Line bdles on the moduli of parahoric bundles

with J. Hong

$\mathrm{Bun}_{\mathcal{G}}$ = $\boxed{\mathcal{G}}$ - bundles on X = smooth proj curve $/\mathbb{C}$

Parahoric Bruhat - Tits group (group scheme over X)

[1] CONSTANT CASE

$\mathcal{G} = G \times X$ $\qquad$ $G/\mathbb{C}$ simple & simply conn

from now on G will always denote such a group

eg: SL_r v.b. of rk r + trivial determinant

[2] PARABOLIC CASE

$p_1 \cdots p_n \in X$ $\qquad \mathring{X} = X \setminus \{p_1 \cdots p_n\}$

$\mathcal{G}|_{\mathring{X}} = G \times \mathring{X}$ $\qquad$ constant

$\mathcal{G}(\mathbb{D}_{p_i}) = G(\underbrace{\mathbb{C}[\![t_i]\!]}_{\text{local coordinate}}) = ev^{-1}(B)$

$ev : G(\mathbb{C}[\![t_i]\!]) \to G(\mathbb{C})$

$t_i \longmapsto 0$

Thm[B.T] (Bruhat - Tits) $\exists!$ smooth affine alg. group $\mathcal{G}$ over X that satisfies $\uparrow$

SL_r 

$\mathrm{Bun}_{\mathcal{G}} = \mathrm{ParBun}_G = \{ \mathcal{E} \in G\text{-bdle} + s_i \in (\mathcal{E}/B)(p_i) \}$

[3] TWISTED CASE

$\pi : \tilde{X} \to X$ $\qquad$ Γ-cover might be ramified

$\Gamma \longrightarrow \mathrm{Aut}(G)$

$\mathcal{G} = (\pi_*(\tilde{X} \times G))^{\Gamma}$

note that $\mathcal{G}|_p = G$ if p not ramified, but at ramification points $\mathcal{G}|_p \neq G$ in general!

Definition $\mathcal{G}$ over X is Parahoric BT (PBT) if

- $\mathcal{G}$ affine & smooth group scheme $/X$
- generically simply conn & simple
- fibers are connected
- $\mathcal{R}$ = "bad points of $\mathcal{G}$:= $\mathcal{G}|_{p_i}$ not reductive"

fiber has root datum which gives G

$\forall p \in \mathcal{R}$ $\quad \mathcal{G}(\mathbb{D}_p)$ is $\boxed{\text{Parahoric sub}}$ of $\mathcal{G}(\mathbb{D}_p^\times)$

$\mathrm{Spec}(\mathbb{C}((t)))$ $\quad$ $\mathrm{Spec}(\mathbb{C}[[t]])$

this means that it has the shape given as follows:

$$G(\mathbb{C}[[t^{1/m}]])^\sigma \to G(\mathbb{C}((t^{1/m})))^\sigma \xleftarrow[\cong]{\mathrm{Ad}_{t^{\frac{\lambda}{m}}}} G(\mathbb{C}((t^{1/r})))^\tau$$

$\mathcal{G}(\mathbb{D}_p) \cong G(\mathbb{C}[[t^{1/m}]])^\sigma$, $\quad \mathcal{G}(\mathbb{D}_p^\times) = G(\mathbb{C}((t^{1/r})))^\tau$

τ diagr. aut of $\tau^r = id$

$\sigma = \mathrm{Ad}_h \circ \tau \quad h \in T_{ad}^\tau$

$\sigma^m = 1 \quad h = \zeta_m^\lambda \quad \lambda \in$ cochar

NOTE the definition of parahoric subgroup given by Bruhat & Tits is different & uses buildings. However we will use this equivalent incarnation of the concept

QUESTIONS [Pappas-Rapoport ~2008]

smooth alg stack loc finite type

(I) can we describe $\mathrm{Pic}(\mathrm{Bun}_\mathcal{G})$ & (II) sections using representation theory?

main TOOL: $\boxed{\text{TWISTED CONFORMAL BLOCKS}}$

UNIFORMIZATION THM:

