more $= R^i_\alpha C^\alpha$ + more; and

$$\text{(iii)}\ \overset{(2)}{S} = \phi^*_\alpha Z^\alpha_A C^A + \text{terms non containing } \phi^*_\alpha C^A \tag{58e}$$

so that the first terms in $\delta\phi^*_\alpha$ and dC^α are appropriately reproduced.

The problem of finding the BRST symmetry $\mathfrak{s}$ – i.e., of finding the extra terms $S^i, S^\alpha, S^A, T_i, T_\alpha, T_A$ in (50) – can thus be reformulated as the problem of finding the solution S of (57c), with the boundary conditions (58). It is remarkable that the first piece in the BRST generator S is just the gauge invariant action S_0.

The equation (57c) is named the "master equation". The boundary conditions (58) define what is known as a "proper solution" [6]. These boundary conditions follow in our presentation from the form of δ (and d). The structure of the Koszul-Tate differential δ was in turn determined by the requirement $H_k(\delta) = 0$ for $k > 0$ (acyclicity of δ at order $k > 0$).

7.5. Solution of the master equation.

The solution of the master equation is derived as follows. As $\overset{(0)}{S}$ and $\overset{(1)}{S}$ are completely fixed by the boundary condition, the first question is to find $\overset{(2)}{S}$, only incompletely given by (58c). One has

$$\overset{(2)}{S} = \phi^*_\alpha(Z^\alpha_A C^A + k^\alpha_{\beta\gamma}C^\beta C^\gamma) + \phi^*_i\phi^*_j(f^{ij}_A C^A + f^{ij}_{\alpha\beta}C^\alpha C^\beta) \tag{59}$$

where $k^\alpha_{\beta\gamma}$, f^{ij}_A and $f^{ij}_{\alpha\beta}$ are unknown.

The condition $(S, S) = 0$ leads, at antighost number one, to the equation

$$2\delta\overset{(2)}{S} + \overset{(2)}{D} = 0 \tag{60a}$$

where $\overset{(2)}{D}$ is given by

$$\begin{aligned}\overset{(2)}{D} &= (\overset{(1)}{S}, \overset{(1)}{S}) = -\phi^*_i[R_\alpha, R_\beta]^i C^\alpha C^\beta = \\ &= -\phi^*_i C^\gamma_{\alpha\beta} R^i_\gamma C^\alpha C^\beta - \phi^*_i M^{ij}_{\alpha\beta}\frac{\delta S_0}{\delta\phi^j}C^\alpha C^\beta.\end{aligned} \tag{60b}$$

We have used here (17) and $(\overset{(0)}{S}, \overset{(1)}{S}) = 0$.

To prove the existence of a solution of (60a), one must check that $\overset{(2)}{D}$ is δ-closed, $\delta\overset{(2)}{D} = 0$. This is easy to do and is left to the reader. The acyclicity of δ (i.e., $H_k(\delta) = 0$ for $k > 0$) implies then that $\overset{(2)}{D}$ is also δ-exact, $\overset{(2)}{D} = -2\delta\overset{(2)}{\overline{S}}$. This equation defines $\overset{(2)}{\overline{S}}$ only up to a δ-exact term. One can use part of this ambiguity to set the

coefficient of $\phi^*_\alpha C^A$ in $\overset{(2)}{\overline{S}}$ equal to Z^α_A. With this adjustement $\overset{(2)}{\overline{S}} = \overset{(2)}{S}$, and $\overset{(2)}{S}$ indeed exists. Actually, the solution $\overset{(2)}{S}$ reads explicitly, in terms of its components $k^\alpha_{\beta\gamma}, f^{ij}_A$ and $f^{ij}_{\alpha\beta}$,

$$k^\alpha_{\beta\gamma} = \frac{1}{2} C^\gamma_{\alpha\beta} \tag{60c}$$

$$f^{ij}_A = \frac{1}{2} C^{ij}_A \tag{60d}$$

$$f^{ij}_{\alpha\beta} = -\frac{1}{4} M^{ij}_{\alpha\beta} \tag{60e}$$

where the structure functions C^{ij}_A are those that appear in (40b).

The terms (60d) and (60c) complete $\delta\phi^*_A$ and dC^α, i.e., are such that $(\phi^*_A, S) = \delta\phi^*_A +$ "higher order" and $(C^\alpha, S) = dC^\alpha +$ "higher order". This is as it should and we could have included (60c,d) as part of the boundary conditions (58e). What our analysis shows is that this is not necessary as the equations (60c,d) are in fact forced by $(S,S) = 0$ (which contains $\delta^2 = 0, d^2 \approx 0$).

Once $\overset{(2)}{S}$ is constructed, the analysis of the remaining terms in the master equation proceeds recursively along similar lines. Assume that S has been constructed up to order $n-1, (n \geq 3)$ and let

$$\overset{(n-1)}{R} = \sum_{i \leq n-1} \overset{(i)}{S}. \tag{61}$$

It is easy to check that for any A of antighost number k_a, the component of antighost number $k_a - 1$ in $(A, \overset{(n-1)}{R})$ reads

$$(A, \overset{(n-1)}{R}) = \delta A + \text{"higher orders"} \quad (n \geq 3) \tag{62a}$$

$$\operatorname{antigh} A = k_a \tag{62b}$$

$$\operatorname{antigh}(\text{"higher orders"}) > k_a - 1 \tag{62c}$$

This is because only the pieces at most linear in the ghosts in $\overset{(n-1)}{R}$ contribute to the antighost number $k_a - 1$ component of $(A, \overset{(n-1)}{R})$. This property selects $\overset{(0)}{S}, \overset{(1)}{S}$ and $\overset{(2)}{S}$ because $\overset{(k)}{S}$ for $k \geq 3$ is at least quadratic in the ghosts. One verifies that $\overset{(0)}{S}, \overset{(1)}{S}$, and the linear piece of $\overset{(2)}{S}$ indeed yields δA.

The equation $(S,S) = 0$ then reads, at antighost number $n-1$,

$$2\delta \overset{(n)}{S} + \overset{(n-1)}{D} = 0 \ . \tag{63}$$

Here $\overset{(n-1)}{D}$ is the component of antighost number $n-1$ of $(\overset{(n-1)}{R}, \overset{(n-1)}{R})$ and depends

only on the functions $\overset{(k)}{S}$, $k \leq n-1$.[8]

Now, for (63) to possess a solution $\overset{(n)}{S}$ given $\overset{(k)}{S}$ with $k \leq n-1$, it is necessary (nilpotency of δ) and sufficient (acyclicity of δ) that $\delta \overset{(n-1)}{D} = 0$. But this simply follows from the Jacobi identity $0 = ((\overset{(n-1)}{R}, \overset{(n-1)}{R}), \overset{(n-1)}{R})$ which implies $\delta \overset{(n-1)}{D} = 0$ at antighost number $n-2$.

We can thus conclude that the solution S of the master equation exists. The solution is not unique because, at each stage, one can add a δ-exact term to S. However, because of (62), this only modifies S by canonical transformation in the antibracket. Hence, the solution of the master equation with the boundary conditions (58) exists and is unique up to a canonical transformations [12]. As a result, the BRST symmetry $\mathfrak{s}A = (A, S)$ also exists.

It is an easy exercise to check that canonical transformations also enable one to pass from one generating set R^i_α to any other generating set $\bar{R}^i_\alpha$ of same dimension. The enlargement of the generating sets by adding trivial gauge transformations and increasing the ghost spectrum requires a further concept, that of "non minimal solution" and will be discussed below (section 7.8).

It should be pointed out that in general, the components $\overset{(3)}{S}$, $\overset{(4)}{S}$, ... are different from zero so that S contains multighost vertices. In the irreducible group case, only $\overset{(0)}{S}$, $\overset{(1)}{S}$ and $\overset{(2)}{S}$ are different from zero, but this is an accident not representative of the general situation. It should also be stressed that nowhere was it necessary to fix the gauge so far, and that the existence of S is global in field space, because the acyclicity of δ is a global statement. Global obstructions may be relevant when discussing gauge fixing conditions (Gribov problem) but do not afflict the gauge independent BRST symmetry in the space of the fields, the ghosts and the antifields.

7.6. Spacetime locality of S.

In order to apply the usual methods of quantum field theory, it is necessary that S be a local functional in spacetime,

$$S = \int \mathcal{L}\, d^D x \tag{64a}$$

where $\mathcal{L}$ is a function of the fields, the ghots, the antifields and their derivatives up to some finite order ("local function"). This is equivalent to

$$\overset{(n)}{S} = \int \overset{(n)}{\mathcal{L}}\, d^D x \tag{64b}$$

where the $\overset{(n)}{\mathcal{L}}$'s are also local functions.

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[8]One has $(\overset{(n-1)}{R}, \overset{(n-1)}{R}) = \overset{(n-1)}{D}$ + higher orders. The lower antighost number components of $(\overset{(n-1)}{R}, \overset{(n-1)}{R})$ vanish as the functions $\overset{(0)}{S}, \dots, \overset{(n-1)}{S}$ obey $(S,S) = 0$ up to order $n-2$.