Assume: $\mathcal{R} \neq \emptyset$ [set of bad points] (or if not $\mathcal{R} = x \in X$)

$\mathring{X} = X \setminus \mathcal{R}$

local object

$\mathrm{Gr}_\mathcal{G}$

Thm: [Heinloth, Drinfeld-Simpson, Beauville-Laszlo]

$$\mathrm{Bun}_\mathcal{G} = \mathcal{G}(\mathring{X}) \Big\backslash \left[\prod_{p \in \mathcal{R}} \mathcal{G}(\mathbb{D}_p^\times) \Big/ \prod \mathcal{G}(\mathbb{D}_p) \right] = LG/L^+G$$

$$\left[= \prod_{p \in \mathcal{R}} \mathrm{Gr}_{\mathcal{G},p} \right]$$

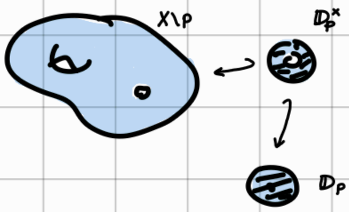
"every $\mathcal{G}$ bdle is trivializable on $\mathring{X}$ & $\amalg \mathbb{D}_p^\times$"

EX: SL_r-bdle : vb + trivial det it is trivial on $X \setminus p$ & on D_p

on Dedekind domains

f.g. + det free $\Rightarrow$ free

$\Rightarrow$ only gluing $SL_r(D_p^\times)$



i.e. $Gr_G \overset{q}{\twoheadrightarrow} Bun_G$ it is a $G(\mathring{X})$-torsor (fpqc top)

$$\mathrm{Pic}(Bun_G) = \underbrace{\text{line bdles on } Gr_G}_{\substack{\text{we'll see how these, and their global}\\ \text{sections, are described using rep thy}\\ \text{of (twisted) affine lie algebras}}} + \underbrace{G(\mathring{X})\text{-linearization}}_{\substack{\text{this is where ALL the geometry}\\ \text{appears, but since } \mathring{X} = X \setminus R \text{, we}\\ \text{got rid of the bad locus} \rightsquigarrow \text{ more}\\ \text{tractable and in fact we get}}},$$

$$q^*: \mathrm{Pic}(Bun_G) \longrightarrow \mathrm{Pic}(Gr_G)$$

PROP [DH] q^* is injective, i.e. at most a line bdle on Gr_G admits one unique linearization.

$$H^0(\mathrm{Bun}_{\mathcal{G}}, \mathcal{L}) = H^0(\mathrm{Gr}_{\mathcal{G}}, q^*\mathcal{L})^{\mathcal{G}(\mathring{X})} \overset{(*)}{=} H^0(\mathrm{Gr}_{\mathcal{G}}, q^*\mathcal{L})^{\mathrm{Lie}\,\mathcal{G}(\mathring{X})}$$

add later!

ⓐ $\mathrm{Pic}(\mathrm{Gr}_{\mathcal{G}})$ ✓ we'll see this in a second

ⓑ what gives a $\mathcal{G}(\mathring{X})$-linearization?

image of q^* — b.1 • given $\mathcal{L} \in \mathrm{Pic}(\mathrm{Bun}_{\mathcal{G}})$, it is not true that $L\mathcal{G} \curvearrowright \mathcal{L}$, but $\exists$ central ext

$$1 \to \mathbb{G}_m \to \widehat{L\mathcal{G}} \to L\mathcal{G} \to 1$$

determined by $\mathcal{L}$ & s.t. $\widehat{L\mathcal{G}} \curvearrowright \mathcal{L}$

$$\Rightarrow 1 \to \mathbb{G}_m \to \widehat{\mathcal{G}(\mathring{X})} \to \mathcal{G}(\mathring{X}) \to 1 \quad (*)$$

the bigger group $\widehat{\mathcal{G}(\mathring{X})} \curvearrowright \mathcal{L}$

a linearization is given by a splitting of $(*)$

inj of q^* — b.2 • even if $L\mathcal{G} \curvearrowright \mathcal{L}$, then we might have, a priori, more than one action of $\mathcal{G}(\mathring{X})$ on $\mathcal{L}$