Is it guaranteed that the equations (64a,b) hold? To investigate this question, let us assume again that the $\overset{(k)}{S}$ have been constructed up to order $n-1$ and are local functionals. [$\overset{(0)}{S}$ is clearly a local functional, as well as $\overset{(1)}{S}$ and the given piece of $\overset{(2)}{S}$ if the gauge transformations and the reducibility equations are local]. The equation (63) reads, in terms of the searched-for local function $\overset{(n)}{\mathcal{L}}$,

$$2\delta \overset{(n)}{\mathcal{L}} + \overset{(n-1)}{d} = \partial_\mu \overset{(n-1)}{k}{}^\mu \tag{65a}$$

where $\overset{(n-1)}{D} = \int \overset{(n-1)}{d} d^D x$ and where k^μ is some local current, which yields a surface term when one integrates both sides of (65). As $\overset{(n-1)}{R}$ is a local functional, $\overset{(n-1)}{D} = (\overset{(n-1)}{R}, \overset{(n-1)}{R})$ is a local functional, so that $\overset{(n-1)}{d}$ in $\overset{(n-1)}{D} = \int \overset{(n-1)}{d} d^D x$ is indeed a local function. Furthermore, because $\delta \overset{(n-1)}{D} = 0$, $\overset{(n-1)}{d}$ obeys

$$\delta \overset{(n-1)}{d} = \partial_\mu j^\mu \tag{65b}$$

The know function in (65a) is $\overset{(n-1)}{d}$, which is really determined from $\overset{(n-1)}{D}$ only up to a local divergence $\partial_\mu \beta^\mu$ but we assume that some definite choice has been made. The unknow functions are $\overset{(n-1)}{\mathcal{L}}$ and $\overset{(n-1)}{k}{}^\mu$ which should be found from (65a) knowing that (65b) holds. Once these are found, S obeys $(S,S) = \int \partial_\mu k^\mu$ with $k^\mu = \sum \overset{(i)}{k}{}^\mu$. The boundary conditions must be such that the surface term is zero.

So, the question of spacetime locality of S can be reformulated as the problem of the local homology of δ: given a local function f such that $\delta f = \partial_\mu j^\mu$ with j^μ local, is it guaranteed that $f = \delta g + \partial_\mu k^\mu$ where both g and k^μ are local?. The function f is also known to be strictly of positive pure ghost number ($\overset{(n-1)}{D}$ involves the ghosts).

It turns out that the answer to this question is positive, provided the gauge transformations $\delta_\epsilon \phi^i = R^i_\alpha \epsilon^\alpha$ obey the following local completeness condition: any local identity on the field equations $\frac{\delta S_0}{\delta \phi^i}$ can be derived from the Noether identity $\frac{\delta S_0}{\delta \phi^i} R^i_\alpha = 0$ by local means, i.e., by differentiation and algebraic manipulations (but no integration). This assumption is very mild as it appears to be always fulfilled by appropriate redefinitions of the gauge transformations if necessary.

To give an example of gauge transformations that do not obey the local completeness condition, consider electromagnetism, invariant under

$$\delta A_\mu = \partial_\mu (\Delta \epsilon). \tag{66a}$$

With appropriate boundaryconditions at spatial infinity, this parametrization of the gauge transformations is equivalent to the standard one, $\delta A_\mu = \partial_\mu \Lambda$, $\Lambda = \Delta \epsilon$, $\epsilon = \Delta^{-1} \Lambda$. The Noether identities that follow from (66) are

$$\Delta \partial_\mu \frac{\delta S_0}{\delta A_\mu} = 0 \tag{66b}$$

The identities $\partial_\mu \frac{\delta S_0}{\delta A_\mu} = 0$ cannot be derived from (66b) by local means, as one needs to invert the Laplacian. So (66a) does not obey the local completeness condition. However, the change of gauge parameters $\Delta\epsilon = \Lambda$ yields a form of the gauge transformations that obey the local completeness assumption.

One can then prove the following.

THEOREM. *If the local function f of antighost number $k > 0$ obeys $\delta f = \partial_\mu j^\mu$ (with j^μ local) and is of strictly positive pure ghost number, then*

$$f = \delta g + \partial_\mu k^\mu$$

where g and k^μ are local functions. In other words, the local cohomology of δ modulo the spacetime exterior differential d is trivial for positive antighost and pure ghost numbers.

The proof of this theorem is given in the reference [20], to which we refer. Let us simply indicate that the restriction on the pure ghost number is important. Consider for example the free particle, $S_0 = \frac{1}{2}\int dt\, \dot{q}^2$. One has one antifield q^* with $\delta q^* = \ddot{q}$ and no ghost. The function $f \equiv q^*$ obeys $\delta f = \frac{d}{dt}(\dot{q})$ but cannot be written as $\delta g + \frac{dk}{dt}$ with local and regular g, k. This does not contradict the theorem because the pure ghost number of f is zero.

If the gauge transformations are reducible, a similar local completeness assumption must be made on the reducibility functions.

The theorem guarantees the spacetime locality of the solution S of the master equation –at least if the rank of the theory, i.e., the highest n for which $\overset{(n)}{S}$ is non vanishing, is finite.

7.7. Antibracket and equivalence classes of BRST invariant observables.

Given two BRST invariant functionals A and B, the antibracket (A, B) depends only on the cohomological classes of A and B: if $A' = A + (K, S)$, $B' = B + (L, S)$, then (A, B) and (A', B') are in the same cohomological class. So, there is a well-defined antibracket structure in the space of cohomological classes of BRST invariant functions.

Because the antibracket (A, B) of two BRST invariant functions of ghost number zero possesses ghost number one, it is clear that the induced antibracket has no direct connection with the Poisson bracket that can be defined among gauge invariant observables [16,19]. Furthermore, although we have no complete proof of this property, there is some evidence that (A, B) is actually cohomologically trivial, i.e., (A, B) is BRST-exact, $(A, B) = (K, S)$. The induced structure appears thus to be completely trivial.

These are the first indications that the antibracket has no obvious physical meaning in spite of its usefulness in the construction of S.

7.8. Non minimal solutions.

The requirement $H^0(\mathfrak{s}) = \{\text{gauge invariant functions}\}$ does not completely fix $\mathfrak{s}$. Indeed, it is always possible to add to a given solution further variables that are cohomologically trivial and hence, that do not modify $H^k(\mathfrak{s})$. The uniqueness theorem given above was derived with the specific set of fields ϕ^i, C^α, C^A and antifields ϕ^*_i, ϕ^*_α, ϕ^*_A and would not apply if this set had been enlarged.

Cohomologically trivial variables can be assumed, with appropriate redefinitions, to fulfill[9]

$$\mathfrak{s}\bar{C} = \pi, \quad \mathfrak{s}\pi = 0, \quad \mathrm{gh}\,\bar{C} = \mathrm{gh}\,\pi - 1. \tag{67a}$$

The condition $\mathfrak{s}F = 0$ eliminates $\bar{C}$. The further passage to the quotient by BRST-exact functions eliminates π. So, $\bar{C}$ and π do not contribute to $H^k(\mathfrak{s})$.

If one requires a canonical action for the BRST symmetry, one must introduce antifields $\bar{C}^*$ and π^* respectively conjugate to $\bar{C}$ and π,

$$(\bar{C}, \bar{C}^*) = 1, \qquad (\pi, \pi^*) = 1 \tag{67b}$$

$$\mathrm{gh}\,\bar{C}^* = -\,\mathrm{gh}\,C - 1, \qquad \mathrm{gh}\,\pi^* = -\,\mathrm{gh}\,\pi - 1. \tag{67c}$$

The term that generates (67a) through the antibracket reads

$$\bar{C}^*\pi. \tag{67d}$$

One has $\mathfrak{s}\pi^* = \bar{C}^*$, $\mathfrak{s}\bar{C}^* = 0$ and so, the pair π^*, $\bar{C}^*$ is also cohomologically trivial.

The general solution of the master equation is given by

$$\bar{S}(\phi^i, C^\alpha, C^A, \phi^*_i, \phi^*_\alpha, \phi^*_A; \pi, \bar{C}, \pi^*, \bar{C}^*) = S(\phi^i, C^\alpha, C^A, \phi^*_i, \phi^*_\alpha, \phi^*_A) + \sum \bar{C}^*\pi \tag{68}$$

where S is the "minimal solution" described above, depending on the "minimal" set of fields ϕ^i, C^α, C^A, ϕ^*_i, ϕ^*_α, ϕ^*_A, and where π, $\bar{C}$, π^*, $\bar{C}^*$ stand for all the trivial variables that are added. The solution $\bar{S}$ containing extra variables is known as a "non minimal solution". Non minimal solutions are unique modulo canonical transformations *and* addition of cohomologically trivial pairs.

Whether extra pairs are required or not depends on the type of gauge fixing condition that is desired. This point will be illustrated on the examples below. Let us simply mention now that the usual "antighosts" are part of the non-minimal sector.

The relation between the reducible and irreducible descriptions of the same gauge symmetry becomes also clear: the corresponding S's are related by a canonical transformation and by the addition of cohomologically trivial pairs.

[9] One could associate odd variables that do not modify $H^0(\mathfrak{s})$, $k > 0$. This possibility will be not explored here.

7.9. Abelian form of S.

As we indicated in section 2.11 it is always possible to abelianize the gauge transformations. Furthermore, one can redefine the field variables $\phi^i \to \chi^i = \chi^i(\phi^j)$, $\chi^i = (\chi^{\bar{a}}, \chi^{\bar{\alpha}})$, in such a way that: (i) the first variables $\chi^{\bar{a}}$ are gauge invariant; (ii) the gauge transformations are just shifts in the last variables $\chi^{\bar{\alpha}}$. This change of variables is generically non local and full of functional subtleties which we will not address here.

The action S_0 depends only on $\chi^{\bar{a}}$ as it is gauge invariant. Together with the boundary conditions, the equations $\frac{\delta S_0}{\delta \chi^{\bar{a}}} = 0$ completely determine $\chi^{\bar{a}}$. The gauge components $\chi^{\bar{\alpha}}$ are completely arbitrary.

The fields $\chi^{\bar{a}}$ may not be all propagating (the equations $\frac{\delta S_0}{\delta \chi^{\bar{a}}} = 0$ may imply $\chi^{\bar{a}} = 0$ for some $\bar{a}$), so that the number of true degrees of freedom is in general smaller than the number of $\chi^{\bar{a}}$'s.