$\boxed{\text{b.2}}$ PROP [DH] q^* is injective

proof: enough to show $L_{\mathring{X}}\mathcal{G}$ has no characters

<u>lem</u> on $\mathring{X} = X \setminus \mathcal{R}$

$$\mathcal{G}|_{\mathring{X}} = \pi_*(G \times \mathring{C})^{\Gamma}$$

for $\mathring{C} \xrightarrow{\Gamma} \mathring{X}$ connected étale & $\Gamma \subseteq \mathrm{Diag}(G)$

$$\mathcal{E} = \mathrm{Iso}(\mathcal{G}, G \times \mathring{X}) \supseteq \mathcal{E}^0$$

$$\begin{tikzcd} \mathcal{E} \arrow[d, "\mathrm{Aut}\,G"] & \mathcal{E}^0 \arrow[d] \\ \mathring{X} & \mathring{C} = \mathcal{E}^0/G_{ad} \arrow[l, "\Gamma"'] \end{tikzcd}$$

$= (\mathcal{E}/G_{ad})^0$

$$0 \to G_{ad} \hookrightarrow \mathrm{Aut}\,G \to \mathrm{Diag} \to 0$$

Γ = stabilizer of $(\mathcal{E}/G_{ad})^0$

[Hong-Kumar] $\pi_*(G\times \mathring{C})^\Gamma$ is an ind-integral group scheme (*)

$\Rightarrow$ only trivial characters

if $G(\mathring{X}) \xrightarrow{\chi} \mathbb{G}_m$, then $d\chi : \mathrm{Lie}(L_{\mathring{x}}G) \to \mathbb{C}$ is induced

but $[\mathrm{Lie}, \mathrm{Lie}] = \mathrm{Lie}$ & so $d\chi = 0 \Rightarrow \chi$ constant by integr.

$\mathfrak{g}$ gen simple

$\square$

$\boxed{a}$ PR describe $\mathrm{Pic}(\mathrm{Gr}_{\mathcal{G}})$ and BW give $H^0(\mathcal{L})$

$$\mathrm{Pic}(\mathrm{Gr}_{\mathcal{G}}) = \prod_{p\in R} \mathrm{Pic}(\mathrm{Gr}_{\mathcal{G},p}) \overset{[PR]}{\cong} \prod_{p\in R} \bigoplus_{i\in Y_p} \mathbb{Z}\Lambda_i$$

Parabolic $\subseteq G$ $\rightsquigarrow$ set of vertices of $I_{\mathfrak{g}}$ = dynkin diagram of Lie G

Parahoric $\subseteq G(\mathbb{C}((t^{1/n})))^\tau$ $\rightsquigarrow$ $\emptyset \neq$ set of vertices of $\hat{I}_\tau$ affine Dynkin diagr of $\hat{G}(\mathbb{C}((t^{1/n})))^\tau$

$\mathcal{G}(\mathbb{D}_p) \longmapsto Y_p$

BW : If $\vec{\Lambda} \in \mathrm{Pic}_+(\mathrm{Gr}_{\mathcal{G}})$ i.e. $\vec{\Lambda} = (\sum a_i \Lambda_i)$ $a_i \in \mathbb{N}$

$H^0(\mathrm{Gr}_{\mathcal{G}}, \mathcal{L}_{\vec{\Lambda}}) = \mathbb{H}_{\vec{\Lambda}}^*$ dual of repr of

$\bigoplus \mathfrak{g}((t^{1/n}))^\tau \oplus \mathbb{C} = \mathrm{Lie}(L\hat{G})$

Q I. When $\mathcal{L}_{\vec{\Lambda}}$ descends to $\mathrm{Bun}_{\mathcal{G}}$

Q II. Can we describe $(\mathbb{H}_{\vec{\Lambda}}^*)^{\mathrm{Lie}(\mathcal{G}(\mathring{X}))}$ more explicitly?