A complete set of gauge transformations is given by

$$\delta \chi^{\bar{a}} = 0, \quad \delta \chi^{\bar{\alpha}} = \epsilon^{\bar{\alpha}}. \tag{69a}$$

The remaining (reducible) gauge transformations can be taken to be

$$\delta \chi^i = O.\epsilon^A \tag{69b}$$

($\alpha = (\bar{\alpha}, A)$). So, one has $R^i_A = 0$ and the reducibility equations read $Z^A_B R^i_A = 0$ with $Z^A_B = \delta^A_B$.

The solution of the master equation is given by

$$S = S_0(\chi^{\bar{a}}) + \chi^*_{\bar{\alpha}} C^{\bar{\alpha}} + C^*_A C^A \tag{70}$$

where C^*_A are the antifields of ghost number -2 conjugate to the ordinary ghosts associated with the ineffective gauge transformations (69b) -and not the antifields conjugate to the ghosts of ghosts C^A. The noticeable feature of S is that it differs from the gauge invariant action $S_0(\chi_{\bar{a}})$ by manifestly cohomologically trivial terms that possess exactly the same structure as the non minimal terms in (68).

8. Path integral

8.1. Gauge invariance of master equation.

We now turn to the problem of writing down the path integral.

If we were working within the Hamiltonian formalism, there would be not much to say because all the work has been done once the BRST symmetry in Hamiltonian

form is constructed. The path integral is simply

$$\int D(\text{Hamiltonian variables}) \exp i \int [\text{"Hamiltonian kinetic term"} - H] dt,$$

where (i) *all* the variables of the Hamiltonian formalism occur in the path integral; (ii) the Hamiltonian kinetic term is the one that yields the Hamiltonian Poisson bracket among the canonical variables (momenta times time derivative of coordinates in canonical coordinates); and (iii) the Hamiltonian H is one representative in the BRST cohomological class associated with the original, gauge invariant Hamiltonian H_0. Different choices of representatives amount to different choices of gauge **[5,9]**. The path integral is formally well-defined because the action that appears in the integrand is not degenerate: the Hamiltonian equations of motion that follow from it are in normal form and hence, possess a unique solution for given initial data.

The same approach cannot be applied to the Lagrangian case, and the straightforward attempt

$$\text{Path integral} = \int D(\text{fields}) D(\text{antifields}) D(\text{ghosts}) \exp \frac{i}{\hbar} S \tag{71}$$

where S is the solution of the master equation, does not work. This is because S is gauge invariant and thus, (71) as it stands is meaningless.

The gauge invariances of the solution S of the master equation are easy to work out. Let us denote collectively the original fields ϕ^i, the ghosts C^α and all the necessary ghosts of ghosts by ϕ^A $(A = 1, \ldots, N)$. These also include the antighosts and the auxiliary fields of the non-minimal sector if any. We will refer to ϕ^A as the "fields". All the remaining variables, i.e., the antifields ϕ^*_i, ϕ^*_α, ..., $\bar{C}^*$, π^*, etc, will be denoted by ϕ^*_A[10]. Finally, let us set

$$z^a = (\phi^A; \phi^*_A), \quad a = 1, \ldots, 2N \tag{72a}$$

and

$$\epsilon(z^a) = \epsilon_a \tag{72b}$$

The antibracket can then be written as

$$(A, B) = \frac{\delta^r A}{\delta z^a} \zeta^{ab} \frac{\delta^l B}{\delta z^b} \tag{73a}$$

where the (inverse of the) "symplectic form" ζ^{ab} reads

$$\zeta^{ab} = \begin{pmatrix} 0 & \delta^A_B \\ -\delta^A_B & 0 \end{pmatrix}, \quad \zeta^{ab} = -\zeta^{ba}. \tag{73b}$$

[10] So, ϕ^*_A stands from now on for all the antifields and not just for the antifields of antighost number three associated with the reducibility equations.

In these notations, the master equation becomes

$$(S,S) = \frac{\delta^r S}{\delta z^a}\zeta^{ab}\frac{\delta^l S}{\delta z^b} = 0 \tag{74a}$$

from which one easily derives, upon differentiation with respect to z^c, that

$$\frac{\delta^r S}{\delta z^a} R^a_c = 0 \tag{74b}$$

Here, we have set

$$R^a_c = \zeta^{ab}\frac{\delta^l \delta^r S}{\delta z^b \delta z^c} \tag{74c}$$

These equations -which express that $\mathfrak{s}^2 z^c$ is zero- indicate that the functional S is gauge invariant under

$$\delta z^a = R^a_c \epsilon^c \quad \left(\Longleftrightarrow \delta z^a(x) = \int R^a_c(x,y)\epsilon^c(y)\,dy\right) \tag{75}$$

where $\epsilon^c(y)$ are arbitrary spacetime functions.

How many gauge invariances does S possess?. Superficially $2N$, which is the total number of fields and antifields. It actually turns out that the action S has less independent gauge invariances, because the matrix R^a_c defining the gauge transformations is nilpotent on-shell,

$$R^a_b R^b_c = 0 \quad \text{(when equations of motion } \frac{\delta S}{\delta z^a} = 0 \text{ hold).} \tag{76}$$

This can be seen by a further differentiation of (74b). The gauge transformations (75) are thus not all independent, there is "on-shell reducibility".

Because $R^2 \approx 0$, the number of independent gauge transformations in (75) is at most equal to N. It is actually precisely equal to N because the general solution of $Rv \approx 0$ is given by $v \approx Rt$, so that the nilpotent matrix R only contains (rank-one) two-dimensional Jordan-blocks $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, making its total rank equal to $2N/2 = N$.

Furthermore, one can also show that (75) exhausts all the gauge symmetries of S. So, the solution of the master equation possesses exactly N independent gauge transformations.

The proofs of these statements are most conveniently derived by making the canonical change of variables $z^a \to \bar{z}^a$ such that $S(\bar{z}^a)$ takes the simple form (70). This is permissible, as the matrix R^a_b transforms on-shell as a $\binom{1}{1}$ tensor under canonical transformations, so its rank properties are unchanged in $z^a \to \bar{z}^a$. One easily checks that the gauge transformations of $S(\bar{z}^a)$ are just the arbitrary shifts in the N variables -say z^A_2- that do not occur in $S(z^a) \equiv S(z^a_1)$, and that these transformations can be written as in (75) because the matrix $\dfrac{\delta^2 S}{\delta z^A_1 \delta z^A_1}$ is invertible.

It is remarkable that the solution S of the master equation contains all its gauge symmetries in the sense that these are just obtained by differentiation of S. Furthermore, for each field-antifield pair, there is one gauge symmetry. This property was the motivation of reference [6].

8.2. BRST invariance as the guiding principle for deriving the gauge fixed action.

Because the solution of the master equation is still gauge invariant, with a gauge algebraic structure that presents no obvious simplification over the original one, it might be felt that nothing has been gained in the construction and that one is exactly back to the original difficulty of writing the correct gauge fixed action, without new insight.

Something has been gained, however, and this is that we now have the BRST symmetry at our disposal. Because BRST invariance can be used as a substitute for gauge invariance, one can completely forget about the gauge symmetries and simply focus on the BRST symmetry. If one can write down a gauge fixed action that incorporates BRST invariance, then, one has also automatically incorporated in the path integral the gauge symmetry of the original action.

This is the point of view developed in the sequel. This means that it will not be attempted to devise an appropriate gauge fixing of the gauge symmetries of S along conventional lines. These gauge symmetries will cease to be of any concern from now on, and were only mentioned to point out that S is not a propagating action. By "propagating action", we mean one without gauge invariance. Our only concern will be to extract a propagating action S_ψ from S in a manner that incorporates all the properties of the BRST formalism. Because the BRST symmetry is a global invariance rather than a local gauge symmetry, one can find BRST invariant actions that are non degenerate.

8.3. Gauge fixed action.

One possibility for getting a non degenerate (= "gauge fixed") action is simply to eliminate N of the $2N$ fields/antifields z^a by means of N equations $\Omega^A(z^a) = 0$ (N "gauge conditions" for the N independent gauge invariances of S).

It turns out that the properties of the BRST formalism are preserved if one takes the functions Ω^A to be in involution, i.e.

$$(\Omega^A, \Omega^B) = 0. \tag{76a}$$

One motivation for (76a) is that these conditions are invariant under canonical transformations. This is important, as canonical transformations account for the ambiguity in S, which should be ultimately irrelevant. Another motivation is that the equations $\Omega^A = 0$ describing the gauge fixing actually involve, with (76a), a single arbitrary function, as in the Hamiltonian formalism [5,9]. This fact will be crucial in proving the independence of physical amplitudes on the choice of Ω^A.

To see that the Ω^A's involve a single function, let us solve $\Omega^A = 0$ for the

antifields (we asume that this can be done). Then, the equations $\phi^*_A - \omega_A(\phi) = 0$ are in involution iff

$$\Omega^A = 0 \quad \Longleftrightarrow \quad \phi^*_A = \frac{\delta \psi}{\delta \phi^A} \tag{76b}$$

for some function $\psi(\phi^A)$ of ghost number -1 and Grassmann parity 1,

$$\psi(\phi^A), \quad \mathrm{gh}\,\psi = -1, \quad \epsilon(\psi) = 1 \tag{76c}$$

The functional $\psi(\phi^A)$ must be local in spacetime , $\psi = \int \rho\, d^D x$, so that the antifields ϕ^*_A are local functions of the fields and their derivatives.

If one inserts (76b) inside the solution $S(\phi, \phi^*)$ of the master equation, one gets the "gauge fixed action" S_ψ

$$S_\psi = S(\phi, \phi^* = \frac{\delta \psi}{\delta \phi}) \tag{77}$$

The remaining part of these lectures will be devoted to showing that the action S_ψ correctly governs the path integral. In particular, the path integral will be proved not to depend on ψ.