(Q1) $e_p : \mathrm{Pic}(Gr_{\mathcal{G},p}) \longrightarrow \mathbb{Z}$ $\qquad \Lambda_i \mapsto \check{a}_i$ dual Kac label

$(\Lambda_0 \mapsto 1$ special vertex$)$

0) 1) 2) 3)

$$\begin{tikzcd}
 & & \mathrm{Pic}\, Gr_{\mathcal{G}} = \prod(\oplus \mathbb{Z}\Lambda_i) \arrow[rr, "e"] & & \prod \mathbb{Z} & \\
1 \arrow[r] & \prod \mathrm{Char}(\mathcal{G}_p) \arrow[r] \arrow[d, equal] & \mathrm{Pic}^{\Delta} = e^{-1}(\mathbb{Z}) \arrow[u, bend left] \arrow[rr] & & c_\Delta \mathbb{Z} \arrow[r] \arrow[u, "\int_\Delta"'] & 0 \quad \text{(PR)} \\
1 \arrow[r] & \prod \mathrm{Char}(\mathcal{G}_p) \arrow[r] & \mathrm{Pic}\, \mathrm{Bun}_{\mathcal{G}} \arrow[u, "q^*", "{[\mathrm{Zhu}]}"'] \arrow[rr, "\text{induced}"'] & & c_{\mathcal{G}} \mathbb{Z} \arrow[r] \arrow[u] & 0 \quad \text{(H)}
\end{tikzcd}$$

CONJ: $\mathrm{Pic}^{\Delta} = q^* \mathrm{Pic}(\mathrm{Bun}_{\mathcal{G}})$ $\qquad c_\Delta = \mathrm{lcm}_{p \in R} \left(\mathrm{GCD}_{i \in Y_p} \check{a}_i \right)$

i.e. $c_\Delta = c_{\mathcal{G}}$

Knew already true in [1] & [2], where $c_\Delta = c_{\mathcal{G}} = 1$

THM • if $\mathcal{G} = \pi_*(G \times C)^{\Gamma \subseteq \Delta}$ then $\mathrm{Pic} = c_{\mathcal{G}}\mathbb{Z}$ and unless $\Gamma = S_3$ (only if $G = D_4$), we have that

$$c_{\mathcal{G}} = c_\Delta = \begin{cases} 2 & \text{if } G = SL_{2r+1} \text{ \& } \pi \text{ ramified} \\ 1 & \text{otherwise} \end{cases}$$

(✩) • If $0 \in Y_p$ $\forall p \in R$ (Iwahori case) and the Γ of $\mathring{C} \to \mathring{X}$ is not S_3, then $c_{\mathcal{G}} = c_\Delta = 1$

how can we prove such result?

enough to show $\exists\, \mathcal{L}$ on $\mathrm{Bun}_{\mathcal{G}}$ with $c(\mathcal{L}) = c_\Delta$

$\mathcal{L}_{\vec{\lambda}}$ on $Gr_{\mathcal{G}}$ + linearization

Q2

Take $\mathcal{L}_{\vec{\lambda}}$ on Pic^{Δ} with Λ_p dominant (i.e. $a_i \in \mathbb{N}$ at least one +)

$$\begin{tikzcd} \mathring{C} \arrow[r, "\mathring{\pi}"] \arrow[d, hook] & \mathring{X} \arrow[d, hook] \\ C \arrow[r, "\pi"] & X \end{tikzcd}$$

$\mathcal{G}|_{\mathring{X}} = \pi_*(\mathring{C} \times G)^{\Gamma}$ ⊛

$\mathcal{H} := \pi_*(C \times G)^{\Gamma} \neq \mathcal{G}$ in general

$H^0(\mathrm{Gr}_{\mathcal{G}}, \mathcal{L}_{\vec{\lambda}})^*$ is a representation of