Before carrying on the analysis, it is necessary to make some comments on (77):

(i) Different choices of ψ effectively correspond not only to different choices of gauge conditions but also to different ways to enforce them in the path integral (delta functions, Gaussian average...). This will be illustrated below in the case of electromagnetism.

(ii) The function(al) $\psi(\phi)$ is required to be such that S_ψ is propagating. That is, S_ψ should have no gauge invariance. [This excludes $\psi = 0$ in the case (76) where it is the antifields that are eliminated, since then S_ψ reduces to $S_0(\phi^i)$. In the path integral, the integration over the gauge directions would yield infinity, while the integration over the ghosts would yield zero. This appears to be a generic feature of bad choices of ψ. The resulting path-integral is intrinsically ill-defined -rather than giving a well-defined, incorrect, answer].

(iii) If one makes the canonical "phase" transformation $\phi^A, \phi^*_A \to \bar{\phi}^A = \phi^A$, $\bar{\phi}^*_A = \phi^*_A - \frac{\delta^r \psi}{\delta \phi^A}$, the "gauge conditions" (76b) can be written as $\bar{\phi}^*_A = 0$. The functional form of S is not invariant under the canonical transformation if ψ defines a propagating action.

(iv) It will be seen that in the path integral, one does not sum over the antifields as these no longer appear in S_ψ. The integration variables ϕ^A obey $(\phi^A, \phi^B) = 0$ and so, reference to the antibracket is completely lost.

(v) One could in principle eliminate some of the fields in favour of the corresponding antifields, i.e., solve $\Omega^A = 0$ for some of the fields. This will be illustrated below. The integration variables that are left over in that more general case are obtained by picking out, from each conjugate pair, either the field or the antifield. These integration variables have again vanishing brackets. Reference to the antibracket

is thus again lost. The symmetry between fields and antifields is not complete however, as the requirement that S_ψ is propagating forces one to keep the gauge invariant fields among the integration variables.

(vi) If some $\overset{(n)}{S}$, $n \geq 3$ are different from zero, S_ψ will contain quartic, sixtic... ghost interactions. These ghost interactions are crucial and follow from BRST invariance. They would be missed by a naive application of the Faddeev-Popov determinant method.

8.4. Gauge fixed BRST symmetry – Gauge fixed BRST cohomology.

The gauge fixed form $\bar{\mathfrak{s}}$ of the BRST symmetry is defined by

$$\bar{\mathfrak{s}}\phi^A = (\mathfrak{s}\phi^A)(\phi, \phi^* = \frac{\delta\psi}{\delta\phi}) \equiv \frac{\delta^l S}{\delta\phi^*_A}(\phi, \phi^* = \frac{\delta\psi}{\delta\phi}) \tag{78}$$

If $\mathfrak{s}\phi^A$ depends on the antifields, i.e., if S is more than linear in the antifields, the gauge fixed BRST symmetry (in Lagrangian form) *depends on the gauge fixing fermion* ψ.

We leave it to reader to check the following straightforward assertions:

(i) The gauge-fixed action is BRST invariant under (78)

$$\bar{\mathfrak{s}}S_\psi = 0 \tag{79}$$

Hence, one can define a conserved Noether charge Ω_ψ that depends in general on ψ.

(ii) The BRST variation $\bar{\mathfrak{s}}\phi^*_A$ of the antifields viewed as functions of the fields differ from $(\mathfrak{s}\phi^*_A)(\phi, \phi^* = \frac{\delta\psi}{\delta\phi})$ by equations of motion terms,

$$\bar{\mathfrak{s}}\phi^*_A = \mathfrak{s}\phi^*_A + \frac{\delta^l S_\psi}{\delta\phi^A}. \tag{80a}$$

Hence

$$\bar{\mathfrak{s}}B = \mathfrak{s}B + \frac{\delta^r B}{\delta\phi^*_A}\frac{\delta^l S_\psi}{\delta\phi^A}. \tag{80b}$$

(iii) Because of this, the gauge-fixed BRST symmetry (in Lagrangian form) is in general only on-shell nilpotent,

$$\bar{\mathfrak{s}}^2\phi^A = \text{field equations} \tag{81}$$

where the field equations in (81) are those of the *gauge-fixed action*. The right-hand side of (81) identically vanishes if and only if $\frac{\delta^2 S}{\delta\phi^A\delta\phi^B}$ is zero. For open algebras, $\frac{\delta^2 S}{\delta\phi^A\delta\phi^B} \neq 0$ and $\bar{\mathfrak{s}}^2 \neq 0$.

(iv) Because of the invariance of the action S_ψ, the surface $\frac{\delta S_\psi}{\delta \phi^A} = 0$ is left invariant under (78). One can thus define the gauge-fixed BRST cohomology as the space of equivalence classes of weakly BRST invariant functions $A(\phi)$ modulo weakly BRST exact ones,

$$\bar{\mathfrak{s}} A = \lambda^A \frac{\delta S_\psi}{\delta \phi^A} \tag{82a}$$

$$A \sim B \quad \text{iff} \quad A = B + \bar{\mathfrak{s}} C + \mu^A \frac{\delta S_\psi}{\delta \phi^A}. \tag{82b}$$

Here, "weakly" means "on the surface of $\frac{\delta S_\psi}{\delta \phi^A} = 0$".

(v) One can define a Koszul resolution $\bar{\delta}$ for the surface $\frac{\delta S_\psi}{\delta \phi^A} = 0$. The antifields $\bar{\phi}^*_A \equiv \phi^*_A - \frac{\delta \psi}{\delta \phi^A}$ can be viewed as the generators of this resolution. No further generator is needed as the equations $\frac{\delta S_\psi}{\delta \phi^A} = 0$ are independent. The relationship betweeen $(\bar{\delta}, \bar{\mathfrak{s}})$ and $(\delta, \mathfrak{s})$ is algebraically identical with the relationship between (δ, d) and $(\delta, \mathfrak{s})$. The same algebraic techniques imply therefore $H^k(\mathfrak{s}) = H^k(\bar{\mathfrak{s}})$ where $\bar{\mathfrak{s}}$ is understood to act on $\frac{\delta S_\psi}{\delta \phi^A} = 0$. Hence, the gauge-fixed BRST cohomology at ghost number zero is again given by the gauge invariant functions.

Given an element $A(\phi, \phi^*)$ in $H^k(\mathfrak{s})$, the corresponding element in $H^k(\bar{\mathfrak{s}})$ can be taken to be

$$A_\psi(\phi) = A(\phi, \phi^* = \frac{\delta \psi}{\delta \phi}). \tag{83a}$$

Indeed, the equation $(A, S) = \mathfrak{s} A = 0$ and (80) imply

$$\bar{\mathfrak{s}} A_\psi = \lambda^A \frac{\delta^l S_\psi}{\delta \phi^A}, \quad \lambda^A = \frac{\delta^r A}{\delta \phi^*_A}. \tag{83b}$$

Conversely, given a solution $\bar{A}(\phi)$ of $\bar{\mathfrak{s}} \bar{A} = \lambda^A \frac{\delta S_\psi}{\delta \phi^A}$, one can recursively construct a solution $A(\phi, \phi^*) = \bar{A}(\phi) - (\phi^*_A - \frac{\delta \psi}{\delta \phi^A})\lambda^A + O((\phi^*_A - \frac{\delta \psi}{\delta \phi^A})^2)$ of $(A, S) = 0$ using the acyclicity of $\bar{\delta}$ and the Jacobi identity for the antibracket.

(vi) When the solution of the master equation is linear in the antifields, the gauge fixed action S_ψ can be written as $S_\psi = S_0 + \mathfrak{s}\psi$. One recovers the familiar formulas of [**31**], applicable to closed algebras.

(vii) As we have just seen, the action S_ψ is not linear in the gauge fixing fermion ψ when the solution of the master equation is not linear in the antifields. Furthermore, the BRST variation $\mathfrak{s}\phi^A$ of the fields involves the antifields. Let $\tilde{\mathfrak{s}}\phi^A$

be the ϕ^*-independent component of $\mathfrak{s}\phi^A$ i.e. $\tilde{\mathfrak{s}}\phi^A = \tilde{\mathfrak{s}}\phi^A(\phi, \phi^* = 0)$, and let $\tilde{S}_\psi = S_0 + \tilde{\mathfrak{s}}\phi^A$. One finds that $\tilde{S}_\phi$ is not invariant under $\tilde{\mathfrak{s}}$. However $\tilde{\mathfrak{s}}\tilde{S}_\phi = O(\psi)$ and the non-vanishing terms in $\tilde{\mathfrak{s}}\tilde{S}_\phi$ are proportional to the functional derivatives of S_0, $\tilde{\mathfrak{s}}\tilde{S}_\phi \sim \frac{\delta S_0}{\delta \phi^i}$. Thus, by modifying $\tilde{\mathfrak{s}}\phi^i$, one can remove these terms. Following Noether lines, one then constructs recursively both $\bar{\mathfrak{s}} = \tilde{\mathfrak{s}} + O(\psi)$ and $S_\psi = \tilde{S}_\psi + O(\psi^2)$ so that $\bar{\mathfrak{s}} S_\psi = 0$. These $\bar{\mathfrak{s}}$ and S_ψ just coincide with the ones obtained by the above methods. The "Noether approach" was followed in the original work [**7**,**8**]. As our remark indicates, this approach has close connections with the method of homological perturbation theory.

8.5. Hamiltonian formulation.

The gauge fixed action S_ψ is a local functional and possesses no gauge invariance. Hence, it can be rewritten in Hamiltonian form without difficulty. If there were problems in going to the Hamiltonian formalism, this would mean -by definition- that the gauge fixation procedure has not been correctly performed and we assume that this is not so[11].