$H^*_{\vec{\lambda}} =$

$\hat{h} = \bigoplus_{p \in R} \mathrm{Lie}(\mathcal{H})(D_p^{\times}) \oplus \mathbb{C}$] BW + constr. of $\mathcal{L}_{\vec{\lambda}}$

$\uparrow$ map of Lie Alg

$\mathrm{Lie}(\mathcal{H})(\mathring{X})$

$\mathbb{V} = \dfrac{\mathcal{H}_{\vec{\lambda}}}{\mathrm{Lie}(\mathcal{H})(\mathring{X})}$

(TWISTED) C.B. assoc with $\vec{\lambda}$ & $\mathcal{H}$ is

$$H^0(\mathrm{Gr}_{\mathcal{G}}, \mathcal{L}_{\vec{\lambda}})^{\mathrm{Lie}(\mathcal{H})(\mathring{X})} =: \underbrace{\mathbb{V}^{+}(\mathcal{H}, \vec{\lambda})_{C \to X}}$$

$\| $ ⊛

$$H^0(\mathrm{Gr}_{\mathcal{G}}, \mathcal{L}_{\vec{\lambda}})^{\mathrm{Lie}(\mathcal{G})(\mathring{X})}$$

Thm [DH, conj by PR]

not sure if family version will hold

$\|$ if $\mathcal{L}_{\vec{\lambda}}$ descends to $\mathrm{Bun}_{\mathcal{G}}$,

$$\underbrace{H^0(\mathrm{Bun}_{\mathcal{G}}, \mathcal{L})}$$

here we use that taking invariants for grp ≡ invariants under Lie Alg if you have integral & only one action of $\mathrm{Lie}(\mathcal{G})(\mathring{X})$ on H^*_{λ}

We can use this backwards actually!

(by work of Hong & Kumar $\mathbb{V}^{+}(\mathcal{H}, \vec{\lambda}) = H^0(\mathrm{ParBun}_{\mathcal{H}}, \mathcal{L})$ where $\vec{\lambda}$ determines $\mathcal{L}$ and parabolic structure)

"$\boxed{b_1}$"

Thm If $\mathbb{V}(\mathcal{H},\vec{\lambda}) \neq 0$,
then $\mathcal{L}_{\vec{\lambda}}$ descends to Bun_G

Idea: nonzero CB allows us to construct a linearization for $\mathcal{L}_{\vec{\lambda}}$. this idea is due to Sorger

repr of $H_{\vec{\lambda}}$ is integrated to proj repr of LG

$$(*)\quad \begin{tikzcd} 1 \arrow[r] & \mathbb{G}_m \arrow[r] \arrow[d, equal] & \hat{L_x^\circ G} \arrow[r] \arrow[d] & L_x^\circ G \arrow[r] \arrow[d] \arrow[ddd, bend left, "\pi"] & 1 \\ 1 \arrow[r] & \mathbb{G}_m \arrow[r] \arrow[d, equal] & \hat{LG} \arrow[r] \arrow[d] & LG \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & \mathbb{G}_m \arrow[r] \arrow[d, equal] & GL(H_{\vec{\lambda}}) \arrow[r] \arrow[d] & PGL(H_{\vec{\lambda}}) \arrow[r] \arrow[d] & 1 \\ 1 \arrow[r] & \mathbb{G}_m \arrow[r] & GL(\mathbb{V}) \arrow[r] & PGL(\mathbb{V}) \arrow[r] & 1 \end{tikzcd}$$

[we obtain this map from Faltings lemma]

and since it is a CB Lie π is zero, but then using integrability we have $\pi = id$, so factors through $\mathbb{G}_m$ & therefore $(*)$ splits

[
$$\begin{tikzcd} \mathcal{H} \arrow[r, "\alpha"] \arrow[d, "u_\gamma"'] & \mathcal{H} \arrow[d, "u_\gamma"] \\ \mathcal{H} \arrow[r, "\gamma\alpha\gamma^{-1}"] & \mathcal{H} \end{tikzcd}$$
$\forall \gamma \in L_x^\circ g$ $\exists u_\gamma$ s.t. $\forall \alpha \in \hat{\mathfrak{h}}$
and so]