We will also assume that the original gauge invariant Lagrangian is non pathological. By this, we mean that the Lagrangian does not provide a counterexample to the Dirac conjecture [**15**,**14**], i.e., that it manifestly exhibits all the relevant gauge symmetries. Under these conditions, the Lagrangian and Hamiltonian gauge transformations are equivalent. For more information, see [**14**]. The usual Lagrangians of physical interest fall into this class. For such Lagrangians, the Lagrangian and Hamiltonian concepts of gauge invariant functions are equivalent and there is a single bracket structure defined among them. The dynamics for the gauge invariant functions is, of course, also the same in either descriptions.

Now, to any local-in-time functional A of the fields and their time derivatives up to some finite order, one can associate, by using the equations of motion if necessary, a well-defined phase space function. In particular, if one expresses the Noether charge Ω_ψ in terms of the canonical variables, one gets a phase space function with the following features: (a) Ω_ψ is off-shell nilpotent because the (Dirac) bracket $[\Omega_\psi, \Omega_\psi]$, which should be zero on-shell, does not contain the time derivatives, i.e., cannot involve the equations of motion. Thus, it must identically vanish, $[\Omega_\psi, \Omega_\psi] = 0$. (b) The canonical transformation generated by Ω_ψ starts like a gauge transformation because $\bar{s}\phi^i = R^i_\alpha C^\alpha +$ "more".

The properties (a) and (b) are just the defining properties of the Hamiltonian BRST charge. From the general theorems on the existence and uniqueness of the BRST charge in the Hamiltonian formalism, one can thus infer that Ω_ψ differs from the gauge independent BRST charge Ω constructed along Hamiltonian lines [**9**] at most by a canonical change of variables (in the Dirac bracket) and the possible addition

[11] The Hamiltonian formalims can be developed even if the Lagrangian contains higher-order time derivatives. One simply needs more conjugate pairs. Also, there could be some second class constraints in the Hamiltonian formalism. But these can be eliminated by means of the Dirac bracket method, and we assume that this has been done. The Hamiltonian formulation is then free of constraints and all the equations of motion are dynamical.

of cohomologically trivial pairs. The canonical transformation relating Ω_ψ to Ω may have a complicated, ψ-dependent structure, but it is nevertheless *canonical.*

Similarly, in each cohomological class of the gauge-fixed BRST cohomology, one can find one function $A_\psi(t)$ that involves only the fields and their independent time derivatives at time t (initial data at t). This is because one can add equations-of-motion-terms in (82b). If one rewrites $A_\psi(t)$ in terms of the canonical variables, one gets a phase space function such that $[A_\psi, \Omega_\psi] = 0$. This implies that the Hamiltonian BRST cohomology and the gauge fixed cohomology are also isomorphic. Therefore, the Hamiltonian BRST cohomology at ghost number zero is given by the gauge invariant observables, a result derived differently, along purely Hamiltonian lines, in [**9, 29, 27**].

We can thus conclude that the Lagrangian and Hamiltonian BRST formalism are equivalent for standard Lagrangians. The equivalence is revealed upon making the Legendre transformation on S_ψ -if that action is not already in first order form.

Further discussion on the comparison between the Lagrangian and Hamiltonian formulations of the BRST symmetry may be found in [**32, 33, 34**].

We can, at this point, develop the path integral formalism along two different lines.

(i) One possibility is to base the whole discussion on the Hamiltonian formalism. The path integral is then clearly related to definite expectation values of operators and yields manifestly unitary answers. The Hilbert space apparatus can be used to define what is meant by the path integral expressions.

This approach has the advantage of being self-contained -at least formally, i.e., if the operator formalism indeed exists. Furthermore, with the introduction of the conjugate momenta, which are quantum-mechanically realized as operators, off-shell nilpotency is achieved even in the open algebra case. This greatly simplifies the discussion and is a key element of the operator formulation of the quantum theory.

(ii) Another possibility is to write down directly the Lagrangian path integral in such a manner that it fulfills the following important requirement: in the abelian representation, it should reduce to a path integral over the gauge invariant degrees of freedom only. The gauge and ghost modes should decouple and drop out from the theory, which becomes manifestly equivalent to the theory in which only the gauge invariant degrees of freedom are present.

This non-Hamiltonian approach possesses a high degree of inner consistency, but it less precise than the Hamiltonain approach. For instance, as we shall see, it fails to yield the complete expression for the integration measure. This is because the measure of the gauge invariant degrees of freedom is not determined by the above requirement. The ambiguity, however, affects only terms that are of formal higher order in $\hbar$ ("quantum corrections"), but which nevertheless may play an important role. It is not unconceivable that this shortcoming could be overcome some day by non-Hamiltonian means.

We will follow here the Lagrangian lines. The Hamiltonian results are mentioned only for the purpose of providing some insight in the Lagrangian derivation.

8.6. The integration measure: the problem.

The gauge-fixed action enables one to compute transition amplitudes as Lagrangian path integrals.

$$Z_\psi = \int [D\mu] \exp \frac{i}{\hbar} S_\psi. \tag{84a}$$

Here, $[D\mu]$ is the integration measure,

$$[D\mu] = [D\phi^A]\mu \tag{84b}$$

$$\mu = \mu_0(1 + \hbar\mu_1 + \hbar^2\mu_2 + \dots). \tag{84c}$$

If desired, one can incorporate the measure in the action by exponentiating it,

$$Z_\psi = \int [D\phi^A] e^{\frac{i}{\hbar}(S_\psi + \frac{\hbar}{i} \ln \mu_0 + O(\hbar^2))} \tag{84d}$$

A correct way to determine the integration measure is to start from the Hamiltonian path integral, for which the measure is known to be the product over time of the Liouville measure $d\phi^A d\pi_A$. Here, the π_A are the momenta canonically conjugate to ϕ^A. So, one has

$$Z_\psi = \int [D\phi^A D\pi_A]\, e^{\frac{i}{\hbar} S^H_\psi} \tag{85a}$$

with

$$S^H_\psi = \int dt(\pi\dot{\phi} - H_\psi). \tag{85b}$$

If it is permissible to evaluate the integral over the momenta by the stationary phase method –and we assume this to be the case, otherwise it would not seem that the path integral can be expressed in Lagrangian form with a local measure–, one gets the expression (84) with a definite expression for the integration measure. This measure is local *in time* because of the structure of the Hamiltonian action (85b). Accordingly, the measure terms in (84d) are generically singular and formally contain $\delta(0)$.

Similarly, the quantum average of phase space observables

$$\langle A \rangle = \int [D\phi^A D\pi_A] e^{\frac{i}{\hbar} S^H_\psi} A(\phi, \pi) \tag{86a}$$

can be rewritten in Lagrangian form as

$$\langle A\rangle = \int [D\mu]\, e^{\frac{i}{\hbar}S_\psi}(A + \hbar\alpha_1 + \hbar^2\alpha_2 + \ldots). \tag{86b}$$

The corrections $\alpha_1, \alpha_2, \ldots$ to the value $A(\phi) = A(\phi, \pi = \pi(\phi))$ of A at the extremum for π are just the higher order terms in the stationary phase method and take again a definite form. For operators that are local in time, these corrections are singular ($\sim \delta(0)$).

Contrary to the Hamiltonian expressions (85a) (86a), the Lagrangian measure and the Lagrangian corrections to A are not universal and depend on the dynamics. These corrections arise because the integration over the conjugate momenta π_A may not simply amount to replacing in the Hamiltonian path integral the momenta by their classical value.

Nevertheless, something can be said about $[D\mu]$ and $\alpha_1, \alpha_2, \ldots$ on general grounds, without using the Hamiltonian formalism.

8.7. Dimensional regularization.

The simplest approach consists in using a regularization method which sets to zero the singular terms proportional to $\delta(0)$ in the local measure $[D\mu]$. Such a method exists, based on dimensional regularization. The local measure is then irrelevant [**35**] and (84a) becomes

$$Z_\psi = \int [D\phi^A]\, e^{\frac{i}{\hbar}S_\psi} \tag{87}$$

Similarly, the simplest regularization of the singular terms $\alpha_1, \alpha_2, \ldots$ proportional to $\delta(0)$ in (86b) is again to set them equal to zero. Thus, one replaces (86b) by

$$\langle A\rangle = \int [D\phi^A]\, e^{\frac{i}{\hbar}S_\psi} A. \tag{88}$$

The expression (88) is usually singular since one has dropped a singular term from $\langle A\rangle$. This singularity appears because A contains products of operators evaluated at coincident times. One regularizes these terms by splitting the times (e.g., $\dot q^2(t) \to \dot q(t+\epsilon)\dot q(t)$ and taking the limit as the times coincides [**36**]. This regularization is, as a rule, compatible with setting $\alpha_1 = \alpha_2 = \cdots = 0$ in (86b).

With these drastic regularization prescriptions, the Lagrangian path integrals (87) or (88) are completely determined. The Lagrangian methods are entirely self-contained.

It is then easy to check that (87) is the correct path integral. Fist of all, the measure is BRST invariant because its variation is proportional to $\delta(0)$ (and $\delta'(0)$ etc ...),

$$\frac{\delta(\bar{s}\phi^A)}{\delta\phi^A} \sim \int \delta(0) = 0.$$

Second, the change of variables

$$\phi^A \to \phi^A - (\bar{s}\phi^A)\mu \tag{89a}$$

where μ is not a constant parameter but a functional of the fields given by

$$\mu = \frac{i}{\hbar}(\psi' - \psi) \tag{89b}$$

shows that

$$Z_\psi = Z_{\psi'}. \tag{89c}$$

The transformation of the measure is proportional to $\frac{\delta\mu}{\delta\phi}$ and accounts for the change $\psi \to \psi'$ in the gauge fixed action.

Third, the gauge fixed BRST cohomology is also seen to be incorporated in the path integral. Indeed, one finds that the quantum average of any BRST invariant operator does not depend on ψ

$$\langle A_\psi \rangle_{S_\psi} = \langle A_{\psi'} \rangle_{S_{\psi'}}. \tag{90a}$$

Here, we have defined

$$\langle A_\psi \rangle_{S_\psi} = \int [D\phi^A]\, e^{\frac{i}{\hbar}S_\psi} A(\phi, \phi^* = \frac{\delta\psi}{\delta\phi}). \tag{90b}$$

Furthermore,

$$\langle \lambda^A \frac{\delta S_\psi}{\delta\phi^A} + \bar{s}B \rangle_{S_\psi} = 0. \tag{90c}$$

In (90), A and λ^A are assumed to be local functions. The path integral associates therefore a well-defined quantum average to any cohomological class of BRST invariant operators, i.e., to any gauge invariant operator.

An important tool in the proof of (90) is the Schwinger-Dyson equation,

$$\langle F \frac{\delta^l S_\psi}{\delta\phi^A} \rangle = -\frac{\hbar}{i} \langle \frac{\delta^l F}{\delta\phi^A} \rangle (-)^{\epsilon_F} \tag{91}$$

This equation is obtained by making a shift of integration variables in the path integral, $\phi^a \to \phi^A + \epsilon^A$ [36]. An alternative, very interesting, derivation of the Schwinger-Dyson equation based on the BRST symmetry has recently been given in [37]

From (91), it follows that

$$\langle \lambda^A \frac{\delta S_\psi}{\delta\phi^A} \rangle = 0$$

if λ^A is a local function since then $\frac{\delta\lambda^A}{\delta\phi^A}$ is singular ($\sim \delta(0)$) and hence, regularized

to zero. Similarly, the use of (91), (83b) and $\frac{\delta A}{\delta\phi^A\delta\phi^*_A} \sim \delta(0) = 0$, combined with the change of variables (89), leads to (90a). Finally, the change of variables

$$\phi^A \to \phi^A + (\bar{\mathfrak{s}}\phi^A)\epsilon, \quad \epsilon = \text{constant}$$

in

$$\int [D\phi^A] B(\phi)\, e^{\frac{i}{\hbar}S_\psi}$$

yields

$$\langle \bar{\mathfrak{s}} B\rangle_{S_\psi} \equiv \int [D\phi^A](\bar{\mathfrak{s}}B)\, e^{\frac{i}{\hbar}S_\psi}.$$

8.8. More careful incorporation of the measure.

Although the use of dimensional regularization provides a consistent and self-contained formalism, it is not always justified: $\delta(0)$ is not always equal to zero and the local measure may be important. So, one needs a formalism that handles more carefully the singular terms, without ascribing any definite value to them.

Such a formalism exists. Because of lack of space, we will not explain it here but will only report the results. We refer to **[6,13]** for the proofs of the main statements. The proofs of the properties not given in **[6,13]** are left as exercises.

The requirement of invariance of the formalism under antibracket canonical transformations enables one to describe the Lagrangian measure and the $\hbar$-corrections to BRST invariant operators in terms of the antibracket structure. One finds that the path integral is given by

$$\langle A_\psi\rangle_{S_\psi} = \int [D\phi^A]\alpha(\phi, \phi^* = \frac{\delta W}{\delta\phi}) \exp \frac{i}{\hbar} W(\phi, \phi^* = \frac{\delta W}{\delta\phi}) \tag{92a}$$

where

$$W(\phi,\phi^*) = S(\phi,\phi^*) + \hbar M_1(\phi,\phi^*) + \hbar^2 M_2(\phi,\phi^*) + \cdots \tag{92b}$$

$$\alpha(\phi,\phi^*) = A(\phi,\phi^*) + \hbar\alpha_1(\phi,\phi^*) + \hbar^2\alpha_2(\phi,\phi^*) + \cdots \tag{92c}$$

obey the equations

$$\frac{1}{2}(W,W) = i\hbar\Delta W \tag{92d}$$

$$(\alpha, W) = i\hbar\Delta\alpha. \tag{92e}$$

Here, Δ is defined by

$$\Delta\alpha = \frac{\delta^r}{\delta\phi^A}\frac{\delta^r\alpha}{\delta\phi^*_A}(-)^{\epsilon_A+1}, \quad \epsilon(\Delta) = 1 \tag{92f}$$

and one has

$$\Delta^2 = 0; \quad \Delta(\alpha, \beta) = (\alpha, \Delta\beta) - (-)^{\epsilon_\beta}(\Delta\alpha, \beta);$$
$$\Delta(\alpha\beta) = \alpha\Delta\beta + (-)^{\epsilon_\beta}(\Delta\alpha)\beta + (-)^{\epsilon_\beta}(\alpha, \beta) \tag{92g}$$

To zeroth order in $\hbar$, the equation (92 d-e) reduce to $(S, S) = 0$ and $(A, S) = 0$. The terms $M_1, M_2, \ldots$ describe the Lagrangian integration measure, while the terms $\alpha_1, \alpha_2, \ldots$ describe the "quantum corrections" to A.

The equations (92e) can be rewritten in terms of a nilpotent operator σ,

$$\sigma\alpha \equiv (\alpha, W) - i\hbar\Delta\alpha, \quad (92e) \quad \Leftrightarrow \quad \sigma\alpha = 0 \tag{93a}$$
$$\sigma^2 = 0 \tag{93b}$$

which coincides with $\mathfrak{s}$ at zeroth order in $\hbar$

$$\sigma = \mathfrak{s} + O(\hbar). \tag{93c}$$

This operator can be thought of as a quantum deformation of s, that takes into account the quantum fluctuations in the integration over the momenta. The deformation preserves nilpotency, but not the Leibnitz rule: in general, σ does not act as a a derivation.

$$\sigma(\alpha\beta) = \alpha(\sigma\beta) + (-)^{\epsilon_\beta}(\sigma\alpha)\beta - i\hbar(-)^{\epsilon_\beta}(\alpha, \beta) \tag{93d}$$

(see 92.g).
The fact that σ does not act as a derivation is not surprising, as the integration over the momenta does not preserve the product structure: the expectation value of the product is in general different from the product of the expectation values.

Provided $H^1(d) = 0$, which we will assume, one can easily show that the cohomology of σ at ghost number zero is isomorphic with the set of $\hbar$-dependent elements in $H^0(\mathfrak{s})$. Hence $H^0(\sigma)$ is also isomorphic with the set of $\hbar$-dependent gauge invariant functions. However, the correspondence between $H^0(\sigma)$ and $H^0(\mathfrak{s})$ is not universal and depends on the dynamics. Given A obeying $(A, S) = 0$, there is no natural element in $H^0(\sigma)$ associated with it. The equation (92e) alone, which just expresses "quantum BRST invariance", allows for the possibility of adding an independent gauge invariant operator at each order in $\hbar$.

Similarly, given S, the equation (92d) for W leaves the same freedom of adding a new, independent gauge invariant term at each order in $\hbar$. *The principle of BRST symmetry alone determines the Lagrangian integration measure only up to BRST invariant terms.* This appears to be the best that one can do if one does not want to analyze the detailed structure of S_ψ and A_ψ.

By making the same change of variables as in the previous section and using the Schwinger-Dyson equation, one can formally prove:

$$\langle A_\psi \rangle = \langle A_{\psi'} \rangle \tag{94a}$$

and

$$\langle \sigma\beta \rangle = 0. \tag{94b}$$

In particular, for $A = 1$, one gets again

$$Z_\psi = Z_{\psi'} \tag{94c}$$

This time, however, it is not necessary to eliminate by hand singular terms to reach (94). So, the Lagrangian path integral incorporates the quantum BRST cohomology and does not depend on the choice of gauge fixing fermion.

Finally, it should be stressed that the full Lagrangian integration measure is not invariant under the original BRST symmetry $\mathfrak{s}$ (or $\bar{\mathfrak{s}}$), even though the action S_ψ and the Hamiltonian integration measure are. The effect of the integration over the momenta amounts in general to more than just replacing the momenta by their on-shell value. There are "quantum fluctuations", which forces one to replace, in the Lagrangian path integral, $\mathfrak{s}$ by σ,

$$\begin{aligned} \mathfrak{s}\phi^A \to \sigma\phi^A &= (\phi^A, W) \\ &= \mathfrak{s}\phi^A + O(\hbar). \end{aligned} \tag{95}$$

It should be kept in mind, however, that the considerations of this section are *very formal* since the correction terms are, as a rule, divergent.

It should also be observed that the possibility of adding equation-of-motion-terms to the classical observables,

$$A(\phi) \to A(\phi) + \lambda^A(\phi)\frac{\delta S_\psi}{\delta\phi^A} \tag{A}$$

is replaced, in the quantum theory, by the possibility of adding "Schwinger-Dyson-equation-terms",

$$\alpha(\phi) \to \alpha(\phi) + \lambda^A(\phi)\frac{\delta S_\psi}{\delta\phi^A} + \frac{\hbar}{i}\frac{\delta\lambda^A}{\delta\phi^A} \tag{B}$$

While the first freedom does not modify the classical "expectation" values, the second freedom does not modify the quantum ones.

That (A) and (B) are indeed incorporated in the formalism is particularly clear in the case of systems without gauge freedom, for which one finds

$$s(F(\phi)\phi_i^*) = F(\phi)\frac{\delta S_0}{\delta\phi^i}$$

while

$$\sigma(F(\varphi)\phi_i^*) = F(\phi)\frac{\delta S_0}{\delta\phi^i} + \frac{\hbar}{i}\frac{\delta F}{\delta\phi^i}$$

so that $\langle\sigma(F\phi_i^*)\rangle = 0$ is just the Schwinger-Dyson equation.

Because the last term in the right-hand side of (B) contains $\hbar$, one can formally think of it as a "quantum correction" to (A). The freedom (A, B) has been implicitly fixed in the previous discussion by assuming that the observables $A(\phi)$ were local in time and depended only on the independent initial data (and not on their time derivatives). Once this is done, the only unknown in α, given A, is related to the integration over the momenta as analysed in section 8.6.