$\boxed{\hat{L_x^\circ G}} = (f, g)$ $\quad g \in L_x^\circ G$ $\quad f : g^* \mathcal{L}_{\vec{\lambda}} \cong \mathcal{L}_{\vec{\lambda}}$

but mumford group of $\mathcal{L}_{\vec{\lambda}}$ wrt $L_x^\circ G$

(*) • If $0 \in Y_p$ $\forall p \in \mathcal{R}$ (Iwahori case) and the Γ of $\mathring{C} \to \mathring{X}$ is not S_3, then $C_g = C_\Delta = 1$

If we find another $\mathring{C} \to \mathring{X}$, then we can show that Γ are conjugated in $\mathrm{Diag}(G)$, so unless D_4 & $|\Gamma| = 2$, Γ is unique. But $|\Gamma|$ is given

close $\mathring{C}'$ & $\mathring{C}$ to C' & C, then $\exists\, \phi : C \to C'$ over X & $\delta \in D$ st $\Gamma = \delta \Gamma' \delta^{-1}$ $\quad \phi(\gamma \cdot p) = (\delta^{-1} \gamma \delta)\phi(p)$

generic splitting degree is an invariant

$\rightarrow$ CB are identified via this ↑

(*) want to show $\vec{\Lambda}_0 = (\Lambda_0)_{p \in \mathcal{R}}$ descends, so enough $\mathbb{V}^{\dagger}(\mathcal{H}, \vec{\Lambda}_0)_{C \to X} \neq 0$. Only case $C^0 \xrightarrow{\Gamma} X^0$ $\Gamma = \mathbb{Z}/2\mathbb{Z}$

If $C^0 \to X^0$ has $\Gamma = \mathbb{Z}/2\mathbb{Z}$, then $C \to X$ ramified exactly at 2ℓ points $(\mathcal{R}_2)$

but now we degenerate $C \to X$ into cover with base ℓ components iso $\mathbb{P}^1$ & marked by x_i, x_{i+1}

$\uparrow 2:1$ $\quad$ ramified at ↑

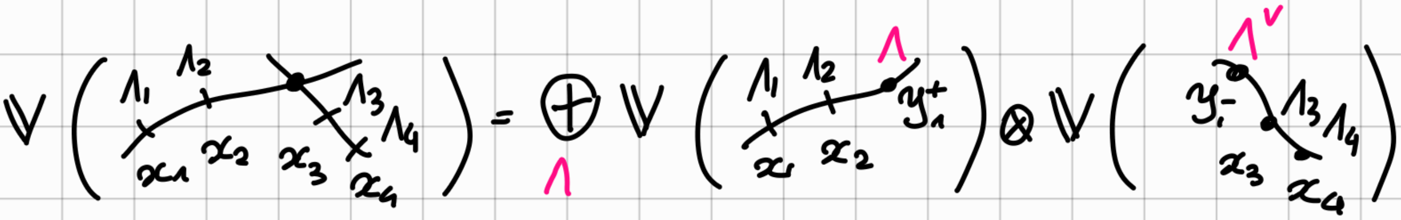
$\mathbb{P}^1$

$\mathbb{V}$ define a V.b of finite rank over moduli of Γ-cover

$$\dim \mathbb{V}(\mathcal{H}, \vec{\Lambda}_0) \boxed{=} \dim \mathbb{V}(\mathcal{H}', \Lambda_0)_{C' \to X'}$$