8.9. Invariance under canonical transformation (in the antibracket).

The canonical covariance of the formalism is straighforward and follows from the fact that the "gauge conditions" Ω^A used to eliminate the antifields are in involution, $(\Omega^A, \Omega^B) = 0$. This is a statement invariant under canonical transformations, so, the conditions $\phi^*_A = \frac{\delta\psi}{\delta\phi^A}$ in one canonical coordinate system are equivalent to the conditions $\bar{\phi}^*_A = \frac{\delta\bar{\psi}}{\delta\bar{\phi}^A}$ in any other canonical coordinate system, with, in general, a different $\bar{\psi}$.

If one rewrites the path integral in terms of the "bare" variables, one finds the same expression, with the only exception that:

(i) ψ is replaced by $\bar{\psi}$; but the physical amplitudes do not depend on ψ;

(ii) there are some Jacobian factors that modify the integration measure.

If the canonical transformation is local in spacetime, the Jacobian factor differs from unity by terms proportional to $\delta(0)$. Within the framework of dimensional regularization, these terms vanish. Therefore, the physical amplitudes take exactly the same form (87), (88) in any canonical basis. This shows in particular that all the representations of the gauge symmetry that are local in spacetime and that can be obtained from one another by local transformations are equivalent [12]. In spacetime local bases, the gauge fixed action is local, the measure is set equal to one by dimensional regularization, and one can use the usual methods of quantum field theory.

Non trivial measure factors appear when one makes non local changes of variables. To handle these, one needs to use the more careful formalism of section 8.8. The effect of the Jacobian is to modify the functional W. One can show that the equation (92d, e), with the new W, are form-invariant under canonical transformations [13]. From this property, it easily follows that the quantum averages are also invariant under canonical transformations. The same conclusions are thus reached as in the case of local transformations.

An interesting application of the invariance of the path integral under canonical transformations is obtained by going to the Abelian representation (70). It is easy to check that the solution (70) of the master equation obeys $\Delta S = 0$. Accordingly, W can be taken to differ from S by a function of the gauge invariant variables $\chi^{\bar{a}}$ only. Similarly, α can also be assumed to depend on $\chi^{\bar{a}}$ only and obeys then $(\alpha, W) = 0$, $\Delta\alpha = 0$.

To evaluate (92a) in the representation (70), one takes a gauge fixing fermion

that does not depend on the gauge invariant variables $\chi^{\bar{a}}$. With that choice, the gauge fixed action takes the form

$$S_\psi = S_0(\chi^{\bar{a}}) + \bar{S}_\psi \tag{96a}$$

where $\bar{S}_\psi$ involves only the gauge degrees of freedom and ghosts. There is complete decoupling between the gauge invariant sector and the gauge/ghost sector. The integration over these latter variables yields a factor independent of $\chi^{\bar{a}}$, and so, the path integral takes the manifestly gauge invariant and correct form

$$\langle A \rangle = \int [D\chi^{\bar{a}}]\mu(\chi^{\bar{a}})\alpha(\chi^{\bar{a}}) \exp \frac{i}{\hbar} S_0(\chi^{\bar{a}}) \tag{96b}$$

for some measure $\mu(\chi^{\bar{a}})$. This gives a justification of the formalism which is not based on the comparison with the Hamiltonian. [The integration over the gauge and ghost modes may be more tricky than what our discussion indicates, but we assume here that there is no subtlety].

Similar arguments show that cohomologically trivial pairs decouple with appropriate choices of ψ and hence, do not modify the path integral.

8.10. Zinn-Justin equation.

The antifield formalism and the master equation finds their roots in developments due to Zinn-Justin [**38**] in the context of the renormalization of Yang-Mills fields.

Let us introduce sourdes j_A and K_A for the fields and their BRST variations, and let us define

$$\begin{aligned} Z[j_A, K_A] &= \int [D\phi^A] \exp \frac{i}{\hbar}[S(\phi, K + \frac{\delta\psi}{\delta\phi}) + j_A\phi^A] \\ &= \int [D\phi^A] \exp \frac{i}{\hbar}[S_\psi(\phi) + K_A(\bar{s}\phi^A) + O(K^2) + j_A\phi^A] \end{aligned} \tag{97}$$

The sources j_A occur linearly, but the dependence on K_A is more complicated unless the gauge algebra is closed. If one constructs the effective action $\Gamma[\langle\phi\rangle, K]$ as the Legendre transform of $\frac{\hbar}{i} \ln Z$ with respect to the sources j_A, one finds

$$(\Gamma, \Gamma) = 0 \tag{98}$$

a result for the master equation (with $(\langle\phi^A\rangle, K_B) = \delta^A_B$). This form of the Ward identity was written for the first time by Zinn-Justin in the case of the Yang-Mills theory [**38**]. It is useful in the analysis of the renormalization of the theory, where the antibracket turns out to play again an important role [**39**].

8.11. Conclusions.

We have shown that the path integral incorporates the BRST cohomology and hence, gauge invariance, in a satisfactory manner. This result (i) holds even in the open algebra case, where the gauge fixed BRST symmetry $\bar{s}$ is only nilpotent modulo

field equations of the gauge fixed action; and (ii) indicates that the operator BRST cohomology at ghost number zero is isomorphic with the set of "transverse", i.e., gauge invariant, operators.

Our conclusions should, of course, be taken with a grain of salt. Formal path integral manipulations may miss importat aspects of the operator formalism (operator ordering, anomalies...) which have not been addressed at all here. A more careful analysis of these subtleties may reveal departures from the above straighforward derivations.

Lastly, we emphasize that only infinitesimal transformations are covered by the BRST formalism developed here. So, in the case of a group, BRST invariance is equivalent to invariance under the gauge transformations in the connected component of the identity, but does not imply invariance under "large" gauge transformations. In spite of this, it should be stressed, as some confusion has arisen, that the BRST transformation is *globally defined*, i.e., it is well defined everywhere in I. This is because the vector fields R^i_α representing the infinitesimal gauge transformations are also well defined everywhere.

9. EXAMPLES

9.1. Electromagnetism.

The action S_0 is

$$S_0 = -\frac{1}{4}\int F_{\mu\nu}F^{\mu\nu}\, d^Dx \tag{99a}$$

and is invariant under

$$\delta_\epsilon A_\mu = \partial_\mu \epsilon. \tag{99b}$$

The gauge transformations are irreducible.

The minimal solution of the master equation reads

$$S = -\frac{1}{4}\int F_{\mu\nu}F^{\mu\nu}\, d^Dx + \int A^{*\mu}\partial_\mu C\, d^Dx \tag{100}$$

To implement the covariant Lorentz gauge,one needs to add to (100) the non-minimal term

$$\int \bar{C}^* b\, d^Dx \tag{101}$$

where $\bar{C}$ is the antighost of ghost number minus one, b is the Takanishi-Lautrup auxiliary field and $\bar{C}^*$, b^* are the corresponding antifields.

If one takes as gauge fermion

$$\psi = -\int \bar{C}\partial^\mu A_\mu\, d^Dx \tag{102}$$

and eliminates all the antifields, the path integral becomes

$$\int [DA_\mu][DC][D\bar{C}] \prod \delta(\partial_\mu A^\mu) \exp \frac{i}{\hbar}[-\frac{1}{4}\int F_{\mu\nu}F^{\mu\nu} d^D x + \int \partial^\mu \bar{C} \partial_\mu C d^D x] \quad (102b)$$

It involves a δ-function of the gauge condition.

If, on the other hand, one adopts

$$\psi = \int \bar{C}(\frac{1}{2\beta} b - \partial^\mu A_\mu)\, d^D x \quad (103a)$$

one finds, after integration over b, the "Gaussian average" representation

$$\int [DA_\mu][DC][D\bar{C}] \exp \frac{i}{\hbar} \int d^D x\, [-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \frac{\beta}{2}(\partial^\mu A_\mu)^2 + \partial^\mu \bar{C} \partial_\mu C]. \quad (103b)$$

Finally, the temporal gauge is reached by sticking to the minimal solution and eliminating C^* and A_0 by means of $\psi = 0$, which is here permissible. One gets

$$S_\psi = -\frac{1}{4}\int F^{\mu\nu} F_{\mu\nu}\, d^D x + \int A^{*0} \partial_0 C\, d^D x \quad (104)$$

where in $F^{\mu\nu}$, A_0 is set equal to zero. The antifield A^{*0} plays the role of the usual antighost. Note that $\psi = 0$ is permissible precisely because one keeps the antifield A^{*0}. If one had eliminated all the antifields in favour of the fields, one would have obtained $S_\psi = S_0$, which leads to an ill-defined path integral.