Factorization $\geq$ $\displaystyle \bigotimes_{i=1}^{\ell} \underbrace{\mathbb{V}\big(\pi^*(G \times \mathbb{P}^1)^{\Gamma}, (\Lambda_0, \Lambda_0)\big)}_{\text{one dimensional by Deshpande-Mukhop.}}{}_{\mathbb{P}^1 \xrightarrow{2:1} \mathbb{P}^1}$

because in general $\forall$ node, we will need to add a new $\vec{\Lambda}$

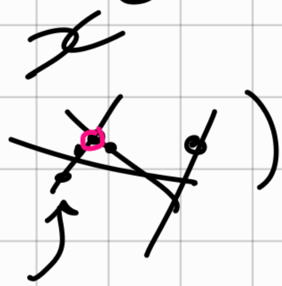


and in the above we chose $\Lambda = \Lambda_0$, which, by POV means that we can remove it

Why is this effective : CB are computable

[D,HK] (a) V.b. on stack of Γ-covers + fixed datum

so dim doesn't change if we degenerate $\tilde{X} \xrightarrow{\Gamma} X$

on degenerate cover (base is )

(b) we can factor $\mathbb{V} = \bigoplus \mathbb{V}$ on each component

$\rightsquigarrow$ if Γ is nice enough ($\mathbb{Z}/2$ or $\mathbb{Z}/3$)

we reduce covers of $\mathbb{P}^1$. • $\mathbb{P}^1 \to \mathbb{P}^1$ 2:1 or 3:1

• $E \to \mathbb{P}^1$ 3:1

In these cases explicit formulas comp. $\mathbb{V}$

[Deshpande-Mukhopadhyay, Hong-Kumar]

Parabolic of G $\rightsquigarrow$ subset of $I_{\mathfrak{g}}$ $\quad \mathfrak{g} = \mathrm{Lie}(G)$

special vertex

$\mathcal{P}_x$ parahoric sub of $G(\mathbb{C}((t^{1/n})))^{\tau}$ $\rightsquigarrow$ $\phi \neq$ subset of $\overset{Y_x}{\hat{I}^{\tau}} \ni 0$ central extension

$\hat{I}^{\tau}$ = Dynkin diagram of ${}^{\vee}\mathfrak{g}(\mathbb{C}((t^{1/n})))^{\tau}$

PR, Zhu: $\mathrm{Pic}(\mathrm{Gr}_{\underbrace{\mathcal{G},x}_{\mathcal{P}_x}}) = \bigoplus_{i \in Y_x} \mathbb{Z}\Lambda_i$

$$\mathrm{Pic}(\mathrm{Gr}_{\mathcal{G},x}) \xrightarrow{\text{central charge}} \mathbb{Z}$$

$\Lambda_i \mapsto \check{a}_i$ dual Kac labels $\in \{1,2,3,6\}$

" $\mathrm{Gr}_{\mathcal{G},x} = \mathcal{G}(\mathbb{D}_p^{\times}) / \mathcal{P}_x = \widehat{\mathcal{G}(\mathbb{D}_p^{\times})} / \mathcal{P}_x \times \mathbb{G}_m$ "

Zhu: $q^* \mathrm{Pic}(\mathrm{Bun}_{\mathcal{G}}) \subseteq \mathrm{Pic}^{\Delta}(\mathrm{Gr}_{\mathcal{G}})$ $\quad \boxed{\text{CONJ: } = \text{ holds}}$

$$c : \prod \mathrm{Pic}(\mathrm{Gr}_{\mathcal{G},x}) \longrightarrow \prod \mathbb{Z}$$

$$\mathrm{Pic}^{\Delta} = c^{-1}(\Delta(\mathbb{Z})) \qquad \mathbb{Z} \overset{\Delta}{\hookrightarrow} \prod \mathbb{Z}$$

$\rightarrow c(\mathcal{L}) = c(q^*\mathcal{L}|_x)$ well defined, $\mathcal{L} \in \mathrm{Bun}_{\mathcal{G}}$

in general $c_{\mathcal{G}} = \min(\{c(\mathcal{L}) \mid \mathcal{L} \in \mathrm{Pic}(\mathrm{Bun}_{\mathcal{G}})\ c(\mathcal{L}) > 0\})$

multiple of c_{Δ} = positive gen of image of Pic^{Δ} via c

CONJ $c_{\mathcal{G}} = c_{\Delta}$