9.2. Abelian 2-form gauge field.

The action S_0 is

$$S_0 = -\frac{1}{12} \int F_{\mu\nu\rho} F^{\mu\nu\rho}\, d^D x \quad (105a)$$

with

$$F_{\mu\nu\rho} = \partial_\mu A_{\nu\rho} + \partial_\rho A_{\mu\nu} + \partial_\nu A_{\rho\mu}. \quad (105c)$$

It is invariant uder

$$\delta A_{\mu\nu} = \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu \quad (105c)$$

The gauge transformations are now reducible: if Λ_ν is equal to $\partial_\nu \epsilon$, (105c) reduces to $\delta A_{\mu\nu} = 0$. One needs the following minimal spectrum of fields and antifields,

$$\begin{array}{cccccc} -3 & -2 & -1 & 0 & 1 & 2 \\ \hline C^* & C^{*\mu} & A^{*\mu\nu} & A_{\mu\nu} & C_\mu & C \end{array} \quad (106a)$$

The minimal solution reads

$$S = \int [-\frac{1}{12} F_{\mu\nu\rho} F^{\mu\nu\rho} + A^{*\mu\nu}(\partial_\mu C_\nu - \partial_\nu C_\mu) + C^{*\mu} \partial_\mu C]\, d^D x. \quad (106b)$$

To mimick the electromagnetic case, one first tentatively introduces antighosts $\bar{C}^\mu$ (for the gauge fixing of $A_{\mu\nu}$) and $\bar{C}$ (for the gauge fixing of C_μ, $C_\mu \to C_\mu + \partial_\mu \epsilon$), and considers the non minimal solution

$$S^{\text{non min}} = S + \int (\bar{C}^*_\mu b^\mu + \bar{C}^* b)\, d^D x \tag{107a}$$

Here, b_μ and b are auxiliary fields. The gauge fixing fermion

$$\psi = \int [\bar{C}^\mu(\partial^\nu A_{\nu\mu}) + \bar{C}\partial^\nu C_\nu]\, d^D x \tag{107b}$$

leads to

$$S_\psi = \int [-\frac{1}{12} F_{\mu\nu\rho}F^{\mu\nu\rho} + \frac{1}{2}(\partial^\mu \bar{C}^\nu - \partial^\nu \bar{C}^\mu)(\partial_\mu C_\nu - \partial_\nu C_\mu) + \partial^\mu A_{\mu\nu} b^\nu + \partial^\nu C_\nu b - \partial^\mu \bar{C} \partial_\mu C]\, d^D x. \tag{107c}$$

This cannot be the final answer, however, because: (i) the integration over b_μ yields $\delta(\partial^\nu A_{\mu\nu})$ in the path integral. This product of delta-functions contains $\delta(0)$ because the arguments $\partial^\mu A_{\mu\nu}$ are not independent $\partial^\mu \partial^\nu A_{\mu\nu} = 0$ (ii) the action (107c) is gauge invariant under $\bar{C}^\mu \to \bar{C}^\mu + \partial^\mu \Lambda$. This formally yields a "compensating" zero in the path integral.

To remedy these problems, one extends the non-minimal sector by adding the term

$$\int \eta^* \pi\, d^D x$$

with $\mathrm{gh}\,\eta^* = -1$, $\mathrm{gh}\,\pi = 1$, $\mathrm{gh}\,\pi^* = -2$, $\mathrm{gh}\,\eta = 0$. The ghost-antighost spectrum is given by

$$\begin{tikzcd}[column sep=small] & & & A_{\mu\nu} \arrow[dl] \arrow[dr] & & & \\ & & \bar{C}^\mu \arrow[dl] \arrow[dr] & & C_\mu \arrow[dl] \arrow[dr] & & \\ & \bar{C} & & \eta & & C & \end{tikzcd}$$

while the auxiliary field spectrum reads

$$\begin{tikzcd}[column sep=small] & b_\mu \arrow[dl] \arrow[dr] & \\ b & & \pi. \end{tikzcd}$$

An appropriate gauge fixing fermion is given by

$$\psi = \int [\bar{C}^\mu(\partial^\nu A_{\nu\mu}) + \bar{C}\partial^\nu C_\nu + \bar{C}^\nu \partial_\nu \eta]\, d^D x. \tag{108a}$$

The gauge fixed action is then

$$S_\psi = \int [-\frac{1}{12} F_{\mu\nu\rho}F^{\mu\nu\rho} + \frac{1}{2}(\partial^\mu \bar{C}^\nu - \partial^\nu \bar{C}^\mu)(\partial_\mu C_\nu - \partial_\nu C_\mu) + (\partial^\nu A_{\nu\mu} + \partial_\mu \eta) b^\mu + (\partial^\nu C_\nu) b - (\partial_\nu \bar{C}^\nu)\pi]\, d^D x \tag{108b}$$

The gauge freedom of the antighost $\bar{C}^{\mu}$ is now fixed. Furthermore, the integration over b_{μ} yields $\delta(\partial^{\nu} A_{\nu\mu} + \partial_{\mu}\eta)$, which is sensible [$\partial^{\mu}\partial^{\nu} A_{\nu\mu} + \partial^{\mu}(\partial_{\mu}\eta) = \Box\eta$ no longer vanishes identically. The delta functions enforce $\Box\eta = 0$, i.e., $\eta = 0$ and hence also $\partial^{\nu} A_{\nu\mu} = 0$. The arguments of the delta functions become independent with the introduction of η].

The expression (108b) has been derived by various authors along various lines **[6,40]**.

To reach a Gaussian average representation, one adds to (108a) the term

$$\int (\alpha \bar{C}^{\mu} b_{\mu} + \beta \bar{C} b + \gamma \eta \pi)\, d^{D}x \tag{109}$$

linear in the auxiliary fields.

9.3. Remark on the Gribov ambiguity.

As the previous examples indicate, the path integral contains a delta-function of the gauge conditions when the gauge-fixing fermion ψ does not depend on the auxiliary fields b. The gauge conditions are just the coefficients of the antighosts in ψ.

The class of available ψ's is much larger, however. For more complicated ψ's, the path integral does not reduce to an integral in a definite gauge. For instance, gauge fixing fermions that are linear in the auxiliary fields lead to a Gaussian average over different gauges[12]. One virtue of the BRST formalism is to incorporate these more general ψ's from the very begining since there is no a priori restriction on the choice of ψ, except that ψ should define an action without gauge invariance through $S_{\psi} = S(\phi, \phi^* = \frac{\delta\psi}{\delta\phi})$.

The important cohomological and invariance features of the BRST formalism do not depend on the existence of global sections transverse to the gauge orbits. We believe that this is a definite advantage for theories afflicted by the Gribov ambiguity, for which no such section exists. As the BRST construction nevertheless goes through in that case, the actions S_{ψ} appear to be still the correct objects to be path-integrated. The only requirement on ψ is that S_{ψ} be propagating. This may force some non trivial dependence of ψ on the auxiliary fields. It would be of interest to completely settle the issue.

The global significance of the BRST symmetry for systems with Gribov horizons has also been pointed out along different lines in **[41]**. That the Gribov ambiguity does not signal a true physical pathology is well known and has been observed earlier. Attempts to overcome the Gribov problem in the path integral may be found in **[42]**. These attempts are consistent with the BRST approach.

[12] In spite of this, we still call ψ the "gauge fixing" fermion, as its purpose is to yield a propagating action without gauge invariance.

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Appendix: Abelianization of the gauge transformations

We will prove the abelianization theorem in the finite dimensional case where functional difficulties are absent.

Let $S(q^i)$ be a function of $q^i \in \mathbb{R}^n$. Assume that the equations $\dfrac{\partial S}{\partial q^i} = 0$ are degenerate. Then $\dfrac{\partial S}{\partial q^i}$ defines a manifold Σ of dimension m, with $0 < m \leq n$. The gauge transformations are

$$\delta_\epsilon q^i = R^i_\alpha \epsilon^\alpha , \quad \delta_\epsilon S = 0 , \quad \alpha = 1, \dots , m \tag{A.1}$$

where the matrix $R^i_\alpha(q)$is of rank m.

Without loss of generality, we can assume that the coordinates $q^i = (q^a, q^\alpha)$ are locally such that R^α_β is invertible. In that case $q^\alpha = \overset{0}{q}{}^\alpha$ are good gauge conditions and the equations $\dfrac{\partial S}{\partial q^\alpha} = 0$ are consequences of the equations $\dfrac{\partial S}{\partial q^a} = 0$,

$$\frac{\partial S}{\partial q^a} R^a_\alpha + \frac{\partial S}{\partial q^\alpha} R^\alpha_\beta = 0 \quad \Rightarrow \quad \frac{\partial S}{\partial q^\alpha} = t^a_\alpha \frac{\partial S}{\partial q^a}. \tag{A.2}$$

By the regularity condition made on the action S, the functions $\dfrac{\partial S}{\partial q^a}$ can be used as first coordinates in the vicinity of $\dfrac{\partial S}{\partial q^i} = 0$. This means that the matrix $T_{ia} \equiv \dfrac{\partial}{\partial q^\alpha} \left(\dfrac{\partial S}{\partial q^a} \right)$ at the stationary point is of rank $n - m$,

$$T_{ia} \mu^a = 0 \quad \Rightarrow \quad \mu^a = 0.$$

But this condition implies in turn that T_{ab} is invertible, because $T_{\alpha a}$ can be expressed in terms of T_{ab} by means of (A.2) at $\dfrac{\partial S}{\partial q^i} = 0$.

If one fixes the gauge variables q^α, the stationary problem $\dfrac{\partial S}{\partial q^a} = 0$ determines uniquely q^a as a function of q^α, $q^a = Q^a(q^\alpha)$. By the above remark, the critical point $Q^a(q^\alpha)$ is furthermore non degenerate. Thus, using Morse lemma, one can make a q^α-dependent, invertible, smooth change of coordinates

$$q^a \longrightarrow x^a = x(q^b, q^\alpha)$$

such that S takes the "canonical form"

$$S = \eta_{ab} x^a x^b\,, \quad \eta_{ab} = \mathrm{diag}(\pm 1)$$

in the vicinity of the critical point $x^a = 0$.

The change of variables $q^a \leftrightarrow x^a$ can be extended to $q^i \leftrightarrow x^a, q^\alpha$. In the new coordinates, S does not depend on q^α. It is thus invariant under the abelian shifts $q^\alpha \to q^\alpha + \epsilon^\alpha$. This exhausts the gauge freedom as $\dfrac{\partial S}{\partial q^i} = 0$ completely determines x^a. The abelianization theorem is thereby proven.

The theorem is easily extended to the case of an action $S(q^i, \alpha^A)$ that depends not only on the dynamical variables q^i, but also, on unvaried extra variables α^A.

An alternative proof of the abelianization theorem is given in [**12,13**]

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